

Project: 1D numerical modelling of melt migration in the Earth's mantle

There are various indications that the asthenosphere underneath mid-oceanic ridges is partially molten at a depth of around 50 km. The melt content here is maybe 5-10 percent. This melt leaves the partially molten zone and is focussed towards a mid oceanic ridge by some mechanism. There has been quite a bit of research over the last 3 decades to better understand the physics of this melt segregation mechanism as well as to better understand how melt is focussed towards the ridges. One of the more popular formulations use a so-called two-phase flow formulation that take into account both the solid matrix and the melt that is contained in it. In this formulation, the partially molten rock will compact slightly once the melt leaves the solid matrix (just as you squeeze a sponge after taking a bath). Here the goal is to solve the two-phase flow equations numerically and to reproduce the results of the attached paper of Vasyliov et al.

Governing equations

The full set of equations are described in Vasyliov et al. [1998]. The resulting (non-dimensional) set of non-linear equations that should be solved are

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot (\phi^n (\nabla P_e + \vec{e}_z)) \quad (1)$$

$$De \frac{\partial P_e}{\partial t} = \nabla \cdot (\phi^n (\nabla P_e + \vec{e}_z)) - \phi^m P_e \quad (2)$$

here ϕ is porosity (melt content which varies from 0-1), P_e the effective pressure, n, m are material constants, De the nondimensional Deborah number which indicates the importance of elasticity (in the mantle, $De = 0$) and \vec{e}_z the unit vector in the z -direction (the direction of the gravity vector).

Here, we will focus on solving the equations in 1-D, which means that we have to solve the following two (coupled) equations

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial z} \left(\phi^n \left(\frac{\partial P_e}{\partial z} + 1 \right) \right) \quad (3)$$

$$De \frac{\partial P_e}{\partial t} = \frac{\partial}{\partial z} \left(\phi^n \left(\frac{\partial P_e}{\partial z} + 1 \right) \right) - \phi^m P_e \quad (4)$$

Both equations are coupled, as ϕ is a nonlinear parameter in eq.4 and P_e occurs in eq. 3. Therefore, nonlinear iterations have to be performed. Moreover, since your code should also work in the case that $De = 0$, you see that equation 4 is essentially a steady-state diffusion equation, which should thus be solved in an implicit manner.

Exercise

1. Discretize and solve the equations (3-4) in the 1D case, using a finite difference discretization. Take into account the nonlinearities, that is add iterations. Assume $n = 3, m = 2$.
2. Reproduce figure 1 of Vasyliov et al., using the same initial and boundary conditions.
3. Perform simulations with an initial step-like perturbation in porosity. How do results change compared to those with an initial gaussian perturbation in porosity? Experiment with different values of n and m .
4. (Bonus question) Solve the governing equations in 2D. Does melt propagate in dikes, waves or in tubes?

Modeling of compaction driven flow in poro-viscoelastic medium using adaptive wavelet collocation method

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Abstract. Different regimes of compaction driven flow have been studied within the framework of a poro-viscoelastic medium. A single dimensionless parameter, the Deborah number De , has been identified, which enables the portrayal of the solution from the purely viscous matrix limit ($De \ll 1$) to the poro-elastic ($De \gg 1$) matrix limit. In viscous limit the evolution of a porosity disturbance (porosity wave) is governed by nonlinear convection-diffusion equation, while in the poro-elastic limit it evolves according to a Burgers-like non-linear advection equation. In both regimes porosity waves of higher amplitude propagate faster. However in the viscous limit porosity waves go through each other in soliton-like fashion, while in poro-elastic limit they coalesce and thus enhance melt segregation. The introduction of other variables, such as chemistry, would induce different responses in the flow for low and high De , allowing for diverse feedback situations.

Introduction

Compaction-driven flow of one phase (fluid) infiltrating through another phase (solid) has received great attention in geological sciences. In the area of magma dynamics most of the previous work has been concerned with the flow through a porous viscous matrix [Mc Kenzie, 1984; Scott and Stevenson, 1986; Spiegelman, 1995]. Recent work by Rubin [1993] has pointed out the importance of viscoelastic matrix in the propagation of magma cracks. Up to now, the effects of viscoelasticity have not been included in the magma segregation problem. Besides magma dynamics, the influences of viscoelasticity on flow compaction can also be important in other environmental circumstances, such as metamorphic reactive flows [Connolly, 1997], petroleum migration [Hunt, 1990], and nuclear waste dispersal [Onishi et al., 1996].

One-dimensional porosity waves (horizontal sheets) within constant viscosity matrix are unstable versus two and three dimensional perturbations [Scott and Stevenson, 1986]. This instability results in formation of propagating cylinders or spheres of high porosity, which requires two or three dimensional modeling [Khodakovskii et al., 1998; Spiegelman, 1995]. Connolly and Podladchikov [1998] in their systematic 1-D and 2-D numerical study of a simplified system of equations for viscoelastic compacting media demonstrated the

stabilization of one-dimensional waves by elasticity and strengthening upward viscosity profile. These authors also observed the evident transition from viscous to elastic mode of wave propagation, given that viscosity of the lithosphere increases towards the earth surface.

In this paper we will derive the equations for fluid extraction in a viscoelastic matrix and show how one single control parameter can be extracted for this system, which will allow the exploration of the different flow regimes, ranging from purely viscous matrix deformation to nearly elastic behavior for the matrix. We will also show the efficacy of the wavelet method in solving this problem endowed with steep gradients.

Equations for Poro-Viscoelastic Flow in Compacting Media

Let us consider a fluid flow through a viscoelastic porous matrix composed of solid grains. Conservation of mass for fluid and solid fractions is given by

$$\frac{\partial(\phi\rho_f)}{\partial t} + \nabla(\phi\rho_f\mathbf{V}_f) = 0, \quad (1a)$$

$$\frac{\partial((1-\phi)\rho_s)}{\partial t} + \nabla((1-\phi)\rho_s\mathbf{V}_s) = 0, \quad (1b)$$

where ρ_f and ρ_s are the density of fluid and solid fractions respectively and ϕ is a porosity of the media. Rheological relations are elastic bulk compressibility of fluid and solid constitutes given by

$$\frac{d\rho_f}{\rho_f} = K_f dP_f, \quad \frac{d\rho_s}{\rho_s} = K_s dP_s \quad (2)$$

and Maxwell volumetric strain-rate effective mean stress law for bulk viscoelastic rheology given by

$$\nabla(\mathbf{V}_s) = -\phi \left(\frac{P_e}{\eta} + K_\phi \frac{dP_e}{dt} \right) - (1-\phi) K_s \frac{dP_s}{dt}, \quad (3)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_s \nabla$ is the ‘‘material’’ time derivative relative to the matrix velocity, $P_e = P_{\text{tot}} - P_f$ is the effective pressure, P_{tot} is the total load, P_f and P_s are fluid and matrix pressures respectively, K_f and K_s are fluid and matrix compressibilities, K_ϕ is the pore compressibility elastic constant, and η is the bulk viscosity of the matrix which is assumed to be approximated by $\eta = \eta_0/\phi^{m-1}$, where η_0 is a constant.

Additional relationships are stress equilibrium condition for fluid phase, described by Darcy’s law

$$\phi(\mathbf{V}_f - \mathbf{V}_s) = -\frac{\kappa}{\mu_f} (\nabla P_f + \rho_f g \vec{e}_z) \quad (4)$$

and stress equilibrium condition for solid matrix $\phi P_f + (1-\phi)P_s = P_{\text{tot}}$, where \vec{e}_z is the unit vector along upward directed z -axis, μ_f is fluid shear viscosity, which is

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assumed to be constant, g is the gravity acceleration, and κ is the matrix permeability which is assumed to be approximated by $\kappa = \kappa_0 \phi^n$, where κ_0 is a constant. There is a considerable variability in the literature concerning the choice for power-law exponents m and n . In this paper we assume $n = 3$ and $m = 2$.

Substituting Eqn. (2) into (1b) and then using (3) we obtain the following equation for evolution of porosity:

$$\frac{d\phi}{dt} = -\phi(1-\phi) \left(\frac{P_e}{\eta} + K_\phi \frac{dP_e}{dt} - K_s \frac{dP_s}{dt} \right) \quad (5)$$

Substituting (2) into (1a) and using (4) and (5) we obtain the following evolution equation

$$\begin{aligned} \frac{dP_f}{dt} = B \frac{dP_{\text{tot}}}{dt} + \frac{1}{Q} \left[\nabla \left(\frac{\kappa}{\mu_f} (\nabla P_f + \rho_f g \vec{e}_z) \right) + \phi \frac{P_e}{\eta} \right] \\ + \frac{K_f \kappa}{Q \mu_f} (\nabla P_f + \rho_f g \vec{e}_z) \nabla P_f, \end{aligned} \quad (6)$$

where $Q = \phi(K_f + K_\phi)$ is effective storativity and $B = \frac{K_\phi}{K_f + K_\phi}$ is the Biot's constant. The above two equations represent the full problem formulation. A limiting form of these equations can be found in [Connolly and Podladchikov, 1998].

Eqns. (5) and (6) contain many different physical mechanisms and for simplification we make the following assumptions:

1. porosity is small, *i.e.* $\phi \ll 1$,
2. total pressure is isostatic, *i.e.* $P_{\text{tot}} = p_0 - \rho_s g z$,
3. compressibility of fluid and solid fractions is negligible,
4. convection of solid matrix is negligible, *i.e.* $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_s \nabla \approx \frac{\partial}{\partial t}$.

Under these assumptions the following set of equations describing the evolution of porosity ϕ and effective pressure P_e is obtained:

$$\frac{\partial \phi}{\partial t} = -\phi \frac{P_e}{\eta} - Q \frac{\partial P_e}{\partial t}, \quad (7a)$$

$$\frac{\partial P_e}{\partial t} = \frac{1}{Q} \left[\nabla \left(\frac{\kappa}{\mu_f} (\nabla P_e + \Delta \rho g \vec{e}_z) \right) - \phi \frac{P_e}{\eta} \right], \quad (7b)$$

where $\Delta \rho = \rho_s - \rho_f$ and $Q = \phi K_\phi$, which is assumed to be constant.

Equations (7) are written for dimensional quantities. There are two natural choices for scaling the effective pressure. One is based on scaling using compressibility Q and is a good choice in the case when poro-elastic effects are dominating. The second one, which is used in this research, is based on balancing of the buoyant and viscous forces and is effective in the viscous limit. Thus the following characteristic scales are chosen: porosity ϕ^* , effective pressure $P_e^* = \Delta \rho g L^*$, time $t^* = \eta_0 / (P_e^* \phi^{*m-1})$, and viscous compaction length $L^* = (\kappa_0 \eta_0 \phi^{*n-m} / \mu_f)^{1/2}$. Using this nondimensionalization equations (7) can be rewritten as

$$\frac{\partial \phi}{\partial t} = -\nabla \left(\phi^n (\nabla P_e + \vec{e}_z) \right), \quad (8a)$$

$$De \frac{\partial P_e}{\partial t} = \nabla \left(\phi^n (\nabla P_e + \vec{e}_z) \right) - \phi^m P_e, \quad (8b)$$

where $De = Q \Delta \rho g L^* / \phi^*$ is the Deborah number used in viscoelastic applications (*e.g.* [Connolly and Podladchikov,

1998]), which determines the character of flow dynamics, ranging from viscous regime ($De \ll 1$) to the poro-elastic regime ($De \gg 1$). For realistic Earth parameters, De may lie between 10^{-2} to 10^2 with corresponding time scales between 10^{-1} and 10^2 Myrs and spatial scales between 5 and 500 km.

An alternative way to understand the importance of Deborah number is to inspect the ratio of time scales T_ϕ and T_{P_e} associated with evolution of porosity and pressure disturbances respectively. The porosity time scale T_ϕ does not depend on value of De and is always of the order $O(1)$, while the effective pressure time scale T_{P_e} changes with De and is $O(De)$. Thus the dynamics of porosity pressure interactions is different for low and large De regimes. In the case of low De porosity disturbance dominates the dynamics of the evolution, while pressure adjusts to the changes in porosity on the time scale $O(De)$ via nonlinear diffusion mechanism shown in Eqn. (8b). In other words, for low values of De pressure behaves as a slave variable to porosity evolution. In the case of $De = 0$ pressure reflects changes instantaneously and the system of equations (8) reduces to the a similar form of the viscous equations used by Khodakovskii *et al.* [1995]. In the limit of large values of De , T_{P_e} is considerably larger than T_ϕ . In most applications we are interested in the evolution of porosity disturbance, and thus the time scale of interest is T_ϕ . However due to pressure porosity coupling in Eqns. (8a,8b) the effective pressure participates in the dynamics of evolution and should also change on the same time scale. This can only occur when the effective pressure is of the order $O(1/De)$. A larger pressure perturbation results in redistribution of porosity which compensate for pressure gradient. The dynamics of time evolution of porosity perturbation is different for viscous and poro-elastic regimes. In the case of low De the porosity follows a non-linear convective-diffusion equation, while in the high De regime the evolution of porosity is described by Burgers-like nonlinear advection equation. For $De \sim 1$, both time scales are comparable and viscous and poro-elastic effects are nonlinearly coupled.

Problem Formulation

For the purpose of simplicity we consider the one-dimensional problem. In this case, the system of equations (8) becomes:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial z} \left(\phi^n \left(\frac{\partial P_e}{\partial z} + 1 \right) \right), \quad (9a)$$

$$De \frac{\partial P_e}{\partial t} = \frac{\partial}{\partial z} \left(\phi^n \left(\frac{\partial P_e}{\partial z} + 1 \right) \right) - \phi^m P_e. \quad (9b)$$

As it can be easily seen the solution $\phi = const$ and $P_e = 0$ is steady state solution of this equation. At time $t = 0$ we introduce local porosity and pressure perturbations at locations z_1 and z_2 , which results in the following initial conditions:

$$\phi(z, 0) = \phi_0 + \Delta \phi_1 f(z, z_1) + \Delta \phi_2 f(z, z_2), \quad (10a)$$

$$P_e(z, 0) = -\Delta P_{e1} f(z, z_1) - \Delta P_{e2} f(z, z_2), \quad (10b)$$

where $f(z, z_k) = \exp\left(-\frac{(z-z_k)^2}{\lambda^2}\right)$. The choice for negative effective pressure perturbation is motivated by the fact that chemical reactions can generate regions of high porosity with high negative effective pressure [Connolly, 1997]. We have

imposed initial disturbances with different amplitudes in order to study their subsequent interaction in the nonlinear evolution. The Deborah number dependence of time evolution of the high porosity region is studied by varying De in the range $10^{-2} - 10^2$ while fixing the initial conditions. In this paper we limit our consideration to the following choice of the parameters: $\phi_0 = 1$, $\Delta\phi_1 = 8$, $\Delta\phi_2 = 1$, $\Delta P_{e1} = \Delta\phi_1/De$, $\Delta P_{e2} = \Delta\phi_2/De$, $z_1 = 0$, $z_2 = 40$, $\lambda = 1$.

The system (9) with initial conditions (10) is solved numerically using a highly accurate dynamically adaptive wavelet collocation algorithm developed by Vasilyev and Paolucci [1996]. The multilevel structure of the algorithm provides a simple way to adapt computational refinements to local demands of the solution. High resolution computations are performed only in regions where singularities or sharp transitions occur. This property of the multilevel wavelet approximation allows local grid refinement up to an arbitrary small scale without a drastic increase of the number of grid points (e.g. 1-D case [Vasilyev and Paolucci, 1996] and 2-D case [Vasilyev and Paolucci, 1997; Vasilyev et al., 1997]). For details of the algorithm we refer to [Vasilyev, 1996] and [Vasilyev and Paolucci, 1997].

Results

The results have been obtained by using the adaptive wavelet collocation method. The correlation function of Daubechies scaling function of order five [Beylkin and Saito, 1993] was employed with the threshold parameter of 10^{-4} , which means that the local relative error of the solution is everywhere less than 10^{-4} .

Fig. 1 shows the spatial and temporal evolution of the porosity and pressure disturbances for a range of De values ranging from 10^{-2} to 10^2 . The time evolution of the initial disturbances is very different in viscous and poro-elastic limit. In viscous limit initial localized disturbance evolves in isolated structure (porosity wave) that travels with constant speed. In poro-elastic limit initial localized disturbance evolves in saw-tooth shape porosity wave, the steepness of which increases with the increase of De . The front of the wave travels faster than the tail, which results in constant widening of the wave and decrease of its amplitude since the fluid mass of the wave remains constant. In both viscous and poro-elastic regimes higher amplitude porosity waves propagate faster. However, the dynamics of the wave interaction is quite different. In the viscous limit (low De) porosity waves go through each other in soliton-like fashion, which is consistent with the earlier observation of Barcion and Richter [1986]. In poro-elastic limit porosity waves coalesce. It has important implications in the context of the melt segregation from the source rocks. Indeed, viscoelastic waves are capable of accumulating melt distributed in small patches ending up with one big pocket of high melt fraction, i.e. viscoelastic waves are effective in the melt segregation. In contrast with viscoelastic waves, viscous soliton-like waves do not accumulate melt at all. However viscous waves may detach from the source and are able to transport the melt over significant distances since they do not leave the melt in the tail behind them.

It should be noted that the poro-viscoelastic wave propagation is highly nonlinear phenomenon. The numerical complexity increases with the increase of porosity contrast. Additional numerical difficulties are associated with increasing

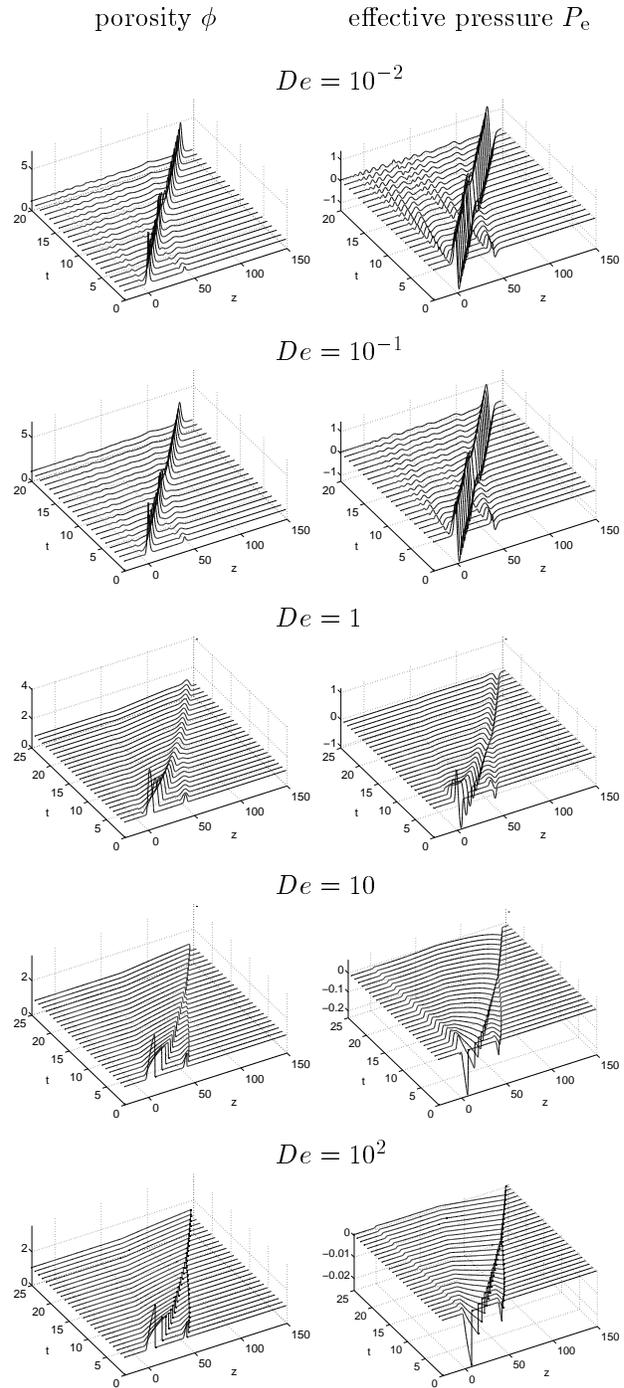


Figure 1. Spatial (z -axis) and temporal (t -axis) evolution of porosity ϕ (left column) and effective pressure P_e (right column) disturbances for Deborah number De ranging from 10^{-2} to 10^2 .

the stiffness of the problem for low values of De and appearance of sharp moving fronts for large values of De . The efficient resolution of shock like structures requires a robust method such as wavelet based numerical algorithm, which can handle sharp gradients without drastically increasing the number of grid points. For example, present calculations at the highest resolution were using less than 300 wavelets,

which is equivalent to an equally spaced finite difference grid of 32,768 or 2^{15} points. Finally, we note that the ripples observed in Fig.1 for the small De case are not a numerical artifact, but they correspond to the train of solitary waves with decreasing amplitude.

Concluding Remarks

We have analyzed and provided a rational way of non-dimensionalizing the equations for the extraction of fluids from porous viscoelastic matrix. One dimensionless parameter De has been found to govern this set of equations. Two different regimes characterizing the behavior of the solutions for porosity and pressure have been identified. For the low De regime the solution is dominated by the non-linear diffusive character of the porosity distribution, while for the high De regime porosity behaves similar to the shock-wave solution of a non-linear Burgers' equation. In viscous limit (low De) the dynamics of porosity wave evolution is solely controlled by the porosity distribution while pressure behaves as a slave variable. For De of order unity, the porosity and the pressure solutions are nonlinearly coupled and this regime is extremely interesting, since it is of direct relevance in some geological situations [Connolly and Podladchikov, 1998]. The introduction of other variables into this nonlinear system, such as chemical species, would have different influences for all regimes, since chemical reactions, which differ significantly with pressure, may not only change the energy balance but may become a porosity source, depending on the environmental circumstances.

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