

TOPICS IN ALGEBRAIC GEOMETRY: FANO VARIETIES

The aim of this seminar is to study Fano varieties and their moduli spaces. We will gather every two weeks in either Bonn or Mainz during the 2015-2016 Wintersemester. We start on October 23rd 2015 and finish January 21st 2016. Every meeting is four hours and consists of two talks (90 minutes each), except for the last meeting where we will have three talks. There will be five meetings in total.

The following is a preliminary incomplete program for this joint seminar. Details will be added during the course of the semester.

1. BASIC PROPERTIES OF FANO VARIETIES AND BOUNDEDNESS OF THE DEFORMATION TYPES

14:15h - 15:45h + 16:15h-17:45h, Friday November 6th 2015, Bonn.

In the first talk we discuss families of Fano varieties and prove their many basic properties as in Chapter 2 of [6]. We also discuss some of the vanishing results proven by Shepherd-Barron in his paper on Fano threefolds [7]. We introduce the three basic invariants: ρ (Picard number), i (index) and d (degree). We prove their basic properties.

In the second talk we present Debarre's Bourbaki on Fano varieties [2]. The aim is to give explicit bounds on the number of deformation types of n -dimensional Fano varieties (in terms of n). Here it is good to have a look at [3, §5.9] as well.

Speakers:

- Ariyan Javanpeykar
- Ronan Terpereau

2. NON-RATIONALITY OF THE GENERAL X_{10} , AND CONSTRUCTING HOLOMORPHIC SYMPLECTIC MANIFOLDS FROM FANO VARIETIES

14:15h - 15:45h + 16:15h-17:45h, Friday November 20th 2015, Mainz.

In the first talk we study the intermediate Jacobian of a Fano threefold. More precisely, this talk will focus on the Fano 3-folds X_{10} ; the Fano threefolds with Picard number one, index one and degree ten. We will prove that all X_{10} are unirational and, by using the Clemens-Griffiths method of the intermediate Jacobian, that it is not rational.

In the second talk we will explain how one constructs holomorphic symplectic manifolds from "Fano varieties of lines/conics/cubics" on certain Fano varieties. We end with a discussion of Kuznetsov's results on semi-orthogonal

decompositions of triangulated categories, and its relation to these constructions. Let us give a more precise description of the content of this second talk.

We first discuss several examples of holomorphic symplectic manifolds:

- (1) Moduli of stable sheaves on K3 surfaces. (Mukai)
- (2) Fano schemes of lines on a smooth cubic fourfold. (Beauville-Donagi)
- (3) Fano scheme of conics in a Grassmannian $\text{Gr}(2, V_5)$ intersected with a generic hyperplane H . This is a smooth projective 5-dimensional variety which admits a smooth fibration onto a 4-dimensional irreducible holomorphic symplectic manifold with \mathbb{P}^1 as fibres. (Iliev-Manivel)
- (4) Fano scheme of cubics on a smooth cubic fourfold. This is a smooth projective 10-dimensional variety which admits a fibration over a 8-dimensional holomorphic symplectic manifold with general fibre \mathbb{P}^2 . This holomorphic symplectic manifold is deformation equivalent to some Hilbert scheme. (Lehn-Lehn-Sorger-van Straten)
- (5) And more...

We then discuss symplecticity according to Kuznetsov. Let S be K3 surface, F a coherent sheaf on S . We explain that the space of infinitesimal deformations of F can be identified with $\text{Ext}_{\mathcal{O}_S}^1(F, F)$. For stable sheaves F , this space is also the tangent space to the moduli space. Furthermore, there is a cup-product. We explain the role of Serre duality here.

We now move on to derived categories, and discuss Serre functors, Calabi-Yau categories and semi-orthogonal decompositions. The philosophy here is that “Objects in a CY category of dimension 2 behave like sheaves on a K3 and have symplectic moduli.” We recall Beilinson’s theorem for the derived category of \mathbb{P}^n , and finish with a theorem of Kuznetsov which “explains” some of the constructions we gave earlier.

Speakers:

- Enrica Floris
- Manfred Lehn

3. MORI FIBRE SPACES, SPECIAL FANO FOURFOLDS OF DEGREE 10 AND RATIONALITY

14:15h - 15:45h + 16:15h-17:45h, Friday December 4th 2015, Mainz.

In the first talk we briefly recall the definition of a Mori fiber space. Then, we show that for a Fano variety to be a general fibre of a Mori fibre space is a rather restrictive condition. More precisely, we will give two criteria (one

sufficient and one necessary) for a \mathbb{Q} -factorial Fano variety with terminal singularities to be realised as a fibre of a Mori fibre space. These criteria give a characterisation of rigid Fano varieties arising as a general fibre of a Mori fibre space. We apply our criteria to Fano varieties of dimension at most three and to rational homogeneous spaces.

In the second talk we give a characterisation of some special Fano fourfolds of degree 10 in terms of their Hodge structure and we will link them to EPW sextics. In addition, we will also analyse rationality problems in relation with certain subcategories of the derived category of these fourfolds.

Speakers:

- Andrea Fanelli
- Giovanni Mongardi

4. FANO CLASSIFICATION AND DIFFERENTIAL EQUATIONS, PERIODS FANO THREEFOLDS OF DEGREE 14 AND THEIR RELATION TO CUBIC THREEFOLDS

14:15h - 15:45h + 16:15h-17:45h, Friday January 8th, Mainz.

In the first talk of this session we discuss Golyshev's work on Fano threefolds with Picard number one and index one. According to the classification result of Iskovskikh, there are precisely 17 families of Fano 3-folds with one-dimensional Picard lattice. Golyshev has shown that there are 17 special third order differential equations describing so called "(N,d)-modular variations". The link between these two completely different classification results is provided by quantum cohomology, in which Gromov-Witten counts of curves are encoded into flat connections. In the first talk Duco will try to explain all this.

In the second talk we discuss a special class of Fano threefolds, namely those with Picard number one, index one and degree 14. It turns out that their moduli space can be related to the moduli space of smooth cubic threefolds via the intermediate Jacobian. Clemens and Griffiths studied the Abel-Jacobi map of the family of lines on a smooth cubic threefold X . They represented its intermediate Jacobian as the Albanese variety of the Fano surface parametrizing lines on X and described its theta divisor. From this description, they deduced the Torelli Theorem and the non-rationality of X . One can easily understand the structure of the Abel-Jacobi maps of some other families of curves of low degree on X . This talk deals with the Abel-Jacobi maps of the families of elliptic quintics and rational quartics lying on a smooth cubic threefold. It is proved that their generic fiber is the 5-dimensional projective space for quintics, and a smooth 3-dimensional variety birational to the cubic

itself for quartics

Speakers:

- Duco van Straten
- Helge Ruddat

5. CANCELLED

14:15h - 15:45h + 16:15h-17:45h, Friday January 22nd, Bonn.

This meeting was cancelled, as all the organizers were absent.

6. THE GENERALIZED MUKAI CONJECTURE FOR SYMMETRIC VARIETIES

14:00h - 18:00h, Friday February 5th, Mainz

In the first talk, we present Serre's construction of a non-liftable variety over $\overline{\mathbb{F}}_p$; see [4, Thm 22.4]. We then explain why curves, abelian varieties, K3 surfaces, hypersurfaces, and complete intersections are liftable (from characteristic p to characteristic zero), in the strongest sense. We deduce from this that many Fano varieties are liftable. Next we present Shepherd-Barron's theorem on liftability of Fano threefolds with Picard number one.

In the last two talks we study Mukai's conjecture. Spherical varieties are normal irreducible varieties with an action of a connected reductive group containing an open orbit for a Borel subgroup and can be considered as a natural generalization of toric varieties. In the first talk, we give a short introduction to the combinatorial description of spherical varieties, which has recently been completed. We introduce the combinatorial description of Gorenstein spherical Fano varieties in terms of certain polytopes (generalizing the reflexive polytopes in the toric case). In this setting, we investigate the generalized Mukai conjecture, an inequality involving the pseudo-index, the dimension, and the Picard number of a smooth Fano variety.

In the third talk we continue the study of the generalized Mukai conjecture from the previous talk, leading us to the formulation of a new conjecture involving a certain rational invariant, which will turn out to be of independent interest. We explain how this conjecture may be restated in a purely combinatorial way. This allows us to deduce a (conjectural) combinatorial smoothness criterion for spherical varieties and, finally, to prove the conjecture in the special case of symmetric varieties. In particular, the generalized Mukai conjecture holds in this case.

Speakers:

- Axel Stäbler
- Johannes Hofscheier
- Giuliano Gagliardi

REFERENCES

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