

## Appendix B

### Legendre Polynomials, Associated

### Legendre Functions and Spherical

### Harmonics

#### 1. Legendre polynomials

Let us consider the real variable  $x$  such that  $-1 \leq x \leq +1$ . We may also set  $x = \cos \theta$ , where  $\theta$  is a real number. The polynomials of degree  $l$  ( $l = 0, 1, 2, \dots$ )

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad (\text{B.1})$$

are known as the *Legendre polynomials*. They satisfy the differential equation

$$\left[ (1 - x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + l(l+1) \right] P_l(x) = 0. \quad (\text{B.2})$$

Furthermore,  $P_l(x)$  has the parity  $(-)^l$  and has  $l$  zeros in the interval  $(-1, +1)$ . A generating function for the Legendre polynomials is

$$\frac{1}{(1 - 2xt + t^2)^{1/2}} = \sum_{l=0}^{\infty} P_l(x) t^l, \quad |t| < 1. \quad (\text{B.3})$$

One also has the recurrence relations

$$(2l+1)xP_l - (l+1)P_{l+1} - lP_{l-1} = 0, \quad (\text{B.4a})$$

$$(x^2 - 1) \frac{dP_l}{dx} = l(xP_l - P_{l-1}) = \frac{l(l+1)}{2l+1} (P_{l+1} - P_{l-1}) \quad (\text{B.4b})$$

(also valid for  $l = 0$  if one defines  $P_{-1} = 0$ ). The orthogonality relations read

$$\int_{-1}^{+1} P_l(x) P_l(x) dx = \frac{2}{2l+1} \delta_{ll'}. \quad (\text{B.5})$$

One also has the closure relation

$$\frac{1}{2} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) P_l(\cos \theta') = \delta(\cos \theta - \cos \theta'). \quad (\text{B.6})$$

Important particular values of the Legendre polynomials are

$$P_l(1) = 1, \quad P_l(-1) = (-1)^l. \quad (\text{B.7})$$

For the lowest values of  $l$  one has explicitly

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned} \quad (\text{B.8})$$

#### 2. Associated Legendre functions

These functions are defined by the relations

$$P_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad m = 0, 1, 2, \dots, l, \quad (\text{B.9})$$

and we see that they are the product of the quantity  $(1 - x^2)^{m/2}$  and of a polynomial of degree  $(l - m)$  and parity  $(-)^{l-m}$ , having  $(l - m)$  zeros in the interval  $(-1, +1)$ . The functions  $P_l^m$  satisfy the differential equation

$$\left[ (1 - x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + l(l+1) - \frac{m^2}{1 - x^2} \right] P_l^m(x) = 0 \quad (\text{B.10})$$

and they are given from a generating function as

$$(2m-1)!(1-x^2)^{m/2} \frac{t^m}{(1-2xt+t^2)^{m+1/2}} = \sum_{l=m}^{\infty} P_l^m(x) t^l, \quad |t| < 1$$

with

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1). \quad (\text{B.11})$$

In particular, one has

$$P_l^0(x) = P_l(x), \quad (\text{B.12})$$

$$P_l^l(x) = (2l-1)!!(1-x^2)^{l/2}. \quad (\text{B.13})$$

The functions  $P_l^m$  satisfy the recurrence relations

$$(2l+1)xP_l^m - (l-m+1)P_{l+1}^m - (l+m)P_{l-1}^m = 0, \quad (\text{B.14})$$

$$\begin{aligned} (x^2 - 1) \frac{dP_l^m}{dx} &= -(l+1)xP_l^m + (l-m+1)P_{l+1}^m \\ &= lxP_l^m - (l+m)P_{l-1}^m, \quad 0 \leq m \leq l-1, \end{aligned} \quad (\text{B.15})$$

$$P_l^{m+2} - 2(m+1) \frac{x}{(1-x^2)^{1/2}} P_l^{m+1} + (l-m)(l+m+1) P_l^m = 0, \quad (B.16)$$

$$P_{l-1}^m - P_{l+1}^m = -(2l+1)(1-x^2)^{1/2} P_l^{m-1}, \quad 0 \leq m \leq l-1 \quad (B.17)$$

and the orthonormality relations

$$\int_{-1}^{+1} P_l^m(x) P_l^n(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ln}. \quad (B.18)$$

Important particular values are

$$P_l^m(1) = P_l^m(-1) = 0, \quad m \neq 0 \quad (B.19)$$

[for  $m = 0$ , see eq. (B.7)]

$$P_l^m(0) = \begin{cases} (-1)^s \frac{(2s+2m)!}{2^s s! (s+m)!}, & \text{if } l-m = 2s \\ 0, & \text{if } l-m = 2s+1. \end{cases} \quad (B.20)$$

The first few associated Legendre functions are given explicitly by

$$\begin{aligned} P_1^1(x) &= (1-x^2)^{1/2}, \\ P_2^1(x) &= 3(1-x^2)^{1/2}x, \\ P_2^2(x) &= 3(1-x^2), \\ P_3^1(x) &= \frac{3}{2}(1-x^2)^{1/2}(5x^2-1), \\ P_3^2(x) &= 15x(1-x^2), \\ P_3^3(x) &= 15(1-x^2)^{3/2}. \end{aligned} \quad (B.21)$$

### 3. Spherical harmonics

The spherical harmonics  $Y_{lm}(\theta, \phi)$  are eigenfunctions of the operators  $L^2$  and  $L_z$ . That is,

$$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}, \quad l = 0, 1, 2, \dots \quad (B.22)$$

$$L_z Y_{lm} = m\hbar Y_{lm}, \quad m = -l, -l+1, \dots, l \quad (B.23)$$

with

$$L_x = i\hbar \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right), \quad (B.24)$$

$$L_y = -i\hbar \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right), \quad (B.25)$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi} \quad (B.26)$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]. \quad (B.27)$$

One has [1]

$$Y_{lm}(\theta, \phi) = (-1)^m \left[ \frac{(2l+1)(l-m)!}{4\pi(l+m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi}, \quad m \geq 0 \quad (B.28)$$

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi). \quad (B.29)$$

The functions  $Y_{lm}$  have the parity  $(-)^l$ . Thus, in a reflection about the origin such that  $(\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi)$ , one has

$$Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi). \quad (B.30)$$

We also note that for  $m = 0$  and  $m = l$  the spherical harmonics are given respectively by the simple expressions

$$Y_{l,0}(\theta) = \left( \frac{2l+1}{4\pi} \right)^{1/2} P_l(\cos \theta) \quad (B.31)$$

and

$$Y_{l,l}(\theta, \phi) = (-1)^l \left[ \frac{2l+1}{4\pi} \frac{(2l)!}{2^{2l}(l!)^2} \right]^{1/2} \sin^l \theta e^{il\phi}. \quad (B.32)$$

The spherical harmonics satisfy the recurrence relations

$$L_{\pm} Y_{lm} = \hbar l(l+1) - m(m \pm 1) Y_{l,m \pm 1},$$

$$L_{+} Y_{l,l} = 0 \quad (B.33)$$

$$L_{-} Y_{l,-l} = 0 \quad (B.34)$$

with

$$L_{\pm} = L_x \pm iL_y = \hbar e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right].$$

The orthonormality relations are

$$\int Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'}, \quad (d\Omega = \sin \theta d\theta d\phi) \quad (B.35)$$

while the closure relation reads

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}^*(\theta, \phi) Y_{lm}(\theta', \phi') = \delta(\Omega - \Omega') \quad (B.36)$$

with

$$\delta(\Omega - \Omega') = \frac{\delta(\theta - \theta') \delta(\phi - \phi')}{\sin \theta}. \quad (B.37)$$

The first few spherical harmonics are given by

$$\begin{aligned} Y_{0,0} &= (4\pi)^{-1/2}, \\ Y_{1,0} &= \left( \frac{3}{4\pi} \right)^{1/2} \cos \theta, \\ Y_{1,1} &= - \left( \frac{3}{8\pi} \right)^{1/2} \sin \theta e^{i\phi}, \end{aligned}$$

$$\begin{aligned}
 Y_{2,0} &= \left(\frac{5}{16\pi}\right)^{1/2} (3 \cos^2 \theta - 1), \\
 Y_{2,1} &= -\left(\frac{15}{8\pi}\right)^{1/2} \sin \theta \cos \theta e^{i\phi}, \\
 Y_{2,2} &= \left(\frac{15}{32\pi}\right)^{1/2} \sin^2 \theta e^{2i\phi}, \\
 Y_{3,0} &= \left(\frac{7}{16\pi}\right)^{1/2} (5 \cos^3 \theta - 3 \cos \theta), \\
 Y_{3,1} &= -\left(\frac{21}{64\pi}\right)^{1/2} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi}, \\
 Y_{3,2} &= \left(\frac{105}{32\pi}\right)^{1/2} \sin^2 \theta \cos \theta e^{2i\phi}, \\
 Y_{3,3} &= -\left(\frac{35}{64\pi}\right)^{1/2} \sin^3 \theta e^{3i\phi}.
 \end{aligned} \tag{B.38}$$

#### 4. Some useful formulae

If  $r_1$  and  $r_2$  are two vectors having polar angles  $(\theta_1, \phi_1)$  and  $(\theta_2, \phi_2)$ , and if we denote by  $\theta$  the angle between these two vectors, the "addition theorem" of spherical harmonics states that

$$P_l(\cos \theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}^*(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) \tag{B.39a}$$

or

$$P_l(\cos \theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}^*(\rho_1) Y_{lm}(\rho_2) \tag{B.39b}$$

where  $\hat{x}$  denotes the polar angles of a vector  $x$ .

Other useful relations are

$$\frac{1}{|r_1 - r_2|} = \sum_{l=0}^{\infty} \frac{(r_<)^l}{(r_>)^{l+1}} P_l(\cos \theta) \tag{B.40}$$

or

$$\frac{1}{|r_1 - r_2|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} \frac{(r_<)^l}{(r_>)^{l+1}} Y_{lm}^*(\rho_1) Y_{lm}(\rho_2) \tag{B.41}$$

where  $r_<$  is the smaller and  $r_>$  the larger of  $r_1$  and  $r_2$ . One also has

$$\exp\{ik|r_1 - r_2|\} = ik \sum_{l=0}^{\infty} (2l+1) j_l(kr_<) h_l^{(1)}(kr_>) P_l(\cos \theta) \tag{B.42}$$

or

$$\frac{\exp\{ik|r_1 - r_2|\}}{|r_1 - r_2|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}^*(\rho_1) Y_{lm}(\rho_2) \tag{B.43}$$

where  $j_l$  and  $h_l^{(1)}$  are respectively the spherical Bessel function and the spherical Hankel function of the first kind (see Appendix C).

The development in spherical harmonics of a plane wave  $\exp(ik \cdot r)$  of wave vector  $k$  is given by

$$\exp(ik \cdot r) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} i^l j_l(kr) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{r}). \tag{B.44}$$

Using the addition theorem (B.39), we may also write

$$\exp(ik \cdot r) = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta) \tag{B.45}$$

where  $\theta$  is the angle between the directions of the vectors  $k$  and  $r$ . In particular, if we choose the  $z$ -axis to coincide with the direction of  $k$ , we have

$$\exp(ik \cdot r) \equiv e^{ikz} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta). \tag{B.46}$$

Finally, we quote the relation

$$\begin{aligned}
 \int Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) Y_{l_3 m_3}(\theta, \phi) d\Omega \\
 = \left[ \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
 \end{aligned} \tag{B.47}$$

where we have introduced the Wigner 3- $j$  symbols (see Appendix E). From eq. (B.47) one also finds that

$$\begin{aligned}
 Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) = \sum_{L=|l_1-l_2|}^{l_1+l_2} \sum_{M=-L}^L \left[ \frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)} \right]^{1/2} \\
 \times \langle l_1 l_2 00 | L 0 \rangle \langle l_1 l_2 m_1 m_2 | L M \rangle Y_{LM}(\theta, \phi)
 \end{aligned} \tag{B.48}$$

where we have used vector addition coefficients (see Appendix E). This last equation may also be written in terms of Wigner 3- $j$  symbols as

$$\begin{aligned}
 Y_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) = \sum_{L=|l_1-l_2|}^{l_1+l_2} \sum_{M=-L}^L (-1)^M \\
 \times \left[ \frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi} \right]^{1/2} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & M \end{pmatrix} Y_{L,-M}(\theta, \phi).
 \end{aligned} \tag{B.49}$$

Additional useful formulae involving the Legendre polynomials, associated Legendre functions and spherical harmonics may be found in the references [2-5].

## References and notes

- [1] The equations (B.22)–(B.23) and (B.35) determine the functions  $Y_{lm}(\theta, \phi)$  up to a phase. The choice of phase made in writing down the relations (B.28)–(B.29) ensures that
  - i) The functions  $Y_{lm}$  obtained in this way verify the recurrence relations (B.33)
  - ii)  $Y_{l,0}(0, 0)$  is real and positive.
 Since different authors choose different phase factor conventions for the spherical harmonics, one should be careful to check this point in dealing with the functions  $Y_{lm}$  used in the physics literature.
- [2] ABRAMOWITZ, M. and I. A. STEGUN (1965), *Handbook of Mathematical Functions* (Dover Publ., New York) Chapter 8.
- [3] MAGNUS, W. and F. OBERHETTINGER (1954), *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea, New York) Chapter 4.
- [4] ERDELYI, A., W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI (1953), *Higher Transcendental Functions* (Bateman Manuscript Project, McGraw-Hill, New York) Vol. I, Chapter 3.

## Appendix C

### Spherical Bessel Functions

Let us consider the differential equation (4.20), namely

$$\left[ \frac{d^2}{dz^2} + \frac{2}{z} \frac{d}{dz} + \left( 1 - \frac{l(l+1)}{z^2} \right) \right] f_l = 0 \quad (\text{C.1})$$

with  $l = 0, 1, 2, \dots$ . Particular solutions of this equation are:

- i) The (genuine) *spherical Bessel functions* (or spherical Bessel functions of the first kind)

$$j_l(z) = \left( \frac{\pi}{2z} \right)^{1/2} J_{l+1/2}(z) \quad (\text{C.2})$$

where  $J_\nu(z)$  is an ordinary Bessel function of order  $\nu$ . The functions  $j_l(z)$  are *regular* at the origin [see eq. (C.11a)].

- ii) The *spherical Neumann functions*

$$n_l(z) = (-1)^{l+1} \left( \frac{\pi}{2z} \right)^{1/2} J_{-l-1/2}(z) \quad (\text{C.3})$$

which are *irregular* solutions of eq. (C.1)

- iii) The *spherical Hankel functions* of the first and second kind

$$h_l^{(1)}(z) = j_l(z) + in_l(z) \quad (\text{C.4})$$

and

$$h_l^{(2)}(z) = j_l(z) - in_l(z) \quad (\text{C.5})$$

which are *irregular* solutions of eq. (C.1). Thus

$$j_A(z) = \frac{1}{2} [h_l^{(1)}(z) + h_l^{(2)}(z)] \quad (\text{C.6})$$

and

$$n_l(z) = \frac{1}{2l} [h_l^{(1)}(z) - h_l^{(2)}(z)]. \quad (\text{C.7})$$

The pairs  $\{j_l(z), n_l(z)\}$  and  $\{h_l^{(1)}(z), h_l^{(2)}(z)\}$  are linearly independent solutions of eq. (C.1) for every  $l$ .

The first three functions  $j_l$  and  $n_l$  are given explicitly by

$$\begin{aligned} j_0(z) &= \frac{\sin z}{z}, \\ j_1(z) &= \frac{\sin z}{z^2} - \frac{\cos z}{z}, \\ j_2(z) &= \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z \end{aligned} \quad (\text{C.8})$$

and

$$\begin{aligned} n_0(z) &= -\frac{\cos z}{z}, \\ n_1(z) &= -\frac{\cos z}{z^2} - \frac{\sin z}{z}, \\ n_2(z) &= -\left(\frac{3}{z^3} - \frac{1}{z}\right) \cos z - \frac{3}{z^2} \sin z. \end{aligned} \quad (\text{C.9})$$

The functions  $j_0(x), j_1(x), j_2(x)$  and  $n_0(x), n_1(x), n_2(x)$ , where  $x$  is real, are plotted in Figs. C.1 and C.2.

The functions  $j_l(z)$  and  $n_l(z)$  may be represented by the ascending series [1-3]

$$j_l(z) = \frac{z^l}{(2l+1)!} \left[ 1 - \frac{\frac{1}{2}z^2}{1!(2l+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2l+3)(2l+5)} - \dots \right] \quad (\text{C.10a})$$

$$n_l(z) = -\frac{(2l-1)!}{z^{l+1}} \left[ 1 - \frac{\frac{1}{2}z^2}{1!(1-2l)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2l)(3-2l)} - \dots \right] \quad (\text{C.10b})$$

where

$$(2l+1)! = 1 \cdot 3 \cdot 5 \cdots (2l+1).$$

We see from eqs. (C.10) that for  $z \rightarrow 0$  one has

$$z^{-l} j_l(z) \xrightarrow{z \rightarrow 0} \frac{1}{(2l+1)!}; \quad j_l(z) \sim \frac{z^l}{(2l+1)!}, \quad (\text{C.11a})$$

$$z^{l+1} n_l(z) \xrightarrow{z \rightarrow 0} -(2l-1)!!; \quad n_l(z) \sim -\frac{(2l-1)!}{z^{l+1}}. \quad (\text{C.11b})$$

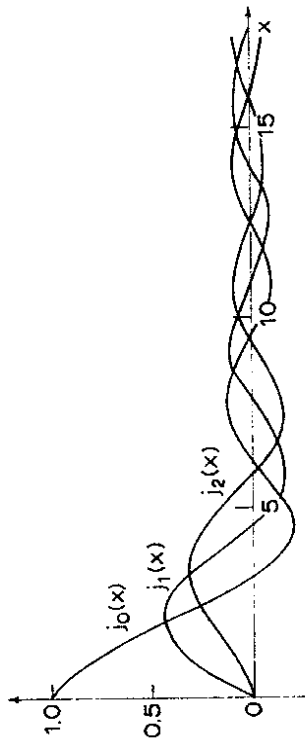


Fig. C.1. The first three spherical Bessel functions.

For real  $x$  somewhat larger than  $l(l+1)$  one may use the asymptotic formulae

$$j_l(x) \xrightarrow{x \rightarrow \infty} \frac{1}{x} \sin\left(x - \frac{1}{2}l\pi\right), \quad (\text{C.12a})$$

$$n_l(x) \xrightarrow{x \rightarrow \infty} -\frac{1}{x} \cos\left(x - \frac{1}{2}l\pi\right), \quad (\text{C.12b})$$

$$h_l^{(1)}(x) \xrightarrow{x \rightarrow \infty} -i \frac{\exp\{i(x - \frac{1}{2}l\pi)\}}{x}, \quad (\text{C.12c})$$

$$h_l^{(2)}(x) \xrightarrow{x \rightarrow \infty} i \frac{\exp\{-i(x - \frac{1}{2}l\pi)\}}{x}. \quad (\text{C.12d})$$

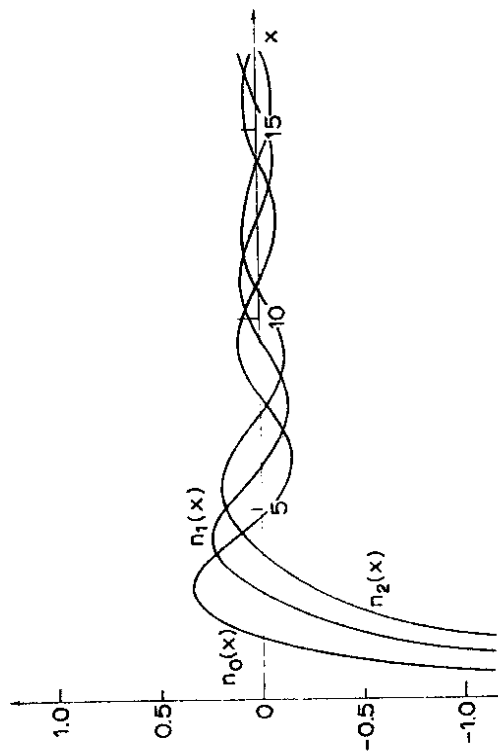


Fig. C.2. The first three spherical Neumann functions.

Some important properties of the spherical Bessel functions  $f_l$  [with  $f_l : j_l, n_l, h_l^{(1)}, h_l^{(2)}$ ] are the recurrence relations (we assume that  $l > 0$ )

$$f_{l-1}(z) + f_{l+1}(z) = \frac{2l+1}{z} f_l(z), \quad (\text{C.13a})$$

$$\frac{d}{dz} f_l(z) = \frac{1}{2l+1} [l f_{l-1}(z) - (l+1) f_{l+1}(z)] \quad (\text{C.13b})$$

$$f_{l-1}(z) = \frac{l+1}{z} f_l(z) + \frac{d}{dz} f_l(z) \quad (\text{C.13c})$$

$$f_{l+1}(z) = \frac{l}{z} f_l(z) - \frac{d}{dz} f_l(z). \quad (\text{C.13d})$$

One also has the differentiation formulae

$$\frac{d}{dz} [z^{l+1} f_l(z)] = z^{l+1} f_{l-1}(z), \quad (\text{C.14a})$$

$$\frac{d}{dz} [z^{-l} f_l(z)] = -z^{-l} f_{l+1}(z) \quad (\text{C.14b})$$

and the analytic continuations (with  $l, m = 0, 1, 2, \dots$ )

$$j_l(z e^{m\pi i}) = e^{m\pi i l} j_l(z), \quad (\text{C.15a})$$

$$n_l(z e^{m\pi i}) = (-1)^m e^{m\pi i l} n_l(z), \quad (\text{C.15b})$$

$$h_l^{(1)}(z e^{(2m+1)\pi i}) = (-1)^l h_l^{(2)}(z), \quad (\text{C.15c})$$

$$h_l^{(2)}(z e^{(2m+1)\pi i}) = (-1)^l h_l^{(1)}(z), \quad (\text{C.15d})$$

$$h_l^{(k)}(z e^{2m\pi i}) = h_l^{(k)}(z), \quad k = 1, 2, \quad (\text{C.15e})$$

In particular, we see that

$$j_l(-z) = (-1)^l j_l(z), \quad (\text{C.16a})$$

$$n_l(-z) = (-1)^{l+1} n_l(z), \quad (\text{C.16b})$$

$$h_l^{(1)}(-z) = (-1)^l h_l^{(2)}(z), \quad (\text{C.16c})$$

$$h_l^{(2)}(-z) = (-1)^l h_l^{(1)}(z). \quad (\text{C.16d})$$

Additional useful properties of the functions  $j_l$  and  $n_l$  are

$$j_l(z) n_{l-1}(z) - j_{l-1}(z) n_l(z) = z^{-2}, \quad l > 0 \quad (\text{C.17a})$$

$$j_l(z) \frac{d}{dz} n_l(z) - n_l(z) \frac{d}{dz} j_l(z) = z^{-2} \quad (\text{C.17b})$$

$$\int j_0^2(x) x^2 dx = \frac{1}{2} x^3 [j_0^2(x) + n_0(x) j_1(x)] \quad (\text{C.17c})$$

$$\int n_0^2(x) x^2 dx = \frac{1}{2} x^3 [n_0^2(x) - j_0(x) n_1(x)] \quad (\text{C.17d})$$

$$\int j_1(x) dx = -j_0(x) \quad (\text{C.17e})$$

$$\int j_0(x) x^2 dx = x^2 j_1(x) \quad (\text{C.17f})$$

$$\int j_l^2(x) x^2 dx = \frac{1}{2} x^3 [j_l^2(x) - j_{l-1}(x) j_{l+1}(x)], \quad l > 0. \quad (\text{C.17g})$$

The last three formulae are equally valid with the  $j$ 's replaced by the corresponding  $n$ 's.

Let us also quote a few definite integrals [4] involving the functions  $j_l$  and which are often used in scattering theory calculations, namely

$$\begin{aligned} \int_0^\infty e^{-ax} j_l(bx) x^{\mu-1} dx &= \frac{\sqrt{\pi} b^l \Gamma(\mu+1)}{2^{l+\frac{1}{2}} a^{\mu+1} \Gamma(l+\frac{3}{2})} \\ &\times {}_2F_1\left(\frac{\mu+l}{2}, \frac{\mu+l+1}{2}; l+\frac{3}{2}; -\frac{b^2}{a^2}\right) \end{aligned}$$

$$(\operatorname{Re}(a+ib) > 0, \operatorname{Re}(a-ib) > 0, \operatorname{Re}(\mu+l) > 0), \quad (\text{C.18a})$$

$$\int_0^\infty e^{-ax} j_l(bx) x^{l+1} dx = \frac{(2b)^l \Gamma(l+1)}{(a^2+b^2)^{l+1}} \quad (\operatorname{Re} a > |\operatorname{Im} b|), \quad (\text{C.18b})$$

$$\int_0^\infty e^{-ax} j_l(bx) x^{l+2} dx = \frac{2a (2b)^l \Gamma(l+2)}{(a^2+b^2)^{l+2}} \quad (\operatorname{Re} a > |\operatorname{Im} b|). \quad (\text{C.18c})$$

Similar integrals involving higher powers of  $x$  may be obtained by differentiating with respect to the quantity  $a$ .

Finally, we note that

$$\int_0^\infty j_l(kr) j_l(k'r) r^2 dr = \frac{\pi}{2k^2} \delta(k-k'). \quad (\text{C.19})$$

## References

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