The gen. Mukai conj. for symm. var.

(joint with Gagliardi)

1. Gen. Mukai conj.

$X/\mathbb{C}$ smooth Fano

Picard number: $\rho_X := \mathrm{rk}(\mathrm{Pic}(X))$

pseudo-index:

$i_X := \min\{(-K_X \cdot C) : C \subseteq X \text{ rat. curve}\}$

Conjecture (Bonavero, Casagrande, Debarre & Druel '03):

$$\rho_X \cdot (i_X - 1) \leq \dim X$$

with "=" iff $X \cong (\mathbb{P}^{i_X - 1})^{\rho_X}$.

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original conj. (Mukai '88): replace i_X by the index r_X of X (recall: $r_X \mid i_X$)

"Problems on characterization of the complex projective space".

(Some) known cases:

(a) $\rho_X = 1, 2$.

(b) $3 i_X > \dim X$

(c) $\dim X \leq 5$

(d) toric
(e) horospherical } both spherical

Goal: If X $\mathbb{Q}$-factorial Gorenstein ~~spherical~~ symm. Fano, then

$\rho_X (i_X - 1) \leq \dim X$ with "=" iff $X \cong (\mathbb{P}^{i_X - 1})^{\rho_X}$

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recall:

* X $\mathbb{Q}$-factorial, if X normal and $\forall$ Weil-div. D $\exists k_D \in \mathbb{Z}$ s.t. $k_D \cdot D$ Cartier.

* Gorenstein, if $-K_X$ Cartier

2. Crash Course in Sph. Geom.

$G/\mathbb{C}$ conn. reductive group (lin. alg. grp. s.t. all lin. rep. completely reducible);

Ex.: * alg. tori: $(\mathbb{C}^*)^n$

* $SL_2 \times \mathbb{C}^*$

Def.: closed subgrp. $H \subseteq G$ spherical, if G/H has open B-orbit for some Borel $B \subseteq G$.

G/H called sph. homogeneous space

Ex.: $G = SL_2 \times \mathbb{C}^* \curvearrowright \mathbb{P}^2 \times \mathbb{P}^1$:

$(A,t)\cdot([x],[y]) = \left(\left[A\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} : tx_2\right], \left[tA\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right]\right)$

$\rightsquigarrow H := \mathrm{Stab}_G([1:0:1],[0:1])$ $\left\{\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t\right) : t \in \mathbb{C}^*\right\}$

$G/H \cong G\cdot([1:0:1],[0:1]) = \{X_2(X_0Y_1 - X_1Y_0) \neq 0\}$

$B = \left\{\left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, *\right)\right\} \subseteq G$ Borel

$B\cdot([1:1:1],[0:1]) = G/H \setminus \{X_1Y_1 = 0\}$ ②

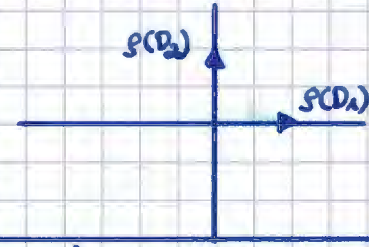
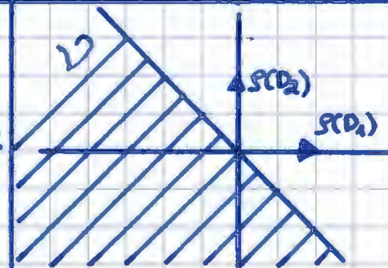
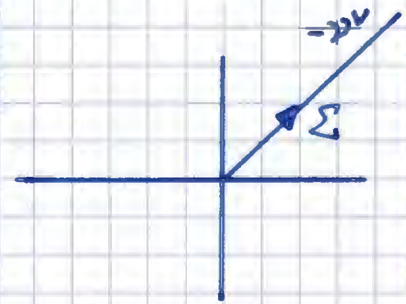
Now: G/H fixed.

Def.: sph. embedding: open G-equiv. embd. $G/H \hookrightarrow X$ (normal irred. G-var.)

X sph. var.

combinatorics:

general	$G/H \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$
Lattice: $\mathcal{M}$: $\{\chi \in \overbrace{\mathfrak{X}(B)}^{\text{char. latt.}}$: $\exists\, 0 \neq f \in \mathbb{C}(G/H) : b\cdot f = \chi(b)\cdot f$ $\forall b \in B\}$ f_χ (unique/$\mathbb{C}^*$)	$\mathfrak{X}(B) = \mathbb{Z}\omega \oplus \mathbb{Z}\varepsilon$ $\omega\left(\begin{pmatrix} b_1 & b_2 \\ 0 & b_1^{-1} \end{pmatrix}, t\right) = b_1$ $\varepsilon\left(\text{— " —}\right) = t$
$\mathcal{N} := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{Z})$	$\frac{X_1}{X_2} \rightsquigarrow \omega+\varepsilon$, $\frac{X_2Y_1}{X_0Y_1 - X_1Y_0} \rightsquigarrow \omega-\varepsilon$ $\rightsquigarrow \mathcal{M} = \mathbb{Z}(\omega+\varepsilon) \oplus \mathbb{Z}(\omega-\varepsilon)$

general	G/H
colors: $\mathcal{D} = \{ D \subseteq G/H :$ B-inv. prime div. $\}$	$\mathcal{D} = \{ D_1 = \{X_1 = 0\}, D_2 = \{Y_1 = 0\} \}$
$\rho : \mathcal{D} \to \mathcal{N}$ $D \mapsto [\chi \mapsto \nu_D(f_\chi)]$	
Valuation cone: $\mathcal{V} := \{ \nu : \mathbb{C}(G/H)^* \to \mathbb{Q}$ G-inv. discr. val. $\}$ geometric interpretation: "boundary div." $i : \mathcal{V} \hookrightarrow \mathcal{N}_\mathbb{Q} = \mathcal{N} \otimes \mathbb{Q}$ similar as above Brion: $\mathcal{V} \subseteq \mathcal{N}_\mathbb{Q}$ cosimplicial solid cone $\rightsquigarrow$ prim. ray gen.s of $-\mathcal{V}^\vee$: Σ, sph. roots	 

3. Gorenstein sph. Fano var.

$G/H \hookrightarrow X$ Gorenstein sph. Fano embd. with

$X_1, \dots, X_n \subseteq X$ boundary div.

Recall: $\nu_1, \dots, \nu_n \in \mathcal{N} \cap \mathcal{V}$

Fact (Brion):

$$-K_X = \sum_{D \in \mathcal{D}} \underbrace{m_D}_{\in \mathbb{Z}_{>0}} D + \sum_{i=1}^{n} X_i$$

(only depend on G/H; ex. formulas)

Def: $Q_X := \mathrm{conv}\left(\nu_1, \dots, \nu_n, \frac{\rho(D)}{m_D} : D \in \mathcal{D}\right) \subseteq \mathcal{N}_\mathbb{Q}$

Ex.: $G/H \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$

boundary div.: $Z_1 = \{X_2 = 0\}$, $Z_2 = \{X_0 Y_1 - X_1 Y_0 = 0\}$

$-K_X = D_1 + D_2 + Z_1 + Z_2 \quad (m_{D_i} = 1)$

Thm. (Alexeev & Brion '03):

For $\varepsilon > 0$, $n \in \mathbb{N}$:

$|\{X \text{ n-dim } \varepsilon\text{-lt sph. Fano }\}| < \infty$

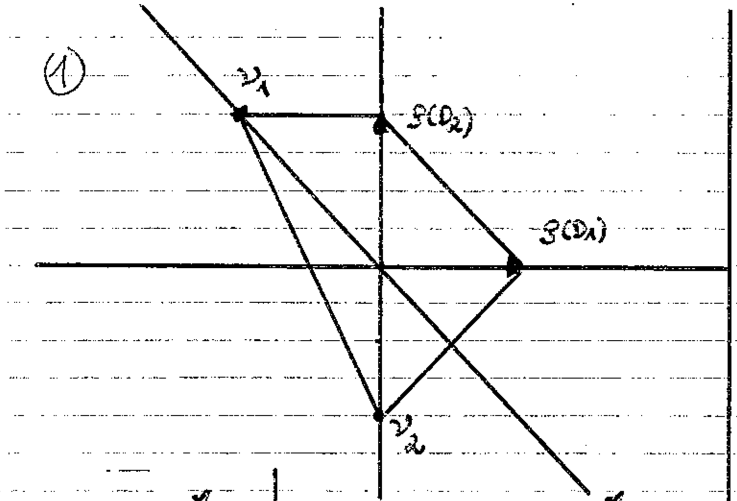
$\rightsquigarrow$ BAB Conj.

Idea: use polytopes Q_X

X ε-lt $\Longleftrightarrow$ $\mathrm{int}(\varepsilon Q_X) \cap \mathcal{N} \cap \mathcal{V} = \{0\}$

and

(1)



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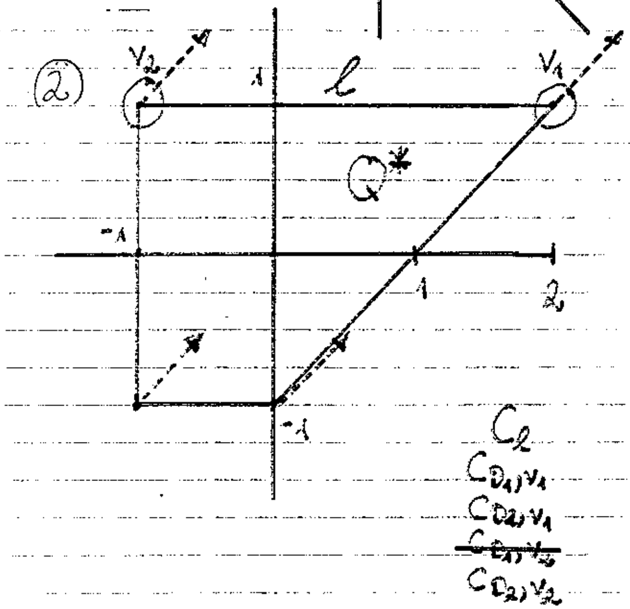

Def. (Pasquier; G. & -)

$Q \subseteq \mathcal{N}_{\mathbb{Q}}$ polytope is G/H-reflexive, if

* $\frac{\rho(D)}{m_D} \in Q \quad \forall D \in \mathcal{D}$
* $0 \in \mathrm{int}(Q)$
* $V(Q) \subseteq \left\{ \frac{\rho(D)}{m_D} : D \in \mathcal{D} \right\} \cup (\mathcal{N} \cap \mathcal{V})$
* $\forall v \in V(Q^*)$: $\quad (Q^* = \{ y \in \mathcal{M}_{\mathbb{Q}} : \langle Q, y \rangle \in \mathbb{Q}_{\geq}$

  $\underbrace{Q^* \cap (v + \mathrm{cone}(\Sigma)) = \{v\}}$ $\Longrightarrow v \in \mathcal{M}$.

  Supported vertices: $V_{supp}(Q^*)$

Ex.: reflexive polytopes

Ex.: ...

Thm. (Pasquier; G. & -)

assignment $X \mapsto Q_X$ induces bijection:

$$\left\{ \begin{array}{l} G/H \hookrightarrow X \text{ Gorenst.} \\ \text{sph. Fano embed.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} Q \subseteq \mathcal{N}_{\mathbb{Q}} \text{ G/H-refl.} \\ \text{polytopes} \end{array} \right\}$$

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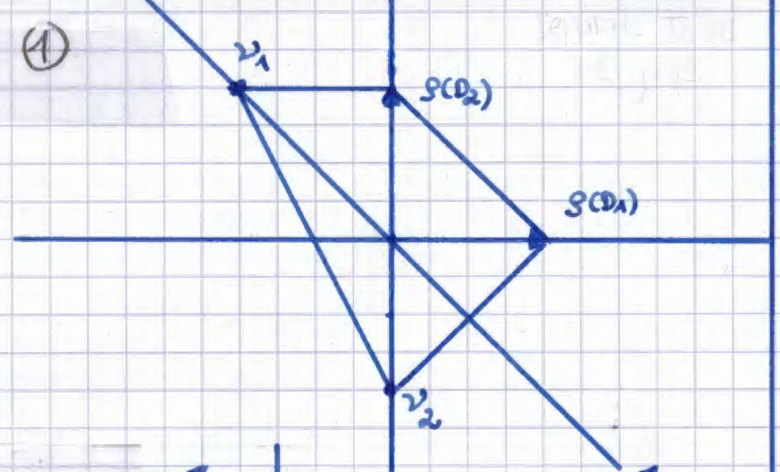

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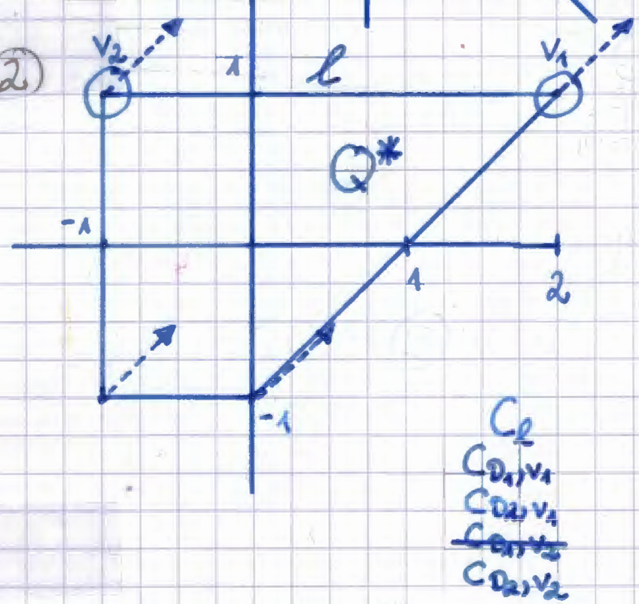

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  $\underbrace{Q^* \cap (v + \mathrm{cone}(\Sigma)) = \{v\}} \Rightarrow v \in \mathcal{M}$.

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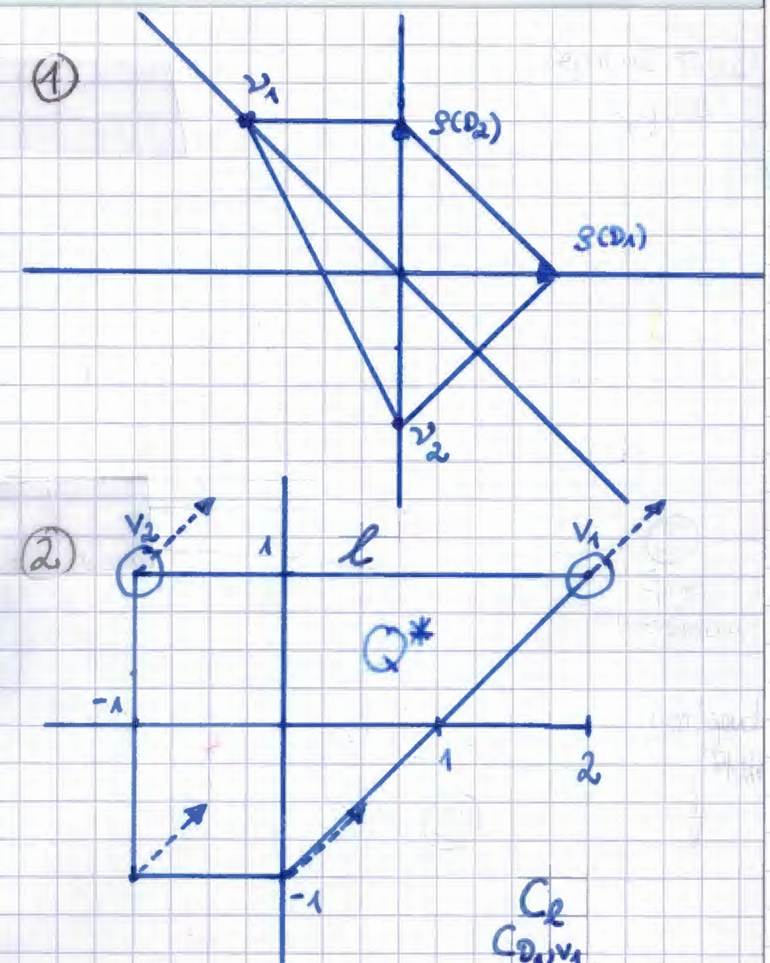

C_ℓ

C_{D_1, v_1}

C_{D_2, v_1}

~~C_{D_1, v_2}~~

C_{D_2, v_2}

4. Curves

$G/H \hookrightarrow X$ Gorenst. sph. Fano with $Q_X \subseteq N_{\mathbb{Q}}$

Brion: all irred. curves $C \subseteq X$ rat. equiv. to B-inv. one.

Comb. descr. B-inv. curves (Brion):

Type 1: $\ell \subseteq Q_X^*$ edge with $V(F) \subseteq V_{supp}(Q_X^*)$

$\rightsquigarrow$ curve $C_\ell \subseteq X$ s.t.

$(-K_X \cdot C_\ell) =$ lattice length of ℓ

Type 2: $v \in V_{supp}(Q_X^*)$ & $D \in \mathcal{D}$ with $\frac{\rho(D)}{m_D} \notin \hat{v}$

$\hat{v} = \{x \in Q_X : \langle x, v \rangle = -1\}$

$\rightsquigarrow$ curve $C_{D,v}$ s.t.

$(-K_X \cdot C_{D,v}) = m_D + \langle \rho(D), v \rangle$

Brion: $\mathrm{cone}([C_F], [C_{D,v}]) = \mathrm{cone}(\text{effective curves}) \subseteq N_1(X) \otimes \mathbb{Q}$.