Does a rotation by an energy dependent overall phase change the pole-positions in a partial wave?

- Some formal aspects -

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L+P-parametrization and energy dependent overall phase

- Consider partial wave T-matrix-element $T(W)$, for example in <u>L + P - parametrization:</u>

$$T(W) = \underbrace{\sum_{j=1}^{N} \frac{a_{-1}^{(j)}}{W_j - W}}_{\uparrow \text{poles}} + \underbrace{B(W)}_{\uparrow \text{composed of Pietarinen-functions}}$$

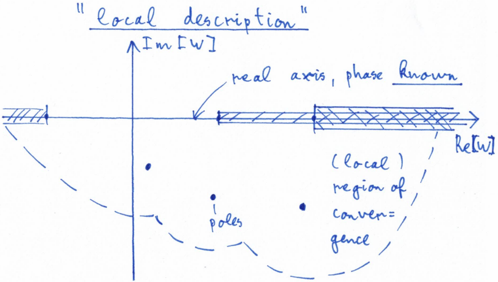
$$Z(W) := \frac{\alpha - \sqrt{x_P - W}}{\alpha + \sqrt{x_P - W}} \, ; \; B(W) = \sum_{n=0}^{N_1} c_n Z^n(W) + \ldots$$

- For <u>real $W_r \in \mathbb{R}$:</u>

$$T(W_r) = |T(W_r)| \, e^{i\phi(W_r)},$$

So, $e^{i\phi(W)}$ is given on the real axis as:

$$e^{i\phi(W_r)} = \frac{T(W_r)}{|T(W_r)|}$$



Toy model

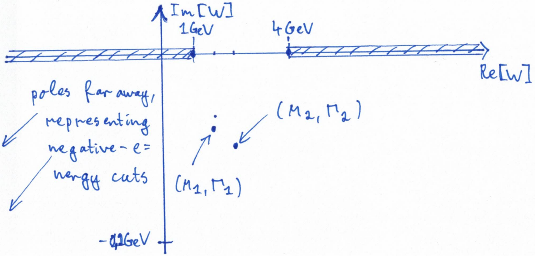
- Pick function from **[A. Švarc et al., Phys. Rev. C 88, 035206 (2013), sec. III]** as a <u>Toy-model:</u>

$$T^{Toy}(W) := \sum_{k=1}^{2} \frac{r_k + i\, g_k}{M_k - W - i\,\Gamma_k/2} + \Phi(W, 0.25) + \Phi(W, 1.) + B(W)$$

$$\Phi(W, a) := \frac{\sqrt{W(-4a+W)}}{2\pi W} \log\left[\frac{2a - W - \sqrt{W(-4a+W)}}{2a}\right]$$

$$B(W) := \frac{10.}{-10. - W - i\,5} + \frac{10.}{-6. - W - i\,4.}$$

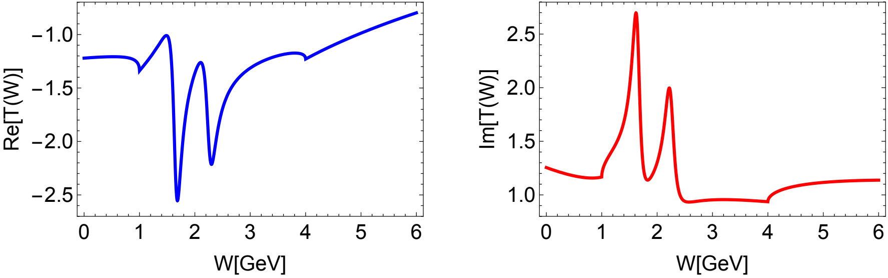
- Sketch of <u>analytic structure:</u>



<u>Parameters:</u>

$r_1 = 0.1$ GeV
$g_1 = 0.09$ GeV
$M_1 = 1.65$ GeV
$\Gamma_1 = 0.165$ GeV
$r_2 = 0.09$ GeV
$g_2 = 0.06$ GeV
$M_2 = 2.25$ GeV
$\Gamma_2 = 0.2$ GeV

Toy model - plot



Central question

▷ **Question:** (motivated by Alfred's results)

"What happens to the pole-parameters $\{W_j\}$ once $T(W)$ is rotated (step-wise) by it's own overall phase:

$$T(W_r) \longmapsto e^{-i\lambda\phi(W_r)} \times T(W_r) \;, \quad W_r \in \mathbb{R} \;,$$

cases $\lambda = \{0, 0.25, 0.5, 0.75, 0.99\}$ studied,

and the result is fitted with L+P ?"

↳ **Thesis / suspicion:** The $\{W_j\}$ stay the same!

However: In order to prove this, one needs the analytic continuation of the (inverse) phase-rotation function

$$F_\lambda(W) := e^{-i\lambda\phi(W)}$$

into the complex energy plane ($W \in \mathbb{C}$), at least reaching the pole-positions!

Analytic continuation of phase rotation function

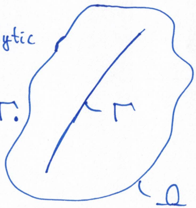
- Does continuation of $F(w)$ exist ?
- Cannot just "plug" complex numbers w into $e^{-i\phi(w_r)} = \frac{|T(w_r)|}{T(w_r)}$, since $|T(w)|$ is non-analytic!

↳ <u>However:</u> One can (at least numerically) try to find an analytic function $F(w)$, which on the real axis "just happens" to be equal to $e^{-i\phi(w_r)} = \frac{|T(w_r)|}{T(w_r)}$.

"<u>Uniqueness-theorem</u>"

f_1 & f_2 analytic on Ω <u>and</u> $f_1(z) = f_2(z)$ on Γ.

$\Rightarrow f_1(z) = f_2(z)$ on Ω.

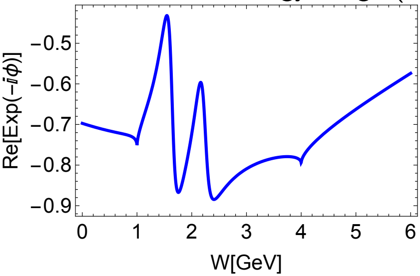


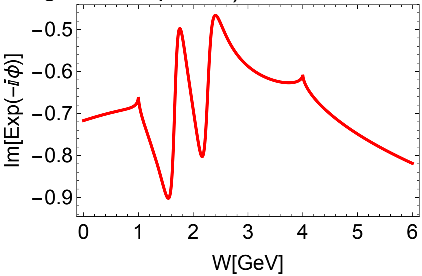
- This $F(w)$ should be unique!

<u>Comment:</u> Ideas w. dispersion relations written but not yet really calculated.

Phase rotation function $e^{-i\Phi(W_r)}$ on the real axis

Full energy range (including branch points):





Phase rotation function $e^{-i\Phi(W_r)}$ on the real axis

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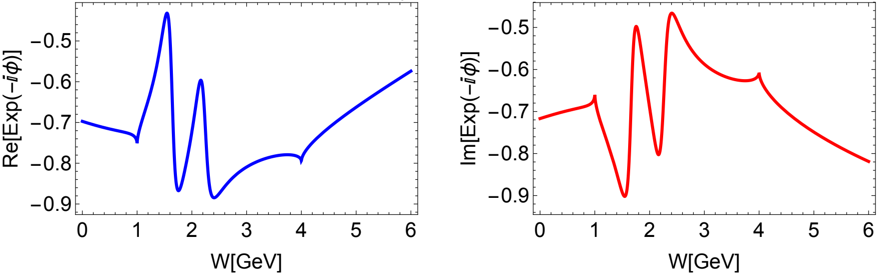
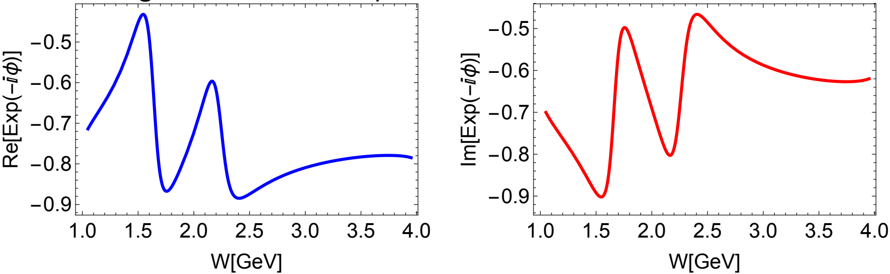


Range between branch points where continuation is done:



Ansatz for numerical analytic continuation

▷ Ansatz for the numerical analytic continuation:

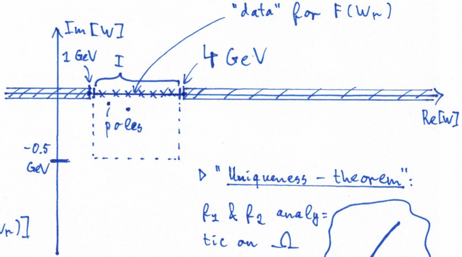
In any point where the function $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$

$f(x,y) = u(x,y) + i\,v(x,y)$ $[x = \mathrm{Re}[z]$ & $y = \mathrm{Im}[z]]$

is analytic, it fulfills the Cauchy-Riemann diff.-equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} .$$

↳ Define box in the complex energy-plane where C.R.-diff. eq.'s are solved numerically ("NDSolve" in MATHEMATICA)



▷ Boundary-conditions on $I \subset \mathbb{R}$:

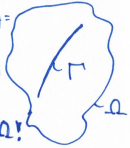
$u(W_r) \equiv \mathrm{Re}[F(W_r)]$ & $v(W_r) \equiv \mathrm{Im}[F(W_r)]$

▷ Choose imaginary width of box large enough to reach poles: $\sim 0.3, \ldots, 0.5$ GeV.

▷ "Uniqueness-theorem":

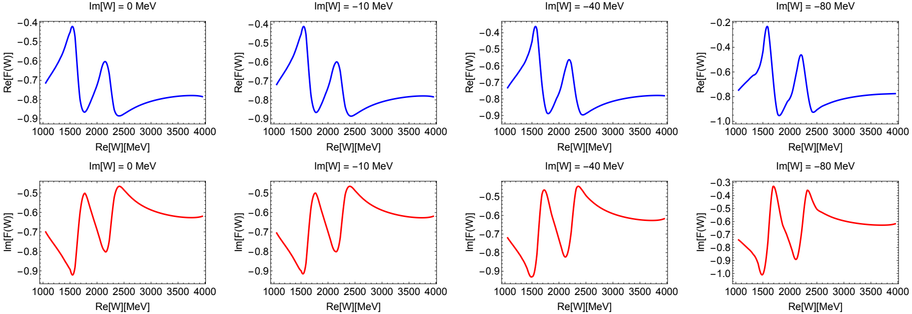
f_1 & f_2 analytic on Ω and $f_1(z) = f_2(z)$ on Γ.

$\Rightarrow f_1(z) = f_2(z)$ on Ω!



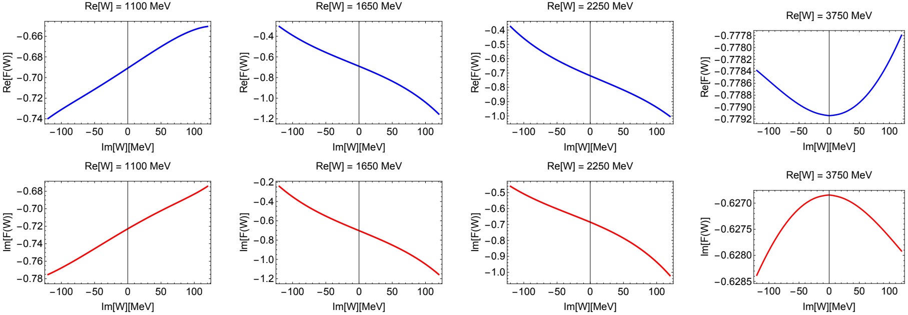
Results for the phase close to the real axis

Dependence on $\mathrm{Re}\,[W]$ for fixed $\mathrm{Im}\,[W]$:



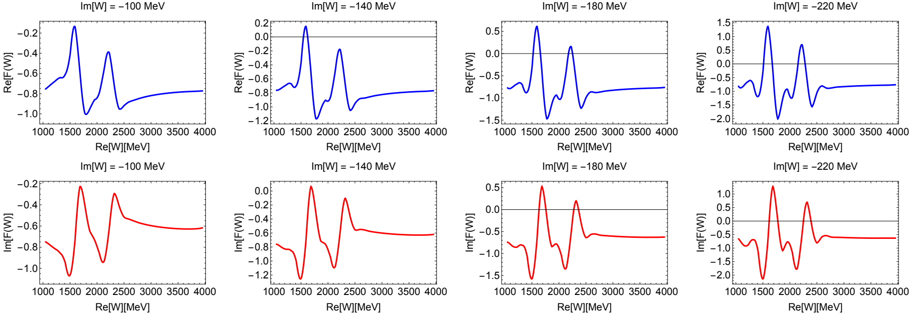
Results for the phase close to the real axis

Dependence on $\mathrm{Im}\,[W]$ for fixed $\mathrm{Re}\,[W]$:



Results for the phase further into the complex plane

Dependence on $\mathrm{Re}\,[W]$ for fixed $\mathrm{Im}\,[W]$:

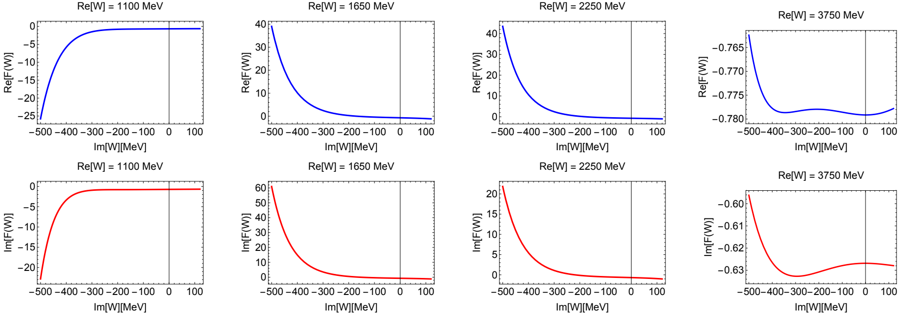


Problem: Numeric analytic continuation is not stable!

Cf.: [C.L. Fu et al., “A simple regularization method for stable analytic continuation”, Inverse Problems 24, 065003 (2008)]

Results for the phase further into the complex plane

Dependence on $\mathrm{Im}\,[W]$ for fixed $\mathrm{Re}\,[W]$:



Problem: Numeric analytic continuation is not stable!

Cf.: [C.L. Fu et al., "A simple regularization method for stable analytic continuation", Inverse Problems 24, 065003 (2008)]

Obtained function still OK at pole positions

▷ However: numerically obtained function $F(w)$ still O.K.

@ Toy-model pole-positions !

↳ The following proof should hold, restricted to $F_\lambda(w)$

functions with $\lambda \in [0,1]$.

($\lambda = 1$ - case not fit-able with L + P → Alfred)

▷ Additional assumptions / presumptions:

$F(w)$ is not allowed to have either poles or zeroes at the pole-positions $\{w_j\}$ of $T(w)$!

zero:

$F(w) \simeq c_j\,(w_j - w)$ & $T(w) \simeq \dfrac{a_{-1}^{(j)}}{(w_j - w)}$

for $|w - w_j| \ll 1$

$\Rightarrow F(w) \times T(w) \simeq c_j \times a_{-1}^{(j)}, \ |w - w_j| \ll 1$

$\Rightarrow$ pole "killed" !

pole:

$F(w) \simeq \dfrac{c_j}{(w_j - w)^n}$ & $T(w) \simeq \dfrac{a_{-1}^{(j)}}{(w_j - w)}$

for $|w - w_j| \ll 1$

$\Rightarrow F \times T \simeq \dfrac{c_j \ a_{-1}^{(j)}}{(w_j - w)^{n+1}}, \ |w - w_j| \ll 1$

$\Rightarrow$ pole-order raised !

Proof of pole position invariance I

▷ $T(w)$ gets mapped to $T^R(w)$ (R: "rotated") according to:

$$T(w) \longmapsto T^R(w) = F(w)\, T(w)$$

$$= F(w) \times \left(\sum_j \frac{a_{-1}^{(j)}}{w_j - w} + B(w) \right)$$

$$= \sum_j \frac{F(w)\, a_{-1}^{(j)}}{w_j - w} + F(w)\, B(w)$$

▷ $F(w)$ is assumed to be analytic at each point $\{w_j\}$

↳ Taylor-expand: $F(w) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(w)\Big|_{w_j} (w - w_j)^n$

▷ Laurent-decompose each individual pole-term:

$$\frac{F(w)\, a_{-1}^{(j)}}{w_j - w} = \frac{1}{0!} \frac{F^{(0)}(w)\big|_{w_j} a_{-1}^{(j)}}{w_j - w} (w - w_j)^0 + \sum_{n=1}^{\infty} \frac{a_{-1}^{(j)}}{n!} F^{(n)}(w)\Big|_{w_j} \frac{(w - w_j)^n}{(w_j - w)}$$

$$= \frac{F(w_j)\, a_{-1}^{(j)}}{w_j - w} + \underbrace{(-)\, a_{-1}^{(j)} \sum_{n=1}^{\infty} F^{(n)}(w)\Big|_{w_j} (w - w_j)^{(n-1)}}_{=:\, r^{(j)}(w)} = \frac{F(w_j)\, a_{-1}^{(j)}}{w_j - w} + r^{(j)}(w)$$

Proof of pole position invariance II

↳ All this yields for $T^R(W)$:

$$T^R(W) = \sum_j \frac{F(W)\, a_{-1}^{(j)}}{W_j - W} + F(W)\, B(W)$$

$$= \sum_j \frac{F(W_j)\, a_{-1}^{(j)}}{W_j - W} + \underbrace{\sum_j r^{(j)}(W) + F(W)\, B(W)}_{=:\, \tilde{B}(W)}$$

$$= \sum_j \frac{\tilde{a}_{-1}^{(j)}}{W_j - W} + \tilde{B}(W)$$

Q.E.D

▷ $T^R(W)$ has poles <u>at the same positions</u> as $T(W)$!

▷ The residues $a_{-1}^{(j)}$ are transformed to "<u>effective residues</u>" $\tilde{a}_{-1}^{(j)}$ under the phase-rotation:

$$a_{-1}^{(j)} \longmapsto \tilde{a}_{-1}^{(j)} = F(W_j)\, a_{-1}^{(j)} .$$

▷ The background $B(W)$ gets <u>heavily modified</u> to become:

$$\tilde{B}(W) = \sum_j r^{(j)}(W) + F(W)\, B(W) .$$

Does this work for the Toy model function?

Does this work?

▷ Multiply numerically obtained $F(W)$ into $T^{T\delta}(W)$ (MATHEMATICA)

↳ Poles stay at the same positions! (cf. next **slide**)

↳ Calculate predictions for residues using $\tilde{a}_{-1}^{(j)} = F(W_j)\, a_{-1}^{(j)}$

as well as $a_{-1}^{(j)} \equiv \frac{1}{2\pi i} \oint_{\partial U_\varepsilon(W_j)} T^R(W)\, dW$

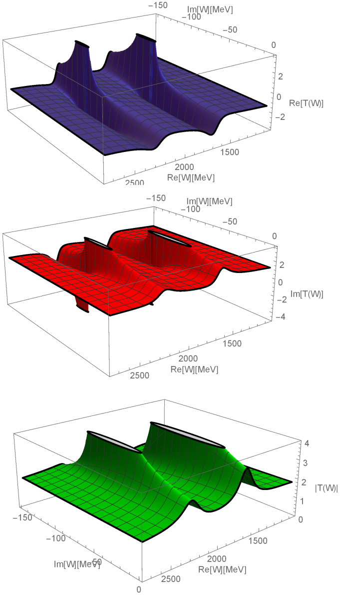
⇒ same numbers using both methods (analyticity!) !

λ	$\lvert a_{-1}^{(1)} \rvert$/GeV	$\lvert a_{-1}^{(2)} \rvert$/GeV	$\phi_{\mathrm{Res.}\,1}$/°	$\phi_{\mathrm{Res.}\,2}$/°
0.	0.1345	0.1082	42	33.7
0.25	0.1176	0.0963	8.3	0.89
0.5	0.1028	0.0857	−25.3	−31.9
0.75	0.0899	0.0763	−59	−64.7
0.99	0.079	0.0682	−91.3	−96.2

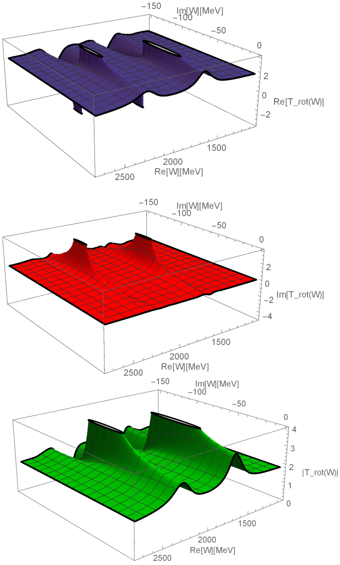
↳ Residues confirmed in FORTRAN − L+P − fit !

Rotated Toy model functions for $\lambda = 0.99$

Non-rotated $T(W)$:



$T^R(W)$ rotated with $\lambda = 0.99$:



Problems

- (FORTRAN-) L+P-fit not 100% perfect if pole-positions are held fixed. Also quite large orders in Pietarinen-expansions needed (around 40 ~ 50 terms).
- If pole-positions are running freely in the fit, something is pushing the poles away from the real axis! (But fit gets better!)

  I.e. $\mathrm{Re}[W_j] \simeq$ const. &

  $|\mathrm{Im}[W_j]|$ grows with λ.

↳ We think: Background is not fitted right for rotated $T(w)$, using the standard L+P-parametrization.

- Maybe, multipl. with $F(w)$ somehow "destroys" (physical) analyticity.
- Is analytic structure of rotated background $\hat{B}(w) := \sum_j r^{(j)}(w) + F(w) B(w)$ too weird / not parametrizable?
- Is the analytic continuation of $e^{-i\phi(w)}$ forbidden?

Issue of angular dependent overall phase I

Write : $T(W,\theta) = \sum_{\ell=0}^{\infty} T_\ell(W) P_\ell(\cos\theta) \Leftrightarrow T_\ell(W) = \frac{2\ell+1}{2} \int_{-1}^{+1} d(\cos\theta) P_\ell(\cos\theta) T(W,\theta)$.

↳ Study what happens to the projection-integral under the "continuum-ambiguity"-transformation :

$$T(W,\theta) \longmapsto \tilde{T}(W,\theta) = e^{+i\Phi(W,\theta)} \times T(W,\theta)$$

One has : $T_\ell(W) \longmapsto \tilde{T}_\ell(W) = \frac{2\ell+1}{2} \int_{-1}^{+1} d(\cos\theta) P_\ell(\cos\theta) \tilde{T}(W,\theta)$

$$= \frac{2\ell+1}{2} \int_{-1}^{+1} d(\cos\theta) P_\ell(\cos\theta) e^{+i\Phi(W,\theta)} T(W,\theta)$$

↳ Expand the phase-rotation into Legendre-polynomials :

$$e^{+i\Phi(W,\theta)} = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) P_{\ell'}(\cos\theta)$$

$$\Rightarrow \tilde{T}_\ell(W) = \frac{2\ell+1}{2} \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \int_{-1}^{+1} d(\cos\theta) P_\ell(\cos\theta) P_{\ell'}(\cos\theta) T(W,\theta)$$

Issue of angular dependent overall phase II

▷ Re-coupling theorem for products of two Legendre-polynomials:

$$P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) = \sum_{j=0}^{M(\ell,\ell')} A_{2j}\, P_{\ell+\ell'-2j}(\cos\theta)$$ **[J. Dougall (1952)]**

$$\hookrightarrow\ \tilde{T}_\ell(W) = \frac{2\ell+1}{2} \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \int_{-1}^{+1} d(\cos\theta)\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\, T(W,\theta)$$

$$= \frac{2\ell+1}{2} \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{j=0}^{M(\ell,\ell')} A_{2j} \underbrace{\int_{-1}^{+1} d(\cos\theta)\, P_{\ell+\ell'-2j}(\cos\theta)\, T(W,\theta)}_{= \frac{2}{2(\ell+\ell'-2j)+1} \times T_{\ell+\ell'-2j}(W)}$$

$$= \underline{\sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{j=0}^{M(\ell,\ell')} A_{2j} \frac{2\ell+1}{2(\ell+\ell'-2j)+1}\, T_{\ell+\ell'-2j}(W)}$$

↪ Angular dependence of $e^{+i\Phi(W,\theta)}$ <u>causes mixing of partial waves!!</u>

▷ Looks physically un-acceptable! (How) Can this be ruled out?

▷ For coupled-channels interpretation ⟼ Alfred

Omelaenko's warning about angular dependent phase

The large amount of experimental information which is needed for the complete experiment does not allow one, however, to obtain values of partial amplitudes f[illegible]e from model assumptions. In fact, in a complete

fre

experiment the amplitudes F_i are determined with accuracy to the transformation

$$\boxed{F_i(E_\gamma, \theta) \rightarrow \exp\left(i\varphi(E_\gamma, \theta)\right) F_i(E_\gamma, \theta).}$$

!!

where $\varphi(E_\gamma, \theta)$ is an independent real function. By choosing $\varphi(E_\gamma, \theta)$ one can vary the angular distributions of the amplitudes F_i, although the observables remain unchanged. Going over then to multipole expansions, one obtains as a result various sets of partial amplitudes differing both in the number of excited waves and in their magnitudes.

Killer Argumen

In a multipole analysis with $l \leq L$ the uncertainty in the phase manifests itself as an ambiguity in the choice of L. In the amplitude corresponding to the solution with some L one can also introduce a phase depending arbitrarily on angle, and the number of terms in the multipole expansions then changes. Having this in mind, obviously it is expedient to use the smallest value L for which one achieves a description of the experimental data.

!

Warning written on [Omelaenko (1981), page 6]