HW 8 (Exam): Algebraische Geometrie II

Handing in: Hand in by March 20th 2016. Grades are from 1 to 10. Keep a copy of your work and include an email address + home address to which we can return the graded exercises. You are strongly encouraged to write your solutions in TeX and hand in a printout.

Exercise 1 (3 points)**.** Let k be an algebraically closed field with $7 \in k^\times$.

1. Compute the singular locus of the projective curve C given by the homogeneous equation $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2(k)$.
2. Show that C has non-trivial automorphisms. [Bonus: Compute $\#\mathrm{Aut}(C)$ when $\mathbb{Q} \subset k$.]
3. Compute the divisors of the functions x/y and x/z.

Exercise 2 (1 point)**.** Let t be an indeterminate, and let k be an algebraic closure of the field $\mathbb{C}(t)$. Show that the elliptic curve E over k given by $y^2 = x^3 + t$ can be defined over $\mathbb{Q}$.

Exercise 3 (2 point)**.** Let $\zeta \in \mathbb{F}_4$ denote a 3rd root of unity. Let E be the elliptic curve over $\mathbb{F}_4$ defined by the equation $y^2 + y = x^3$. Let $f : E \to E$ be given by $f(x, y) = (\zeta x, y)$, and let $g : E \to E$ be given by $g(x, y) = (x + 1, y + x + \zeta)$

1. Show that f and g are automorphisms.
2. Show that f and g do not commute in the ring $\mathrm{End}(E)$, i.e., $f \circ g \neq g \circ f$.

Exercise 4 (2 points)**.** Let k be an algebraically closed field with $2 \in k^\times$, and let E be the elliptic curve given by $y^2 = x^3 - x$ over k.

1. Show that the map $[i] : E \to E$ given by $[i](x, y) = (-x, iy)$ defines an isogeny $[i] : E \to E$ and that $[i]$ satisfies $[i]^2 + [1] = 0$ in $\mathrm{End}(E)$.
2. For a, b in $\mathbb{Z}$, show that the degree of the endomorphism $a + b[i]$ of E is equal to $a^2 + b^2$.

Exercise 5 (2 points)**.** Let $7 \leq p$ be a prime number, and let E_p be the elliptic curve $y^2 = x^3 + p$ over $\mathbb{F}_5$. Compute the zeta function $Z(E_p, t)$ of E_p as a rational function

$$Z(E_p, t) = \frac{P(t)}{(1 - t)(1 - 5t)}.$$