HW 2: Algebraische Geometrie II

Handing in: Hand in by November 16th 2015. Grades are from 1 to 10. Keep a copy of your work and include an email address + home address to which we can return the graded exercises. You are strongly encouraged to write your solutions in TeX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game: Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Exercise 1. Let Y be the common zero set of the polynomials $x^2 - yz$ and $xz - x$ in $\mathbb{A}^3(k)$, where k is an algebraically closed field. Show that Y is the union of three irreducible components. Describe them and find their prime ideals (in $k[x, y, z]$).

Exercise 2. Let k be an algebraically closed field. Let d and n be positive integers, and consider $\mathbb{P}^{\binom{n+d}{d}-1}(k)$ as the space of all hypersurfaces of degree d in $\mathbb{P}^n(k)$, by associating to any hypersurface $\{F(X_0, \ldots, X_n) = 0\}$ in $\mathbb{P}^n(k)$ with F homogeneous of degree d in $k[X_0, \ldots, X_n]$ the projective vector of all $\binom{n+d}{d}$ coefficients of f. Show that the locus of *smooth* hypersurfaces in $\mathbb{P}^{\binom{n+d}{d}-1}(k)$ is non-empty, open and dense.

Exercise 3. Let p and n be positive integers with p a prime number, and let k be an algebraically closed field. Let X be the hypersurface given by

$$X_0^p + \ldots + X_n^p = 0$$

in $\mathbb{P}^n(k)$. Compute the singular locus of X.

Exercise 4. Prove or disprove (by means of a counterexample) the following statements.

1. A connected closed subset of affine space is irreducible (in the Zariski topology).
2. All non-empty open subsets of an irreducible topological space are dense.
3. Let k be an algebraically closed field. If one identifies $\mathbb{A}^2(k)$ with $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$, then the Zariski topology on $\mathbb{A}^2(k)$ is the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1(k)$.
4. The variety $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ is isomorphic to a smooth hypersurface of degree two in $\mathbb{P}^3(k)$.