HW 6: Algebraische Geometrie II

Handing in: Hand in by February 1st 2016. Grades are from 1 to 10. Keep a copy of your work and include an email address + home address to which we can return the graded exercises. You are strongly encouraged to write your solutions in TeX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game: Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Let k be an algebraically closed field.

Exercise 1 (3 points)**.** Let $n \in \mathbb{Z}_{\geq 0}$.

1. Compute a k-basis for $\mathrm{H}^1(\mathbb{P}^1(k), -n \cdot 0)$.
2. Compute a k-basis for $\mathrm{H}^0(\mathbb{P}^1(k), \Omega^1(n \cdot 0))$.
3. Give the Serre duality pairing explicitly.

Let $f : X \to \mathbb{P}^1(k)$ be a surjective morphism of smooth projective irreducible curves over k. We define $\deg f$ of f to be the degree of the divisor

$$(f)_0 := \sum_{P \in X, f(P)=0} v_P(f)[P].$$

Note that $\deg f$ is a positive integer.

Exercise 2 (4 points)**.** Let X be a smooth projective irreducible curve of genus g over k. Prove or disprove (by means of a counterexample):

1. There exists a non-constant rational function $f : X \to \mathbb{P}^1(k)$ such that $\deg f \leq g + 1$.
2. If P and Q are distinct points of X, then there exist an integer $n \geq 1$ and a non-constant rational function f on X such that $\mathrm{div}(f) = nP - nQ$.
3. If D is an effective divisor on X, then $\dim_k \mathrm{H}^0(X, D) = \deg D + 1 - g$.

4. If $g = 2$, then there exists a non-constant rational function $f : X \to \mathbb{P}^1(k)$ of degree precisely 2.

Exercise 3 (3 points)**.** Assume that k is of characteristic zero and let $X = \mathbb{P}^1(k)$. For f and g non-constant rational functions on X, we say that f and g are *equivalent* if there is an automorphism σ of X such that $f = \sigma g$. (Here we consider f and g as surjective morphisms $X \to \mathbb{P}^1(k)$.) Clearly, this defines an equivalence relation on the set of surjective morphisms $X \to \mathbb{P}^1(k)$. Show that, for all integers $d \geq 2$, the set of equivalence classes of surjective morphisms $f : X \to \mathbb{P}^1(k)$ of degree d is infinite.