I Reminder

Conjecture (BCDD)

X smooth Fano

Then

$$\rho_X(i_X - 1) \leq \dim X$$

eq. $\Leftrightarrow X \cong (\mathbb{P}^{i_X-1})^{\rho_X}$

Goal

Prove for X $\mathbb{Q}$-factorial Gorenstein spherical Fano

Johannes told you what is known

$$\begin{tikzcd}
\text{toric (Casagrande)} \arrow[r, hook] \arrow[d, hook] & \text{Symmetric } (-\&H) \arrow[d, hook] \\
\text{horospherical (Pasquier)} \arrow[r, hook] & \text{spherical } (?) \\
\text{flag (BCDD)} \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] &
\end{tikzcd}$$

Definition

G connected reductive $/\mathbb{C}$ $\quad\theta: G \to G$ involution

$$(G^\theta)^0 \subseteq H \subseteq N_G(G^\theta) \qquad G/H \hookrightarrow X$$

Then X symmetric $\quad$ historically first

e.g.: $\operatorname{diag}(G) \subseteq G \times G$

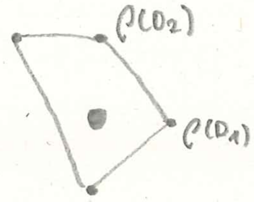
II First step of proof

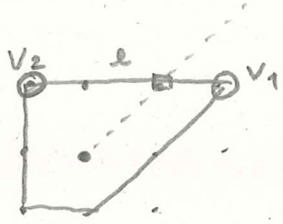
$$G/H \hookrightarrow X \qquad \mathbb{Q}\text{- fact. Gorenst. sph. Fano}$$

$$Q \qquad\qquad Q^*$$

Example

$\mathbb{P}^2 \times \mathbb{P}^1$





$$(-K_X \cdot C_{D_1,V_1}) = 3$$
$$\cancel{C_{D_1,V_2}}$$
$$(-K_X \cdot C_{D_2,V_1}) = 2$$
$$(-K_X \cdot C_{D_2,V_2}) = 2$$
$$(-K_X \cdot C_l) = 3$$

$$\Rightarrow \quad l_{\mathbb{P}^2\times\mathbb{P}^1} = 2$$

$$\Delta := \mathcal{D} \cup \{\text{boundary divisors}\} \qquad -K_X = \sum_{D\in\mathcal{D}} m_D D + \sum_{\text{bound.}} X_i = \sum_{D\in\Delta} m_D D$$

$m_D = 1$
for boundary div.

Properties

Example

$$\rho_X = |\Delta| - \underset{\substack{\parallel \\ \operatorname{rank} \mathcal{M}}}{\operatorname{rank} X} \qquad\qquad \rho_{\mathbb{P}^2\times\mathbb{P}^1} = 4 - 2 = 2$$

$\mathbb{Q}$-factoriality

$\Leftrightarrow$

$$\forall v \in V_{supp}(Q^*) : \#\{D \in \Delta \mid \rho(D) \in V(\hat{v})\} = \operatorname{rank} X \qquad\qquad \text{yes}$$

Gorenstein

$\Leftrightarrow$

$$V_{supp}(Q^*) \subseteq \mathcal{M} \qquad\qquad \text{yes}$$

$$l_X \leq \min\{ m_D + \langle \rho(D), v\rangle : D \in \Delta,\ v \in V_{supp}(Q^*),\ \langle \rho(D), v\rangle \neq -m_D \} \qquad\qquad l_{\mathbb{P}^2\times\mathbb{P}^1} = 2$$

Proposition

Let $\vartheta \in \operatorname{conv}(V_{\text{supp}}(Q^*))$

Then

$$\ell_X(l_X - 1) \leq \sum_{D \in \Delta} (m_D - 1 + \langle \rho(D), \vartheta \rangle) + \operatorname{rank} X$$

Definition

For arbitrary spherical X define

$$\wp(X) := \sup \left\{ \sum_{D \in \Delta} (m_D - 1 + \langle \rho(D), \vartheta \rangle) \;\middle|\; \vartheta \in Q^* \cap \operatorname{cone}(\Sigma) \right\}$$

Conjecture A

X complete spherical

Then

$$\wp(X) \leq \dim X - \operatorname{rank} X$$

eq. $\Leftrightarrow$ $X \cong$ toric variety

Proposition

Conj. A $\Rightarrow$ GMC

Example

$\sum_{D \in \Delta} \rho(D) = D_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ $\wp(X) = 0 + \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle = 1 = 3 - 2 = \dim X - \operatorname{rank} X$

$\vartheta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\mathbb{A}^2 \times \mathbb{P}^1$ toric $\rightsquigarrow$ Conj. A ok.

III Spherical skeletons and multiplicity-free spaces

X spherical G-variety

R root system of G describes G very roughly

Corresponding rough description of X?

Definition

$$\mathcal{R}_X := (\Sigma, \Delta) \qquad \Sigma \subseteq \mathcal{X}(B) \text{ in fact } \Sigma \subseteq \mathcal{X}(R)$$

with $\qquad D \mapsto P_D \subseteq G \leftrightarrow A \subseteq S \subseteq R$

$\qquad D \mapsto \rho(D)|_{\operatorname{span}\Sigma} \qquad$ recall $\rho(D) : \mathcal{M} \to \mathbb{Z}$

spherical R-skeleton of X

Remark

$\mathcal{R}_X$ determines $p(X)$, write $p(\mathcal{R}_X)$

$P \subseteq G$ stabilizer of open B-orbit in X

Definition

$G \curvearrowright V$ linearly and V sph. variety

Then V multiplicity-free space

$$\mathbb{C}[V] = \bigoplus_\lambda V_\lambda \quad \leftarrow \text{simple } G\text{-module multiplicity one}$$

Remark

Classified (Benson-Ratcliff, Leahy), sph. skeletons known (Knop)

Conjecture B

$\mathcal{R}$ complete sph. skeleton

Then

$$p(\mathcal{R}) \leq \dim G/P$$

eq. $\Leftrightarrow$ $\mathcal{R} \cong$ sph. skeleton of multiplicity-free space

Theorem

Conj. A and B are equivalent.

Why conj. B?

$(\Sigma, \Delta) = (\Sigma, \mathcal{D}) +$ boundary

sph. system $\quad \mathcal{D} \quad \rho \quad <0 \quad \rightsquigarrow$ finitely many per R

IV Cox Rings

$$\begin{tikzcd} \operatorname{Spec}(\mathbb{C}[x_0,\dots,x_n]) = \mathbb{A}^{n+1} \supseteq \mathbb{A}^{n+1}\setminus\{0\} \arrow[d, "/\mathbb{C}^*"] \\ \mathbb{P}^n \end{tikzcd}$$

$$\mathbb{C}[x_0,\dots,x_n] = \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

X irred., normal

Cox Ring $\quad \mathcal{R}(X) := \bigoplus_{[D]\in Cl(X)} \Gamma(X, \mathcal{O}_X(D))$

finitely generated? (MDS)

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{R}(X)) =: \overline{X} \supseteq \hat{X} \arrow[d, "/S"] \\ X \end{tikzcd}$$

S quasitorus $\quad \mathcal{X}(S) = Cl(X)$

Birkar, Cascini, Hacon, McKernan

log terminal $\mathbb{Q}$-factorial Fano $\rightsquigarrow$ MDS

Brion

- spherical X have finitely generated $\mathcal{R}(X)$
- $\mathcal{R}_{X_1} = \mathcal{R}_{X_2} \Rightarrow \mathcal{R}(X_1) \underset{\uparrow}{\cong} \mathcal{R}(X_2)$

  w/o grading

Facts

- X complete $\Rightarrow$ $\mathcal{R}(X) \cong$ poly $\Leftrightarrow$ $X \cong$ toric.
- X spherical $\Rightarrow$ $\overline{X}$ spherical $\overline{G}$-variety for some $\overline{G} \twoheadrightarrow G$.

  $\mathcal{R}(X) \cong$ poly $\Rightarrow$ $\overline{X}$ mult.-free space

Proposition

$$X \cong \text{toric} \Rightarrow \mathcal{R}_{\overline{X}} = \mathcal{R}_X$$

Conj. A and B are equivalent.
Explain.

Theorem

Let R be the sph. skeleton of a symmetric variety.
Then Conj. B holds.

V A smoothness criterion

Theorem

Let X be a locally factorial affine spherical variety with a fixed point.
If Conj. B holds, then X smooth $\Leftrightarrow$ $p(X) = \dim X - \operatorname{rank} X$.

Explain using Cox Ring.