Theoretical Elementary Particle Physics
Exercise 4

June 27, 2015

-to be handed in on Monday, 6th of July or, alternatively, not later than Wednesday afternoon (8th of July) in my office (Staudingerweg 9, 02-121)

1. **Running mass and anomalous dimension in QCD (50 points)**

We define a running quark mass $m(\mu)$ through the equation

$$\mu \frac{d}{d\mu} m(\mu) = \gamma_m \, m(\mu), \tag{1}$$

where γ_m is the so-called anomalous dimension of the mass.
(a)(10 points) Show first that γ_m is equivalently given through the renormalization group equation

$$\gamma_m = -\frac{1}{Z_m(\mu)} \mu \frac{d}{d\mu} Z_m(\mu), \tag{2}$$

where Z_m is the mass renormalization constant as calculated in the course.
(b)(15 points) The anomalous dimension of the mass can be expanded in a perturbation series in the strong coupling constant as :

$$\gamma_m = \gamma_m^0 \, \frac{g^2(\mu)}{(4\pi)^2} + O(g^4). \tag{3}$$

Use the known expression for Z_m ($Z_m = 1 - \frac{g^2}{(4\pi)^2}\frac{4}{\epsilon}$, this was calculated in the lectures $\rightarrow$ see Skript 4) to derive the value of γ_m^0. (**hint :** Formula

$$\beta(g) = -\epsilon g \left[1 + \frac{1}{\epsilon} \frac{\beta_0}{(4\pi)^2} g^2 + \mathcal{O}(g^4) \right] \tag{4}$$

is useful and the expression for β_0 is shown in eq. (7) (**solution :** γ_m^0=-8)
(c)(25 points) By inserting eq. (3) into eq. (1), derive the one-loop expression of the running quark mass :

$$m(Q^2) = \frac{\hat{m}}{(\ln(Q^2/\Lambda_{QCD}^2))^{d_m}}, \tag{5}$$

with $d_m = 4/(11 - 2/3 N_f)$ and where $\hat{m}$ is an integration constant (the mass analog of Λ_{QCD}) that equals $\hat{m} = m(Q^2 = e\Lambda_{QCD}^2)$ where e stands for Euler's number.

2. **Running coupling constant in QCD in two loop order (50 points)**

In QCD, the β-function in two loop order is of the form

$$\beta(g) = -\frac{\beta_0}{(4\pi)^2}\, g^3 - \frac{\beta_1}{(4\pi)^4}\, g^5 + ..., \tag{6}$$

where

$$\beta_0 = 11 - \frac{2}{3}N_f, \tag{7}$$

and

$$\beta_1 = 102 - \frac{38}{3}N_f. \tag{8}$$

Show that the running coupling constant can be written as :

$$\begin{aligned}\alpha_s(Q^2) &\equiv \frac{g^2(Q^2)}{4\pi} \\ &\approx \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2_{QCD})}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(Q^2/\Lambda^2_{QCD})}{\ln(Q^2/\Lambda^2_{QCD})}\right].\end{aligned} \tag{9}$$

hints : The strategy is generally similar to the one where only 1-loop contribution is considered (Skript 5 /pages 8 & 9). Integrate between two mass scales and separate the dependence on each scale which implies that both LHS and RHS are equal to a constant that is chosen to be $\frac{-\beta_0}{4\pi}\ln\left(\Lambda^2_{QCD}\right)$. Note that the 4π "mismatch" between this expression and the one in the script is present because in this exercise we are dealing with α rather than g^2.
In general, we are interested in $Q^2 \gg \Lambda^2_{QCD}$ because perturbative expansion is valid in that case. Therefore, the following approximations can be applied :

$$\ln\left[\frac{\beta_0^2}{4\pi}\ln\left(\frac{Q^2}{\Lambda^2_{QCD}}\right) + \frac{\beta_1}{4\pi}\right] \approx \ln\ln\left[\frac{Q^2}{\Lambda^2_{QCD}}\right] \tag{10}$$

$$\left[1 + \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(Q^2/\Lambda^2_{QCD})}{\ln(Q^2/\Lambda^2_{QCD})}\right]^{-1} \approx \left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(Q^2/\Lambda^2_{QCD})}{\ln(Q^2/\Lambda^2_{QCD})}\right] \tag{11}$$