6 Nonlinear problems

All of the equations we considered sofar have been linear partial differential equations, which means that coefficients in the equations are either constant (or spatially variable), but are independent on the result of the equation itself. If the coefficients are dependent on the result of the equation, we call it a nonlinear problem.

There are a number of ways to solve such nonlinear problems. The easiest way, which works in many cases is to replace the nonlinear PDE by a linear one and perform iterations until the solution converges (also called Picard iterations). Other tricks exist of which the most important is linearization of the nonlinear terms and solving the (more complicated) PDE (the keyword here is Newton-Rhapson iterations). This method is more robust and converges quadratically. I's however way more complicated to implement and will therefore not be discussed here.

To illustrate the problem, we consider a case of fluid flow in a porous media (governed by the Darcy equation) whose diffusivity κ is a function of the fluid pressure (high fluid pressure creates it's own permeability). In 1-D the governing equation is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(P)\frac{\partial P}{\partial x}\right) \tag{1}$$

where P is the fluid pressure [Pa], and $\kappa(P)$ the hydraulic diffusivity [m^2s^{-1}]. The equation is nonlinear because the diffusivity is a function of the fluid pressure P, which is related to the effect of dilation and cracking under enhanced fluid pressure. Let's assume that the hydraulic diffusivity is given by

$$\kappa(P) = \kappa_0 + cP^m \tag{2}$$

where κ_0 is the background diffusivity, c a constant and m a power exponent.

Discretization is done similar to the 1-D thermal diffusion problem with an implicit scheme:

$$\frac{P^{n+1} - P^n}{\Delta t} = \frac{\kappa_{i+1/2}^{n+1}\frac{P_{i+1}^{n+1}-P_i^{n+1}}{\Delta x} - \kappa_{i-1/2}^{n+1}\frac{P_i^{n+1}-P_{i-1}^{n+1}}{\Delta x}}{\Delta x} \tag{3}$$

where

$$\kappa_{i\pm 1/2}^{n+1} = \frac{\kappa_i^{n+1} + \kappa_{i\pm 1}^{n+1}}{2} \tag{4}$$

If, additionally, we have zero flux boundary conditions on top and bottom, we can arrange the equations in matrix form

$$\mathbf{Ap} = \mathbf{rhs} \tag{5}$$

and solve for P^{n+1}. The problem, however, is that κ depends on P^{n+1}. Therefore we have to perform iterations. The general recipe is

1. Use the pressure P^n to compute the diffusivities $\kappa_{i\pm 1/2}^{n+1}$ using equations 2 and 4.
2. Determine the coefficients in $\mathbf{A}$ using the estimated diffusivities.
3. Solve the system of equations to find the new pressure P^{n+1}.
4. Use this new pressure estimate to recalculate diffusivities and the coefficients in $\mathbf{A}$.
5. Return to step 2 and continue until the pressure P_{n+1} stops changing, whic indicates that the solution has converged. Use as an indication of convergence the following error estimate:

$$error = \frac{max(abs(\mathbf{p}^{it} - \mathbf{p}^{it-1}))}{max(abs(\mathbf{p}^{it}))} \tag{6}$$

If convergence is reached (i.e. $error < 1e-4$), continue to the next timestep.

6.0.1 Exercise

- Write a program that solves the equations described above. Take as values $\kappa_0 = 10^{-4}$ m2/s, $c = 1e - 8$, $m = 2$. The initial pressure perturbation is 10 MPa, and the model domain is on the order of 10 km. Compare the nonlinear solutions to the linear ones, obtained by setting $c = 0$. Create an initial step-like profile in pressure on the LHS of the model.
- Bonus question: write a 2D version, think of an application and submit a paper to Nature.