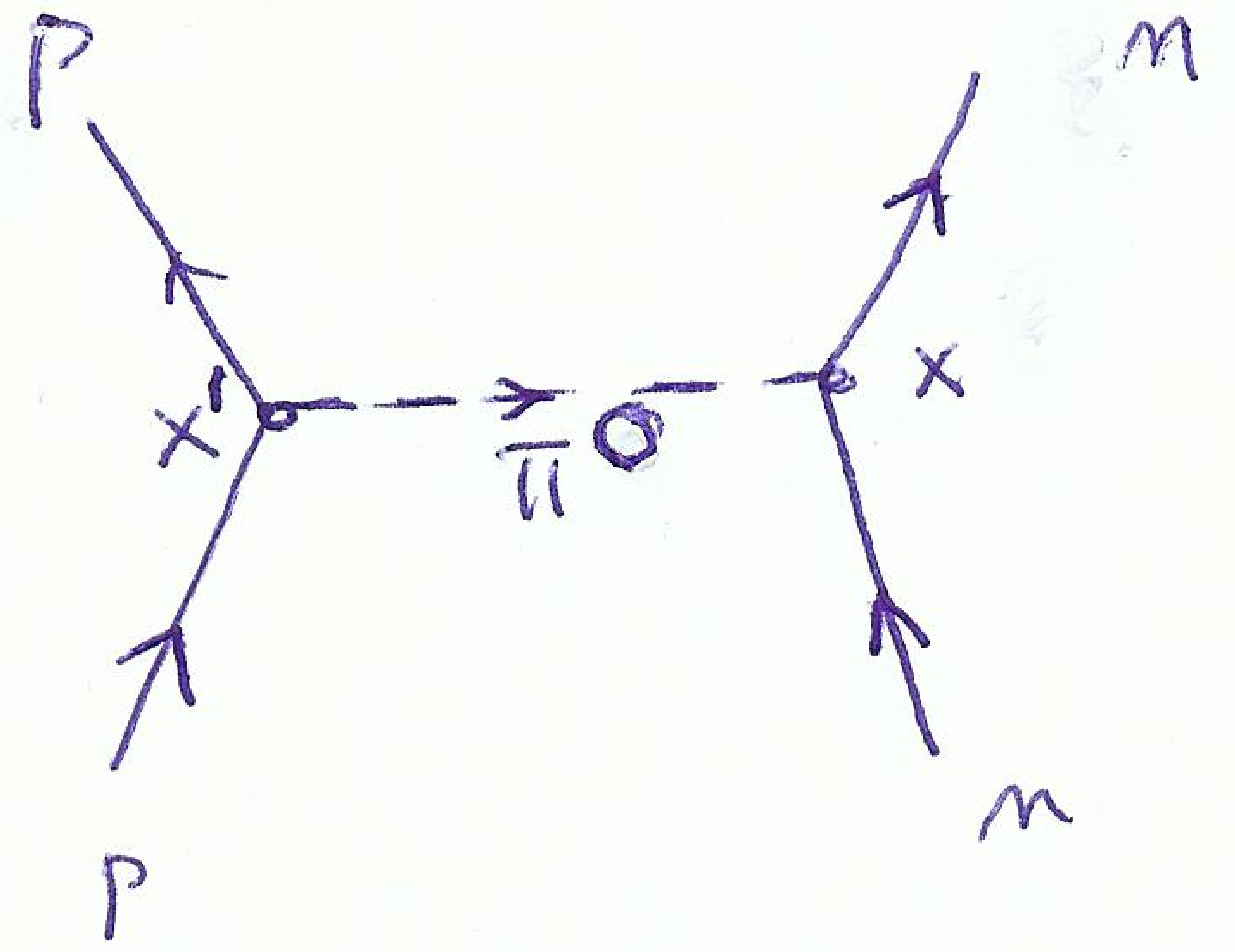
SUPPLEMENTARY NOTES ON FEYNMAN RULES

⇒ SPIN-0 PROPAGATOR

↳



SPIN-0 PARTICLE (π^0) PROPAGATING FROM SPACE-TIME POINT x' TO SPACE-TIME POINT x

IN EACH POINT SPIN-0 FIELD $\phi(x)$ IS REPRESENTED IN SECOND QUANTIZATION AS:

$$\phi(x) = \int \frac{d^3\bar{k}}{\sqrt{(2\pi)^3 2E_{\bar{k}}}} \left\{ a(\bar{k})\, e^{-ikx} + a^+(\bar{k})\, e^{+ikx} \right\}$$

$$\equiv \underset{\substack{\uparrow \\ \text{ANNIHILATION TERM} \\ \sim a(\bar{k})}}{\phi_+(x)} + \underset{\substack{\uparrow \\ \text{CREATION TERM} \\ \sim a^+(\bar{k})}}{\phi_-(x)}$$

$$E_{\bar{k}} = \sqrt{\bar{k}^2 + m^2}$$

$$e^{-ikx} = e^{-iE_{\bar{k}}t + i\bar{k}.\bar{x}}$$

$$e^{+ikx} = e^{+iE_{\bar{k}}t - i\bar{k}.\bar{x}}$$

$a(\bar{k})$, $a^+(\bar{k})$ SATISFY COMMUTATION RELATIONS

$$\left[a(\bar{k}),\ a(\bar{k}')\right] = 0$$

$$\left[a^+(\bar{k}),\ a^+(\bar{k}')\right] = 0$$

$$\left[a(\bar{k}),\ a^+(\bar{k}')\right] = \delta^3(\bar{k}-\bar{k}')$$

↳ COMMUTATORS OF $\phi_+(x)$, $\phi_-(x')$ FIELDS AT DIFFERENT SPACE-TIME POINTS x, x'

- NOTE IF $x^0 = x'^0$

$$\Downarrow$$

ETCR

EQUAL

TIME

COMMUTATION

RELATIONS

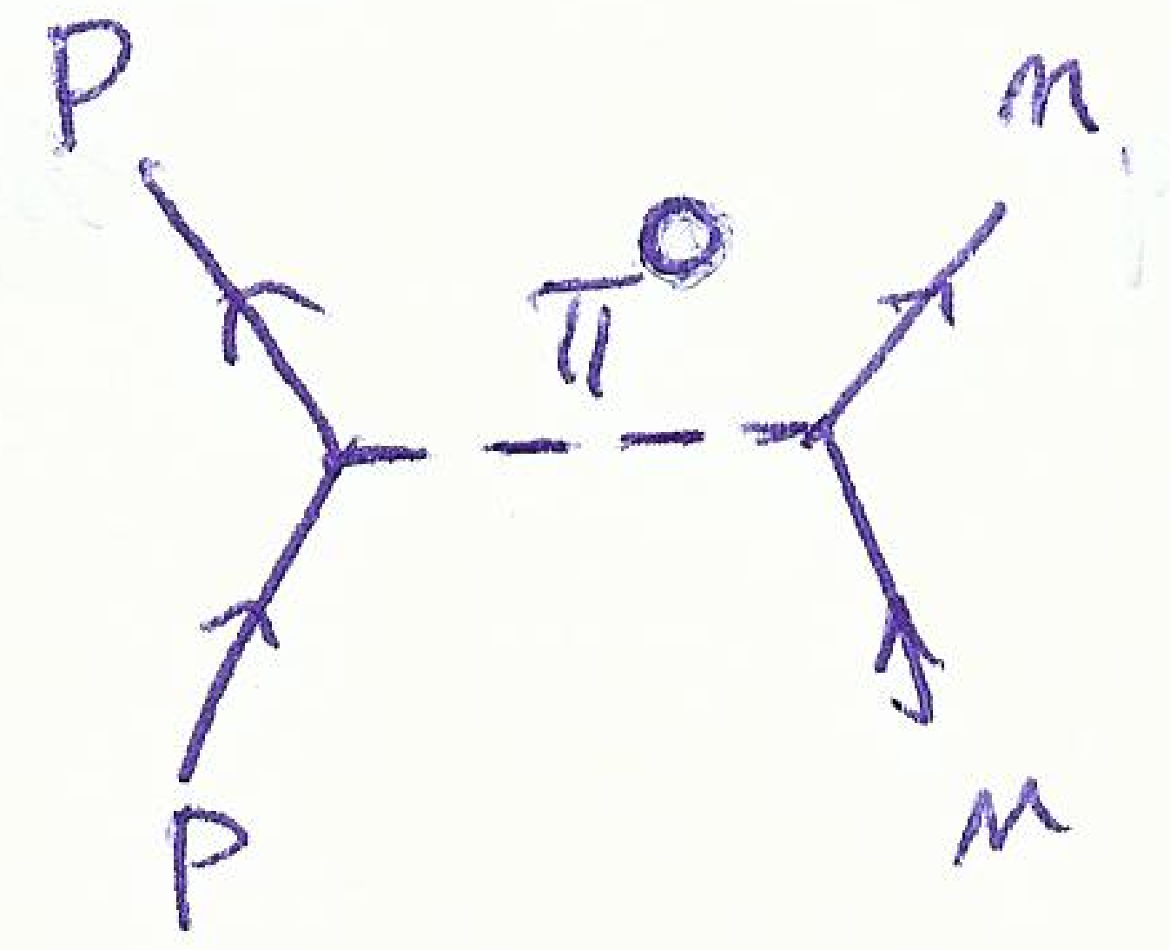
$$\left[\phi(x^0,\bar{x}),\ \phi(x^0,\bar{x}')\right] = 0$$

- IF $\underset{\sim\sim\sim}{x^0 \neq x'^0}$

THESE COMMUTATORS IN GENERAL DON'T VANISH

THEY ENTER MATRIX ELEMENTS OF PHYSICAL PROCESSES WHERE A SPIN-0 PARTICLE PROPAGATES

e.g.



$\rightsquigarrow$ WE WANT TO CALCULATE

$$<0|\left[\phi_+(x),\ \phi_-(x')\right]|0>$$

CALCULATE FIRST :

$$\left[\phi_+(x),\ \phi_-(x')\right]$$

$$= \int \frac{d^3\bar{k}}{\left[(2\pi)^3 2E_{\bar{k}}\right]^{1/2}} \int \frac{d^3\bar{k}'}{\left[(2\pi)^3 2E_{\bar{k}'}\right]^{1/2}}\ e^{-ik.x}\ e^{+ik'.x'} \cdot \underbrace{\left[a(\bar{k}),\ a^+(\bar{k}')\right]}_{\delta^3(\bar{k}-\bar{k}')}$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}}\ e^{-ik.(x-x')}$$

$$\langle 0 | [\phi_+(x), \phi_-(x')] | 0 \rangle$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} e^{-ik\cdot(x-x')}$$ WITH $k^0 = E_{\bar{k}}$

$$\equiv i\Delta^+(x-x')$$ ONLY FUNCTION OF DIFFERENCE BETWEEN SPACE-TIME POINTS $x-x'$ (SPACE-TIME TRANSLATIONAL INVARIANCE)

<u>NOTE</u>: D^+ SATISFIES KLEIN-GORDON EQ.: $(\Box + m^2)\Delta^+(x) = 0$
BECAUSE $(k^0)^2 - \bar{k}^2 = m^2$

$\rightsquigarrow$ ANALOGOUSLY

$$\langle 0 | [\phi_-(x), \phi_+(x')] | 0 \rangle$$

$$= \int \frac{d^3\bar{k}}{[(2\pi)^3 2E_{\bar{k}}]^{1/2}} \int \frac{d^3\bar{k}'}{[(2\pi)^3 2E_{\bar{k}'}]^{1/2}} e^{+ikx} e^{-ik'x'} \cdot \underbrace{[a^+(\bar{k}), a(\bar{k}')]}_{-\delta^3(\bar{k}-\bar{k}')}$$

$$\downarrow$$

$$\langle 0 | [\phi_-(x), \phi_+(x')] | 0 \rangle$$

$$= - \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} e^{ik\cdot(x-x')}$$

$$= i\Delta^-(x-x') = -i\Delta^+(x'-x)$$

NOTE: CHANGE IN ORDER OF ARGUMENTS

↳ TIME - ORDERED PRODUCT

$$T\{\phi(x).\phi(x')\} \equiv \theta(x^0 - x'^0)\,\phi(x)\,\phi(x') + \theta(x'^0 - x^0)\,\phi(x')\,\phi(x)$$

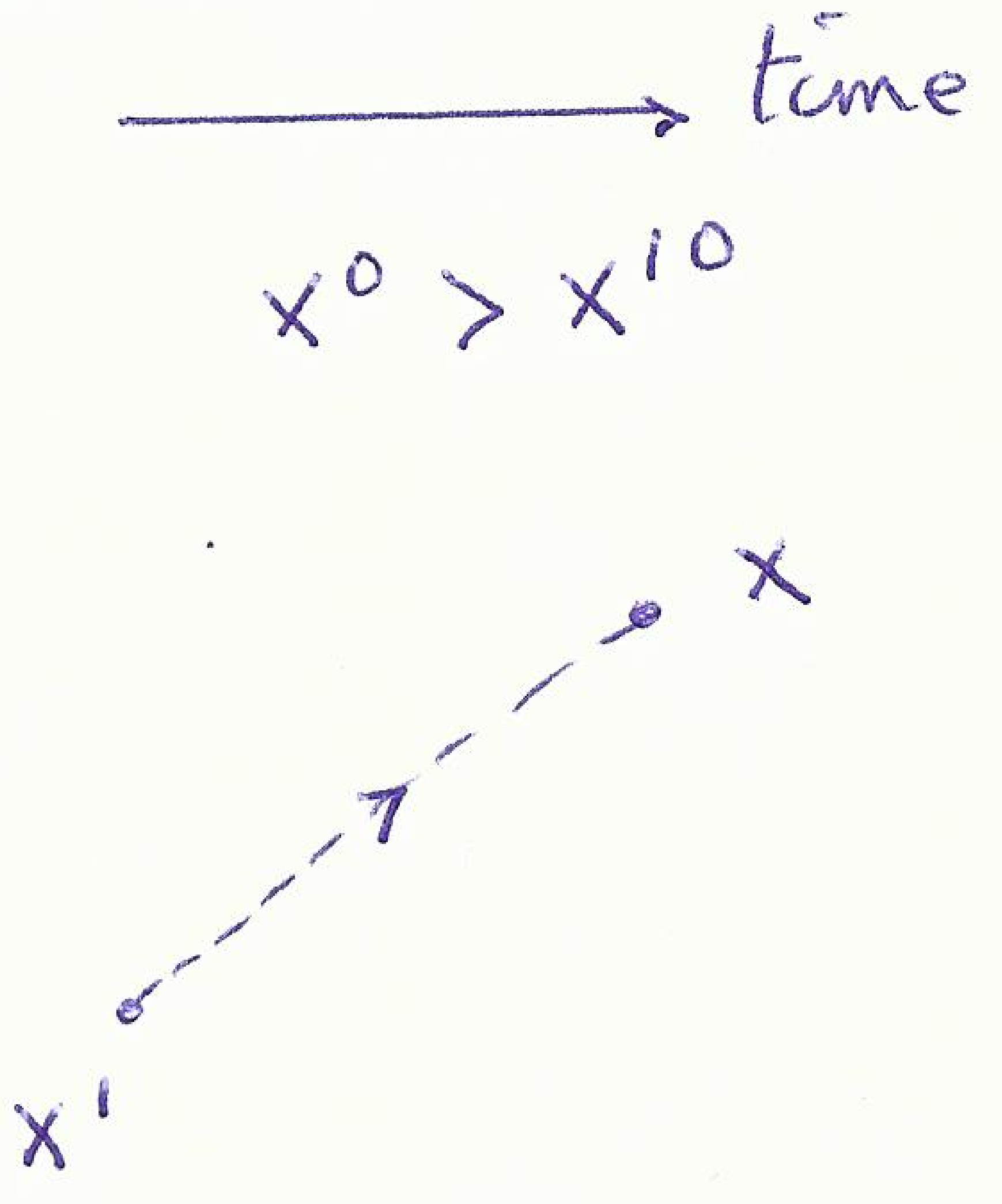
$$\theta(\alpha) = \begin{cases} 1, & \alpha > 0 \\ 0, & \alpha < 0 \end{cases}$$

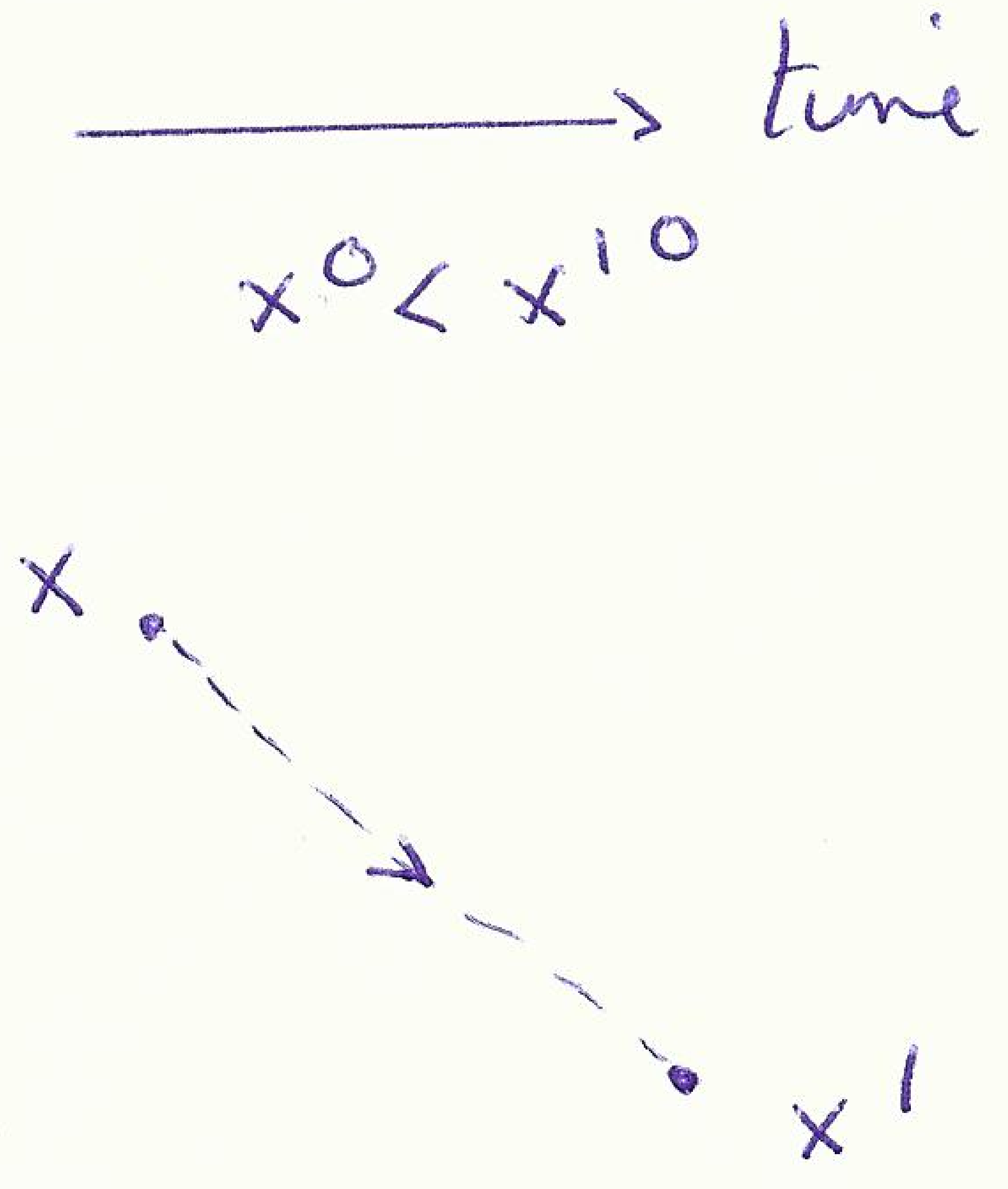
IDEA OF TIME - ORDERED PRODUCT :

FIELD WITH LARGEST TIME ARGUMENT

STANDS ON LEFT

SCHEMATICALLY





INTERPRETATION :
PARTICLE
PROPAGATING
FROM x' TO x

PARTICLE
PROPAGATING
FROM x TO x'

DEFINE A PROPAGATOR FUNCTION Δ_F AS → FEYNMAN PROPAGATOR

$$\boxed{\langle 0 | T\{\phi(x).\phi(x')\} | 0 \rangle \equiv i\Delta_F(x - x')}$$

↳ THIS FUNCTION WILL ENTER PHYSICAL PROCESSES (→ FEYNMAN DIAGRAMS).

WE CAN EXPRESS Δ_F USING Δ^+ AND Δ^- AS

$$i\Delta_F(x-x') = \theta(x^0-x'^0)\,\langle 0|\phi(x)\,\phi(x')|0\rangle$$

$$+\,\theta(x'^0-x^0)\,\langle 0|\phi(x')\,\phi(x)|0\rangle$$

AND USING

$$\langle 0|\phi(x)\,\phi(x')|0\rangle$$

$$= \langle 0|\big(\phi_+(x)+\phi_-(x)\big)\cdot\big(\phi_+(x')+\phi_-(x')\big)|0\rangle$$

$\downarrow$ $\quad \phi_+(x')|0\rangle = 0$ PARTICLE CANNOT BE ANNIHILATED FROM VACUUM

$$\langle 0|\phi_-(x) = 0$$

$$= \langle 0|\phi_+(x)\,\phi_-(x')|0\rangle$$

$$= i\Delta^+(x-x')$$

$$\therefore\quad i\Delta_F(x-x') = \theta(x^0-x'^0)\, i\Delta^+(x-x')$$

$$+\,\theta(x'^0-x^0)\, i\Delta^+(x'-x)$$

$\downarrow$ USING $\Delta^-(x-x') = -\Delta^+(x'-x)$

$$\Delta_F(x-x') = \theta(x^0-x'^0)\,\Delta^+(x-x') - \theta(x'^0-x^0)\,\Delta^-(x-x')$$

⤳ USING OUR EXPRESSIONS FOR Δ^+ & Δ^-,

WE OBTAIN

$$i\,\Delta_F(x-x') = \theta(x^0-x'^0)\int\frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}}\,e^{-ik.(x-x')} + \theta(x'^0-x^0)\int\frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}}\,e^{+ik.(x-x')}$$

$$\boxed{i\,\Delta_F(x-x') = \int\frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}}\left\{\theta(x^0-x'^0)\,e^{-ik.(x-x')} + \theta(x'^0-x^0)\,e^{+ik.(x-x')}\right\}}$$

WITH $k^0 = E_{\bar{k}}$

↳ PROPAGATOR IN MOMENTUM SPACE

$$\boxed{\Delta_F(x-x') \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{-ik.(x-x')}\, \Delta_F(k^2)}$$

FOURIER TF. $\Delta_F(k^2)$ IS GIVEN BY

$$\boxed{\Delta_F(k^2) = \frac{1}{k^2 - m^2 + i\varepsilon}}$$

$\varepsilon > 0$
INFINITESIMAL

PROOF : BY CONTOUR INTEGRATION IN COMPLEX k^0 PLANE

$\Delta_F(k^2)$ HAS 2 POLES

$$\Delta_F(k^2) = \frac{1}{(k^0 - E_{\bar{k}} + i\varepsilon)(k^0 + E_{\bar{k}} - i\varepsilon)}$$

↓ POLE AT $k^0 = E_{\bar{k}} - i\varepsilon$

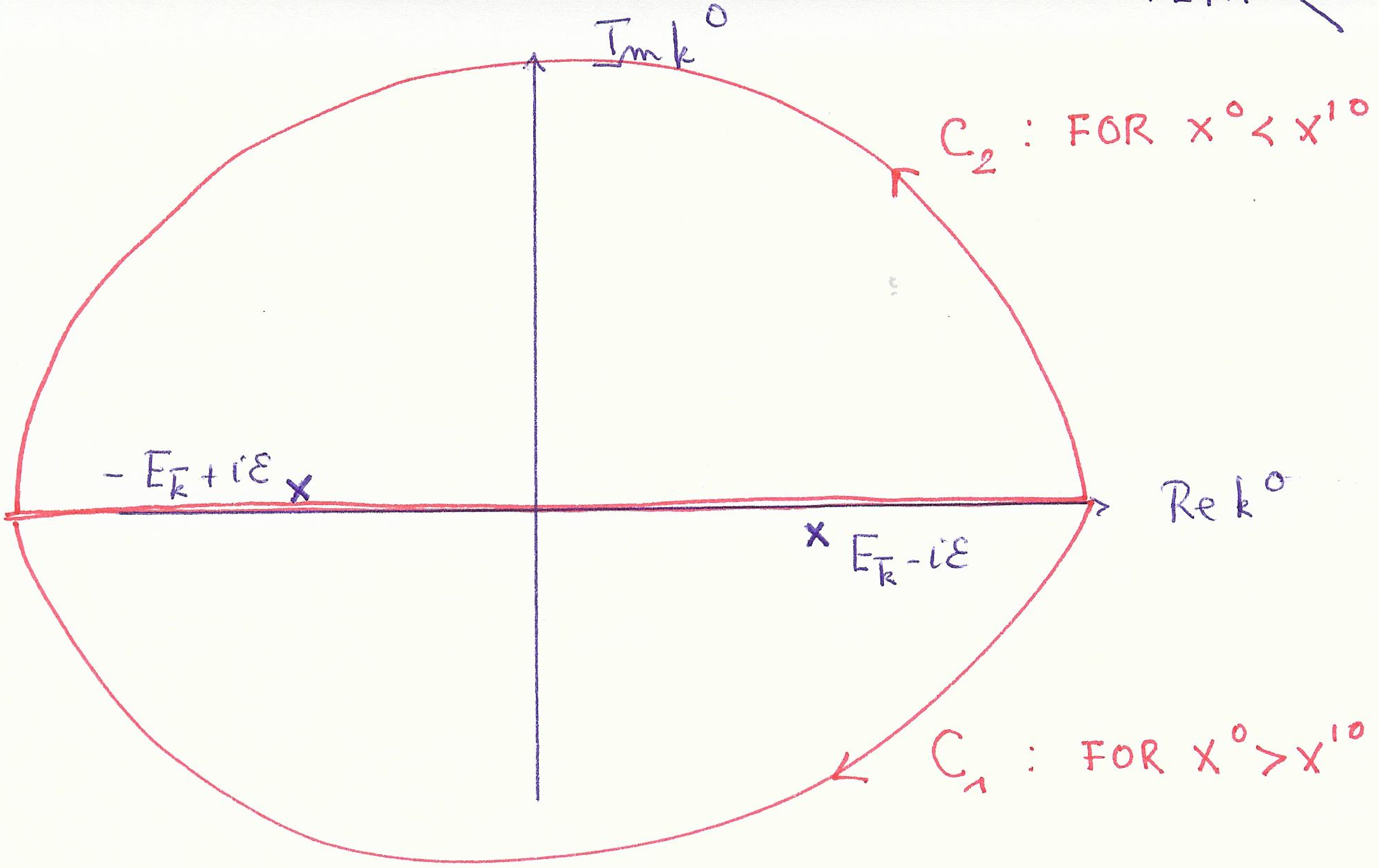
↓ POLE AT $k^0 = -E_{\bar{k}} + i\varepsilon$

NOTICE $(k^0 - E_{\bar{k}} + i\varepsilon)(k^0 + E_{\bar{k}} - i\varepsilon)$

$$= (k^0)^2 - \underbrace{E_{\bar{k}}^2}_{\bar{k}^2 + m^2} + 2i E_{\bar{k}}\,\varepsilon$$

↓ FOR ε INFINITESIMAL

$$= k^2 - m^2 + i\varepsilon$$



- FOR : $\underline{\underline{x^0 - x'^0 > 0}}$: CLOSE CONTOUR IN LOWER HALF-PLANE

$$e^{-ik_0\cdot(x^0-x'^0)}$$

REQUIRES $\text{Im}\,k^0 < 0$

IN ORDER TO NEGLECT THE SEMI-CIRCLE AT INFINITY

$$x^0 > x'^0 : \quad \Delta_F(x-x') = \int \frac{d^3\bar k}{(2\pi)^4}\, e^{i\bar k\cdot(\bar x - \bar x')} \cdot \underbrace{\oint_{C_1} dk^0 \, \frac{e^{-ik^0(x^0-x'^0)}}{(k^0 - E_{\bar k} + i\varepsilon)(k^0 + E_{\bar k} - i\varepsilon)}}_{\text{RESIDUE THEOREM} \;\downarrow}$$

$$-2\pi i \, \frac{1}{2E_{\bar k}}\, e^{-iE_{\bar k}\cdot(x^0-x'^0)}$$

$$x^0 > x'^0 : \quad \Delta_F(x-x') = -i \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} e^{-ik\cdot(x-x')}$$

WITH $k^0 = E_{\bar{k}}$

- FOR $\underline{\underline{x^0 - x'^0 < 0}}$: CLOSE CONTOUR IN UPPER HALF-PLANE IN ORDER TO NEGLECT THE SEMI-CIRCLE AT INFINITY

$$x^0 < x'^0 : \quad \Delta_F(x-x') = \int \frac{d^3\bar{k}}{(2\pi)^4} e^{i\bar{k}\cdot(\bar{x}-\bar{x}')}$$

$$\underbrace{\cdot \int_{C_2} dk^0 \frac{e^{-ik^0(x^0-x'^0)}}{(k^0 - E_{\bar{k}} + i\varepsilon)(k^0 + E_{\bar{k}} - i\varepsilon)}}_{}$$

RESIDUE THEOREM ↓

$$+ 2\pi i \cdot \frac{1}{(-2E_{\bar{k}})} e^{+iE_{\bar{k}}(x^0-x'^0)}$$

$$\Delta_F(x-x') = -i \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} e^{iE_{\bar{k}}(x^0-x'^0)} e^{+i\bar{k}\cdot(\bar{x}-\bar{x}')}$$

↓ CHANGE INT. VARIABLE $\bar{k} \to -\bar{k}$

$$= -i \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} e^{iE_{\bar{k}}(x^0-x'^0) - i\bar{k}\cdot(\bar{x}-\bar{x}')}$$

$$= -i \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} e^{ik\cdot(x-x')}$$

WITH $k^0 = E_{\bar{k}}$

- IN TOTAL :

$$\Delta_F(x-x') = -i \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}} \left\{ \theta(x^0-x'^0)\, e^{-ik.(x-x')} + \theta(x'^0-x^0)\, e^{+ik.(x-x')} \right\}$$

WITH $k^0 = E_{\bar{k}}$

↳ THIS AGREES WITH EXPRESSION OBTAINED BEFORE, WHICH COMPLETES THE PROOF THAT FOURIER TF $\Delta_F(k^2) = \frac{1}{k^2-m^2+i\varepsilon}$

↳ $$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{1}{k^2-m^2+i\varepsilon}$$

$$(\Box_x + m^2)\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{(-k^2+m^2)}{k^2-m^2+i\varepsilon}$$

$$= -\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}$$

$$\boxed{(\Box_x + m^2)\Delta_F(x) = -\delta^4(x)}$$

PROPAGATOR FUNCTION HAS INTERPRETATION OF GREEN'S FUNCTION

↳ SOLUTION OF FIELD EQUATION WITH $-\delta^4(x)$ SOURCE TERM

⇒ SPIN 1/2 PROPAGATOR.

$$\hookrightarrow \quad \psi(x) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \sum_{s=\pm\frac{1}{2}} \Big\{ b(\bar{p},s)\, u(\bar{p},s)\, e^{-ipx} + d^{+}(\bar{p},s)\, v(\bar{p},s)\, e^{+ipx} \Big\}$$

$$= \underset{\substack{\downarrow \\ \text{ANNIHILATION}}}{\psi^{+}(x)} + \underset{\substack{\downarrow \\ \text{CREATION}}}{\psi^{-}(x)}$$

$$\psi^{+}(x)\,|0\rangle = 0$$

$$\hookrightarrow \quad \Big[\psi^{+}_{\alpha}(x), \bar{\psi}^{-}_{\beta}(x')\Big]_{+}$$

ANTI-COMMUTATOR.

$$= \int \frac{d^3\bar{p}}{\left[(2\pi)^3 2E_{\bar{p}}\right]^{1/2}} \int \frac{d^3\bar{p}'}{\left[(2\pi)^3 2E_{\bar{p}'}\right]^{1/2}} \sum_{s} \sum_{s'} \cdot$$

$$\cdot\; e^{-ipx}\, e^{ip'x'}\, u_{\alpha}(\bar{p},s)\, \bar{u}_{\beta}(\bar{p}',s')$$

$$\cdot\; \underbrace{\Big[b(\bar{p},s), b^{+}(\bar{p}',s')\Big]_{+}}_{\delta_{ss'}\,\delta^3(\bar{p}-\bar{p}')}$$

$$[\psi_\alpha^+(x), \bar{\psi}_\beta^-(x')]_+$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} \, e^{-ip\cdot(x-x')} \underbrace{\sum_s u_\alpha(\bar{p},s)\,\bar{u}_\beta(\bar{p},s)}_{(\not{p}+m)_{\alpha\beta}}$$

$$= (i\gamma^\mu\partial_\mu + m)_{\alpha\beta} \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} \, e^{-ip\cdot(x-x')}}_{i\Delta^+(x-x')}$$

SCALAR THEORY

$$\therefore \boxed{\begin{array}{l} [\psi_\alpha^+(x), \bar{\psi}_\beta^-(x')]_+ \equiv i S^+_{\alpha\beta}(x-x') \\ \text{WITH} \\ S^+(x) = (i\gamma^\mu\partial_\mu + m)\,\Delta^+(x) \end{array}}$$

↳ ANALOGOUSLY

$$[\psi_\alpha^-(x), \bar{\psi}_\beta^+(x')]_+ \equiv i S^-_{\alpha\beta}(x-x')$$

WITH

$$S^-(x) = (i\gamma^\mu\partial_\mu + m)\,\Delta^-(x) \quad \& \; \Delta^-(x) = -\Delta^+(-x)$$

↳ **FEYNMAN PROPAGATOR**

$$\leadsto \boxed{\begin{aligned} &\langle 0 | T\{\psi_\alpha(x)\,\bar{\psi}_\beta(x')\} | 0 \rangle \\ &\equiv i\big(S_F(x-x')\big)_{\alpha\beta} \end{aligned}}$$

$$T\{\psi_\alpha(x)\,\bar{\psi}_\beta(x')\}$$

$$= \Theta(x^0 - x'^0)\;\psi_\alpha(x)\,\bar{\psi}_\beta(x')$$

$$- \Theta(x'^0 - x^0)\;\bar{\psi}_\beta(x')\,\psi_\alpha(x)$$

↑
FERMIONS

$$\leadsto \quad i\big(S_F(x-x')\big)_{\alpha\beta}$$

$$= \Theta(x^0 - x'^0)\;\langle 0 | \psi_\alpha(x)\,\bar{\psi}_\beta(x') | 0 \rangle$$

$$- \Theta(x'^0 - x^0)\;\langle 0 | \bar{\psi}_\beta(x')\,\psi_\alpha(x) | 0 \rangle$$

$$= \Theta(x^0 - x'^0)\;\langle 0 | \psi^{+}_\alpha(x)\,\bar{\psi}^{-}_\beta(x') | 0 \rangle$$

$$- \Theta(x'^0 - x^0)\;\langle 0 | \bar{\psi}^{+}_\beta(x')\,\psi^{-}_\alpha(x) | 0 \rangle$$

$$i\left(S_F(x-x')\right)_{\alpha\beta}$$

$$= \Theta(x^0 - x'^0)\; i\left(S^+(x-x')\right)_{\alpha\beta}$$

$$- \Theta(x'^0 - x^0)\; i\left(S^-(x-x')\right)_{\alpha\beta}$$

$$= \left(i\gamma^\mu \partial_\mu + m\right)_{\alpha\beta} \Big\{ \Theta(x^0 - x'^0)\; i\Delta^+(x-x') - \Theta(x'^0 - x^0)\; i\Delta^-(x-x') \Big\}$$

$$= \left(i\gamma^\mu \partial_\mu + m\right)_{\alpha\beta}\; i\Delta_F(x-x')$$

$$\therefore \quad \boxed{S_F(x-x') = \left(i\gamma^\mu \partial_\mu + m\right) \Delta_F(x-x')}$$

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.(x-x')}\, S_F(p)$$

$$\boxed{S_F(p) = \frac{(\not{p} + m)}{p^2 - m^2 + i\varepsilon}}$$

FEYNMAN PROPAGATOR FOR SPIN 1/2

IN MOMENTUM SPACE

⇒ S-MATRIX EXPANSION

↳ TIME EVOLUTION : SCHRÖDINGER vs INTERACTION PICTURE

- SCHRÖDINGER PICTURE

$$\hat{H}\,|\Psi(t)\rangle_S = i\hbar\frac{d}{dt}|\Psi(t)\rangle_S$$

↳ SCHRÖDINGER

ALL TIME DEPENDENCE IN STATE VECTOR $|\Psi(t)\rangle_S$

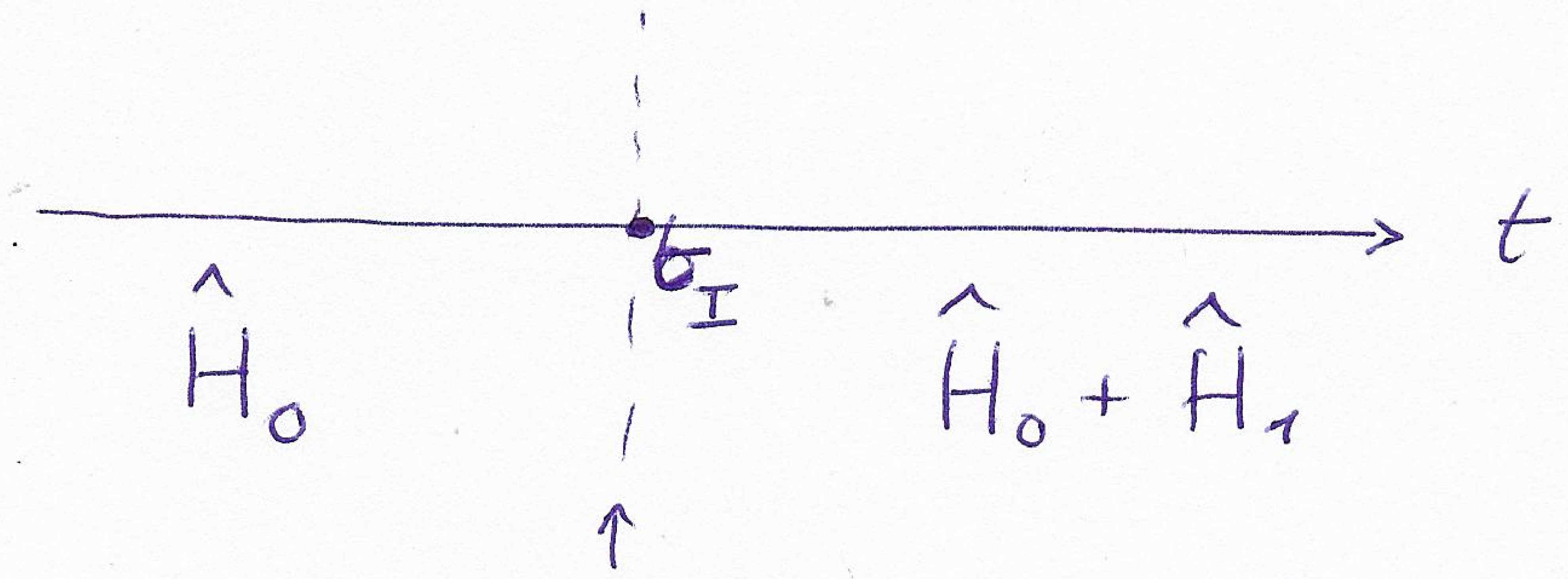
$$|\Psi(t)\rangle_S = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\,|\Psi(t_0)\rangle_S$$

$$= \underbrace{U(t,t_0)}_{\substack{\text{TIME EVOLUTION OPERATOR}\\ \text{(UNITARY)}}}\,|\Psi(t_0)\rangle_S$$

- INTERACTION PICTURE

⇝ $\hat{H} = \hat{H}_0 + \hat{H}_1$

↳ PERTURBATION



INTERACTION
ADIABATICALLY
SWITCHED ON

FOR $t < t_I$

$$|\Psi(t)\rangle_S = e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)} \, |\Psi(t_0)\rangle_S$$

$\rightsquigarrow$ WE CAN ABSORB t- DEPENDENCE DUE TO H_0 PART OF HAMILTONIAN BY DEFINING A STATE $\underline{\underline{\forall t}}$ THROUGH A UNITARY TF.

$$\boxed{\begin{array}{c} |\Psi(t)\rangle_I \equiv U_0^+ \, |\Psi(t)\rangle_S \\ U_0^+ = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} \end{array}}$$

WITH $|\Psi(t_0)\rangle_I = |\Psi(t_0)\rangle_S$

<u>NOTE</u> : FOR UNPERTURBED HAMILTONIAN

i.e. $\hat{H} = \hat{H}_0 \quad (\hat{H}_1 = 0)$

$$|\Psi(t)\rangle_I = |\Psi(t_0)\rangle_S = |\Psi(t_0)\rangle_I$$

STATE $|\Psi(t)\rangle_I$ IS CONSTANT IN TIME

(i.e. WE HAVE TAKEN THE TIME DEPENDENCE DUE TO H_0 OUT OF THE STATE BY THIS UNITARY TF WHICH CORRESPONDS WITH INTERACTION PICTURE).

- EQUIVALENCE OF MATRIX ELEMENTS.

$$
\begin{aligned}
&{}_S\langle \Psi(t) | \hat{O}^S | \Psi(t) \rangle_S \\
&= {}_I\langle \Psi(t) | \hat{O}^I(t) | \Psi(t) \rangle_I
\end{aligned}
$$

MATRIX ELEMENTS SHOULD BE INVARIANT UNDER UNITARY TF.

$$
\begin{aligned}
&{}_S\langle \Psi(t) | \hat{O}^S | \Psi(t) \rangle_S \\
&= {}_I\langle \Psi(t) | U_0^+ \hat{O}^S U_0 | \Psi(t) \rangle_I .
\end{aligned}
$$

$$\therefore \quad \boxed{\hat{O}^I(t) \equiv U_0^+ \hat{O}^S U_0 .}$$

$\rightsquigarrow$ IN INTERACTION PICTURE : SOME OF THE TIME DEPENDENCE (DUE TO H_0) IS TRANSFERED INTO THE OPERATOR $\hat{O}^I(t)$

$$i \frac{d}{dt} \hat{O}^I(t) = - \hat{H}_0 \hat{O}^I(t) + \hat{O}^I(t) \hat{H}_0$$

$$\boxed{i \frac{d}{dt} \hat{O}^I(t) = \left[\hat{O}^I(t), \hat{H}_0 \right]}$$

TRIVIAL CASE $\hat{O}^I = \hat{H}_0$

$\hat{H}_0^S = \hat{H}_0^I = \hat{H}_0$ (CONSTANT).

~> TIME DEPENDENCE OF $|\Psi(t)\rangle_I$

$$i\frac{d}{dt}|\Psi(t)\rangle_I = i\frac{d}{dt}\left(\underbrace{e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}}_{U_0^+}|\Psi(t)\rangle_S\right)$$

$$= -\hat{H}_0|\Psi(t)\rangle_I$$

$$+\, U_0^+\underbrace{\left(i\frac{d}{dt}|\Psi(t)\rangle_S\right)}_{(\hat{H}_0+\hat{H}_1)|\Psi(t)\rangle_S}$$

$$= -\hat{H}_0|\Psi(t)\rangle_I + \underbrace{U_0^+(\hat{H}_0+\hat{H}_1)U_0}_{\hat{H}_0+\hat{H}_1^I(t)}|\Psi(t)\rangle_I$$

WITH $\underline{\underline{\hat{H}_1^I(t) = U_0^+\hat{H}_1U_0}}$

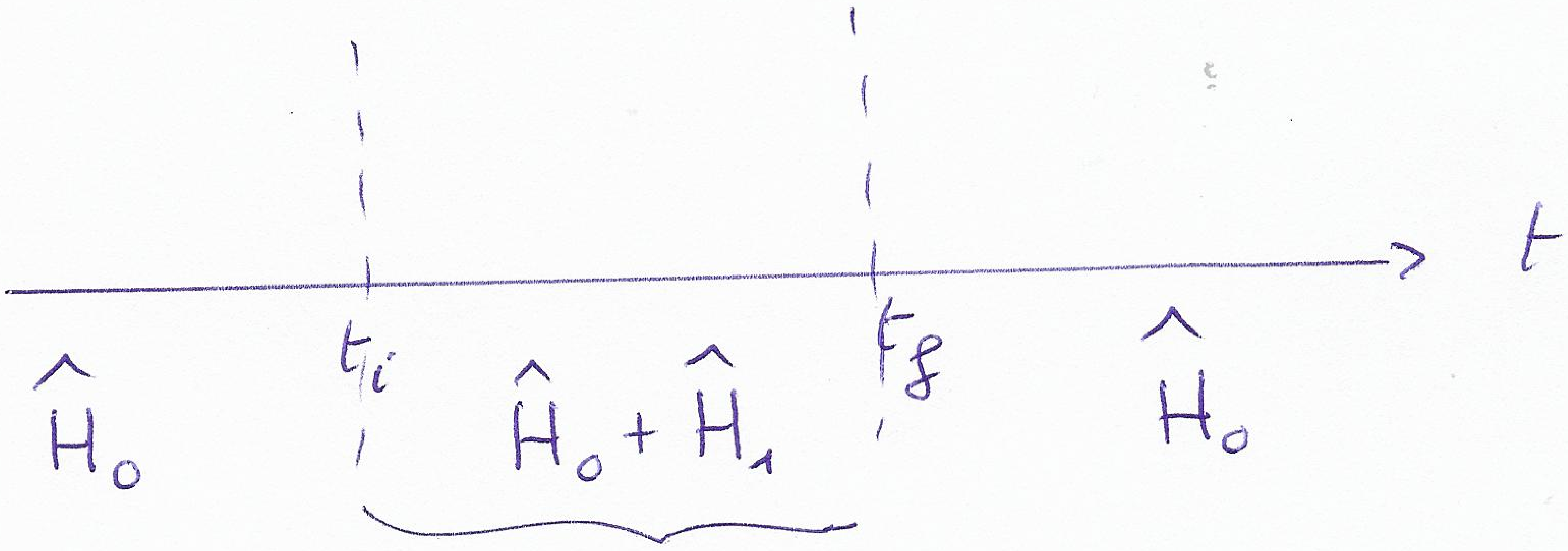
$$\therefore\quad \boxed{i\frac{d}{dt}|\Psi(t)\rangle_I = \hat{H}_1^I(t)|\Psi(t)\rangle_I}$$

IN INTERACTION PICTURE:

TIME DEPENDENCE ASSOCIATED WITH H_1 IS IN STATES $|\Psi(t)\rangle_I$

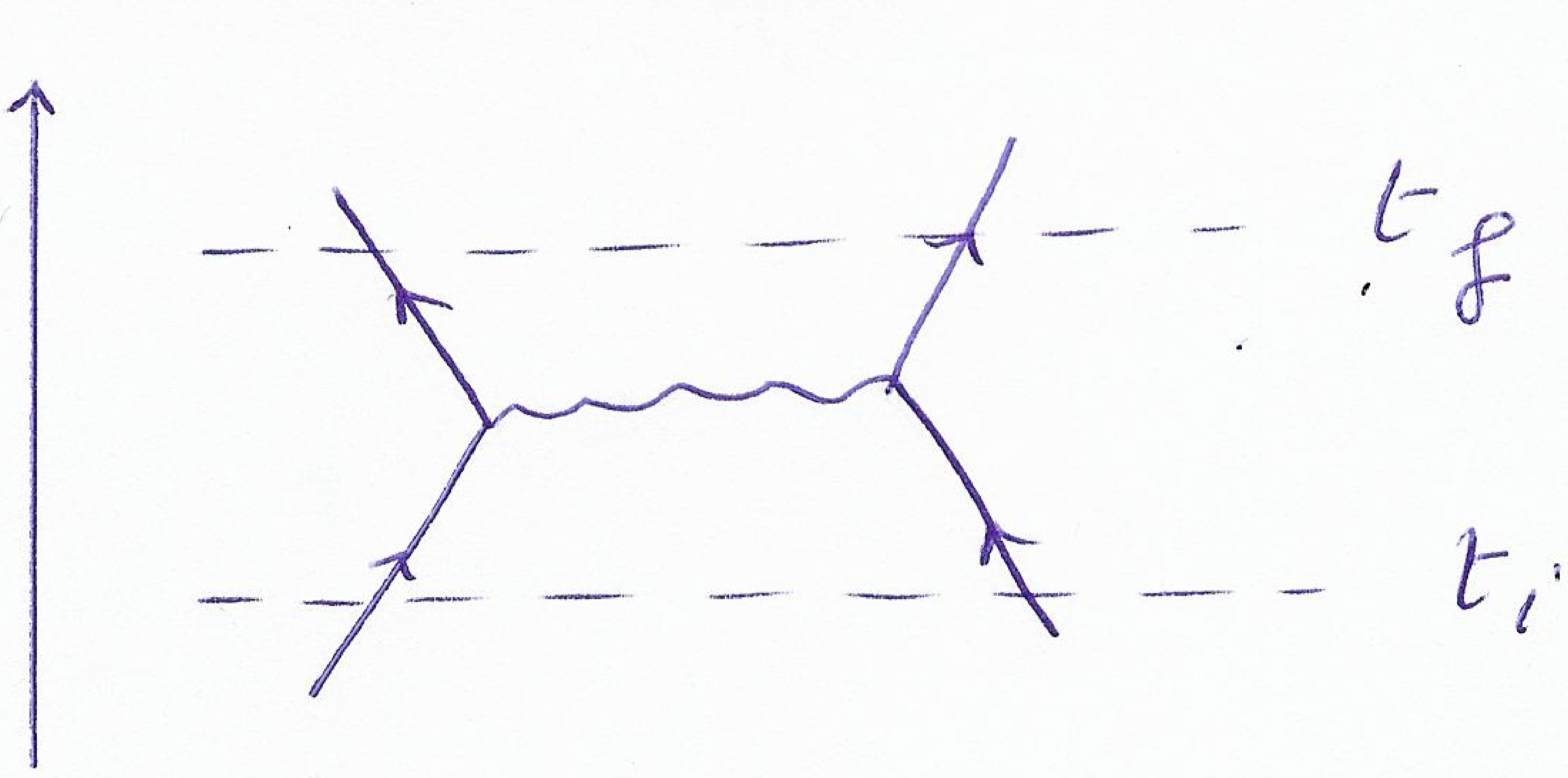
" " " " H_0 " " OPERATORS

↳ TIME EVOLUTION IN INTERACTION PICTURE / S-MATRIX



INTERVAL IN TIME
DURING WHICH
FIELDS INTERACT

e.g.



⤳ FOR $t \ll t_i$, i.e. $t = -\infty$
SYSTEM IS IN INITIAL STATE $|i\rangle$

$$|\Psi(t=-\infty)\rangle_I = |i\rangle \quad (\text{e.g. } |e_1^- e_2^-\rangle)$$

⤳ DUE TO INTERACTION : STATE EVOLVES ACCORDING TO UNITARY TF INTO

$$\underline{\underline{|\Psi(t=+\infty)\rangle_I \equiv S\, |\Psi(t=-\infty)\rangle_I}}$$

↳ S-MATRIX

$$S^\dagger S = S S^\dagger = \mathbb{1}$$

$\rightsquigarrow$ TRANSITION PROBABILITY AMPLITUDE TO A SPECIFIC FINAL STATE $|f\rangle$ (e.g. $e_3^- e_4^-$)

$$\langle f | \Psi(t=+\infty) \rangle = \langle f | S | i \rangle \equiv S_{fi}$$

$\rightsquigarrow$ UNITARITY OF S - MATRIX

SUM OF PROBABILITIES OF SCATTERING INTO ALL POSSIBLE FINAL STATES SHOULD BE 1 :

$$\sum_f |\langle f | S | i \rangle|^2 = \sum_f |S_{fi}|^2 = 1$$

BECAUSE

$$= \sum_f \langle i | S^+ | f \rangle \langle f | S | i \rangle$$

$\Rightarrow$ COMPLETENESS

$$= \langle i | S^+ S | i \rangle$$

$\Rightarrow$ UNITARITY OF S - MATRIX

$$= \langle i | i \rangle$$

$$= 1.$$

⇝ ITERATIVE SOLUTION

$$i \frac{d}{dt} |\Psi(t)\rangle_I = H_1^I(t) \, |\Psi(t)\rangle_I$$

$$\Big\downarrow \text{ INTEGRATE } \int_{-\infty}^{t}$$

WITH $|\Psi(-\infty)\rangle = |i\rangle$

$$\boxed{|\Psi(t)\rangle_I = |i\rangle - i \int_{-\infty}^{t} dt_1 \; H_1^I(t_1) |\Psi(t_1)\rangle_I}$$

INTEGRAL EQUATION CAN BE SOLVED ITERATIVELY IF H_1^I IS A PERTURBATION (SMALL EXPANSION PARAMETER)

$$\begin{aligned} |\Psi(t)\rangle_I &= |i\rangle \\ &\quad + (-i) \int_{-\infty}^{t} dt_1 \; H_1^I(t_1) \; |i\rangle \\ &\quad + (-i)^2 \int_{-\infty}^{t} dt_1 \; H_1^I(t_1) \int_{-\infty}^{t_1} dt_2 \; H_1^I(t_2) |\Psi(t_2)\rangle_I \\ &= |i\rangle \\ &\quad + (-i) \int_{-\infty}^{t} dt_1 \; H_1^I(t_1) |i\rangle \\ &\quad + (-i)^2 \int_{-\infty}^{t} dt_1 \; H_1^I(t_1) \int_{-\infty}^{t_1} dt_2 \; H_1^I(t_2) \; |i\rangle \\ &\quad + (-i)^3 \int_{-\infty}^{t} dt_1 \; H_1^I(t_1) \int_{-\infty}^{t_1} dt_2 \; H_1^I(t_2) \int_{-\infty}^{t_2} dt_3 \; H_1^I(t_3) |i\rangle \\ &\quad + \cdots \end{aligned}$$

FOR $t \to +\infty$

$$|\Psi(t=+\infty)\rangle_I = S\,|i\rangle$$

∴ S-MATRIX EXPANSION

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n$$

$$\cdot\; H_1^I(t_1)\, H_1^I(t_2) \cdots H_1^I(t_n)$$

NOTE : IN THIS PRODUCT, THE INTEGRATION RANGES IMPLY THAT

$$t_1 > t_2 > t_3 > \cdots > t_n$$

∴ TIME-ORDERED PRODUCT

$$T\left\{ H_1^I(t_1)\, H_1^I(t_2) \cdots H_1^I(t_n) \right\}$$

$$= H_1^I(t_1)\, H_1^I(t_2) \cdots H_1^I(t_n)$$

IF $t_1 > t_2 > \cdots > t_n$

USING T-PRODUCT, WE CAN FINALLY WRITE

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_n \; T\left\{ H_1^I(t_1) \cdots H_1^I(t_n) \right\}}$$

$n!$ PERMUTATIONS

↳ IN QFT : HAMILTONIAN DENSITY

$$H_1(t) \equiv \int d^3\bar{x} \quad \mathcal{H}_1(t,\bar{x}) = \int d^3\bar{x}\, \mathcal{H}(x)$$

FROM NOW ONWARDS, WE DROP INDEX I (INTERACTION PICTURE) WHICH IS ALWAYS UNDERSTOOD HOWEVER

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots \int d^4x_n \, T\left\{ \mathcal{H}_1(x_1) \ldots \mathcal{H}_1(x_n) \right\}}$$

DYSON EXPANSION OF THE S-MATRIX

↳ EXAMPLES :

- ϕ^4 THEORY , ϕ : SCALAR FIELD

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2}_{\mathcal{L}_0} \underbrace{- \frac{1}{4!}\lambda\phi^4}_{\mathcal{L}_1}$$

$\mathcal{L}_0 \quad + \quad \mathcal{L}_1$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$$

$$\mathcal{H} = \pi\dot\phi - \mathcal{L}$$

$$= \underbrace{\frac{1}{2}\left(\pi^2 + (\bar\nabla\phi)^2 + m^2\phi^2\right)}_{\mathcal{H}_0} + \underbrace{\frac{\lambda}{4!}\phi^4}_{\mathcal{H}_1}$$

$\mathcal{H}_0 \quad + \quad \mathcal{H}_1$

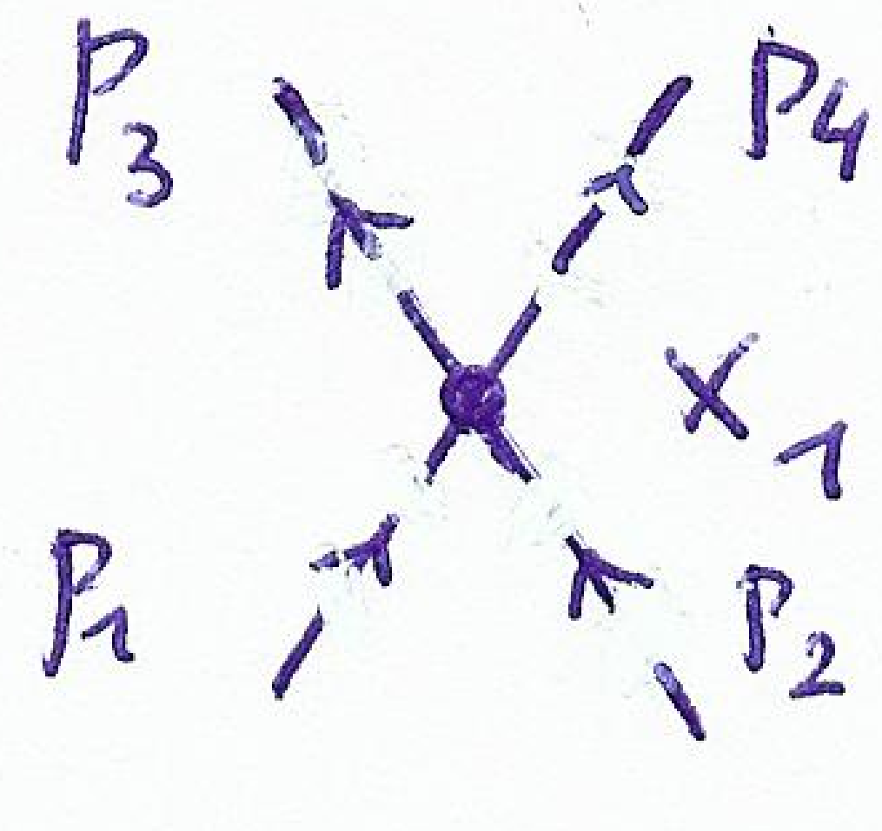
$$\mathcal{H}_1 = -\mathcal{L}_1$$

IF λ IS SMALL PARAMETER : PERT. EXPANSION IN λ

⇝ LOWEST ORDER : $O(\lambda)$ ⇒ DENOTE TERM IN S-MATRIX BY S_1

$$S_1 = i\int d^4x_1\, \mathcal{L}_1(x_1)$$

GRAPHICALLY



MOMENTUM SPACE MATRIX ELE

$$\langle f | S_1 | i \rangle$$

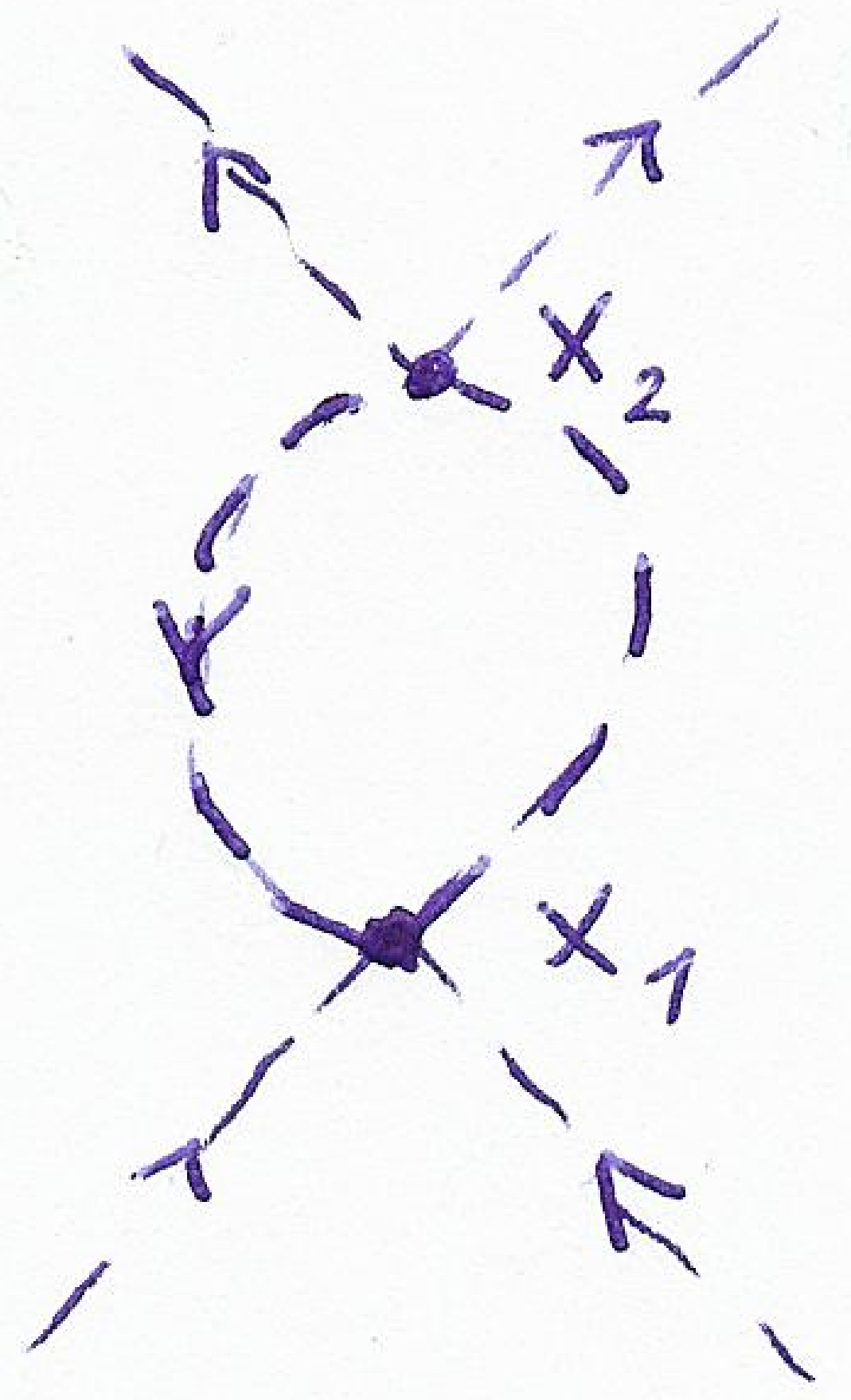
$$= \underbrace{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)}_{\text{ENERGY-MOMENTUM CONSERVATION}}(-i\lambda)$$

$\rightsquigarrow$ SECOND ORDER : $O(\lambda^2)$

$$S_2 = i^2 \int d^4x_1 \int d^4x_2 \; T\left\{ \mathcal{L}_1(x_1)\, \mathcal{L}_1(x_2) \right\}$$

$$= i^2 \frac{\lambda^2}{4!\,4!} \int d^4x_1 \int d^4x_2 \; T\left\{ \phi^4(x_1)\, \phi^4(x_2) \right\}$$

e.g. $\langle f | S_2 | i \rangle$ CORRESPONDING WITH



WE HAVE TO WORK OUT T- PRODUCT

↓

WICK'S THEOREM

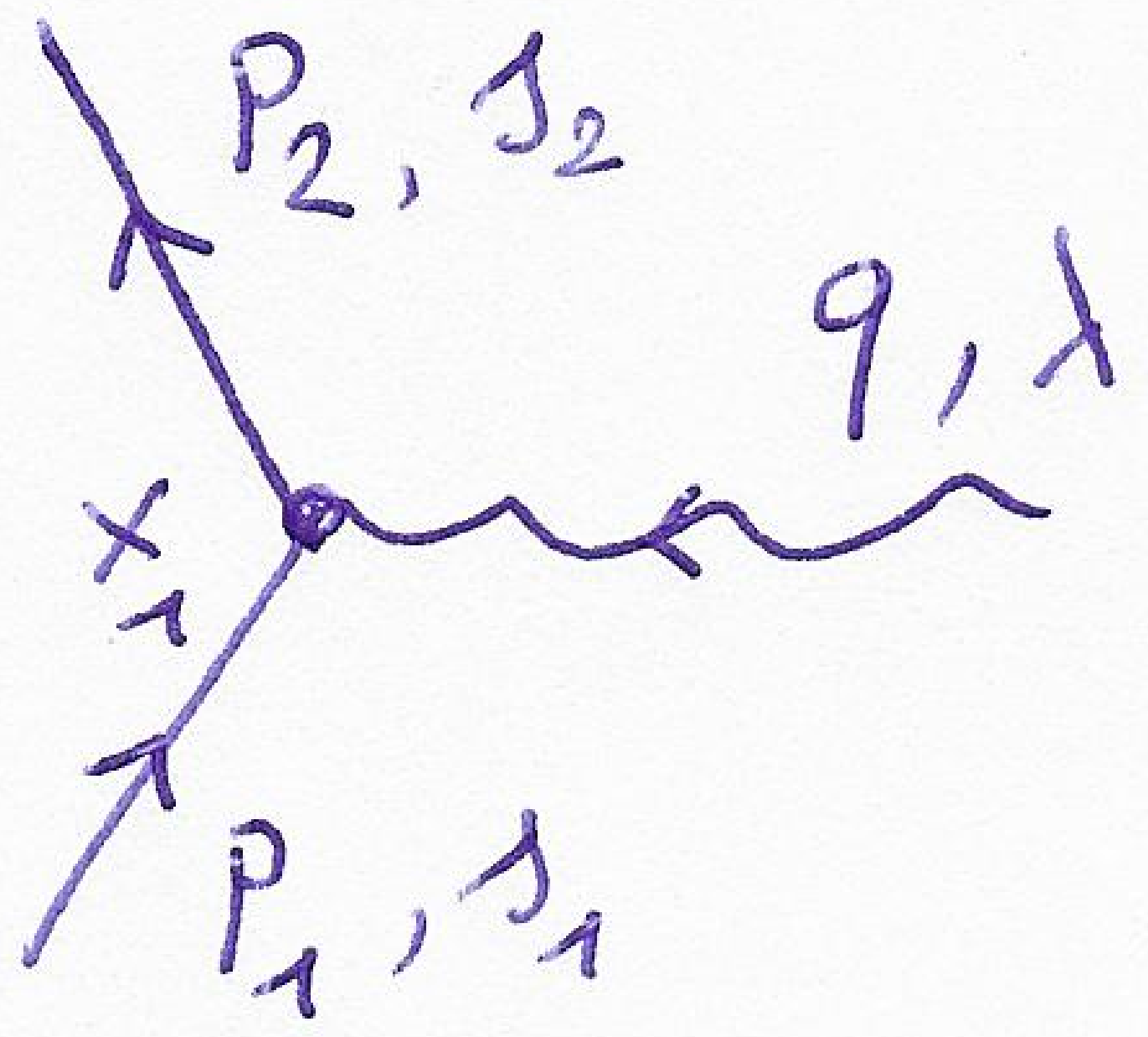
- QED

$$\mathcal{L}_{QED} = \underbrace{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_0} \underbrace{- e\bar{\psi}\gamma^\mu\psi A_\mu}_{+ \quad \mathcal{L}_1}$$

$$\mathcal{H}_1 = -\mathcal{L}_1 = e\,\bar{\psi}\gamma^\mu\psi A_\mu$$

⇝ LOWEST ORDER IN e

$$S_1 = -ie\int d^4x_1\ \bar{\psi}(x_1)\gamma^\mu\psi(x_1)A_\mu(x_1)$$



FEYNMAN RULE FOR VERTEX

$$\underline{\underline{-ie\gamma^\mu}}$$

$$\langle f | S_1 | i \rangle$$

$$= \bar{u}(p_2, s_2)\left[-ie\gamma^\mu\right]u(p_1, s_1)\cdot \underbrace{\varepsilon_\mu(q,\lambda)}_{\text{PHOTON POLARIZATION VECTOR}}$$

$$\cdot (2\pi)^4\,\delta^4(p_1 + q - p_2)$$

↳ WICK'S THEOREM

⤳ WRITE $\mathcal{H}_1(x)$ FIRST AS NORMAL ORDERED PRODUCT (CREATION OP. LEFT OF ANNIHILATION OP.)

$$\mathcal{H}_1(x) = N\{A(x)\,B(x)\ldots\}$$

e.g. ϕ^4 THEORY $\qquad \mathcal{H}_1(x) = \frac{\lambda}{4!}\,N\{\phi(x)\,\phi(x)\,\phi(x)\,\phi(x)\}$

QED $\qquad \mathcal{H}_1(x) = e\,N\{\bar{\psi}(x)\,\gamma^\mu\psi(x)\,A_\mu(x)\}$

⤳ T- ORDERED PRODUCT

$$\boxed{T\{A(x_1)\,B(x_2)\} = N\{A(x_1)\,B(x_2)\} + \langle 0|\,T\{A(x_1)\,B(x_2)\}\,|0\rangle}$$

NOTE : $\langle 0|\,N\{A(x_1)\,B(x_2)\}\,|0\rangle = 0$

NOTATION : CONTRACTION OF $A(x_1)$ & $B(x_2)$

$$\underbrace{A(x_1)\,B}(x_2) \equiv \langle 0|\,T\{A(x_1)\,B(x_2)\}\,|0\rangle$$

$$T\{A\,B\} = N\{A\,B\} + \underbrace{A\,B}$$

$\rightsquigarrow$ CONTRACTIONS CORRESPOND WITH <u>PROPAGATORS</u>

- $$\underbracket{\phi(x_1)\,\phi}(x_2) = \langle 0 | T\{\phi(x_1)\,\phi(x_2)\} | 0 \rangle$$
  $$= i\,\Delta_F(x_1 - x_2)$$

- $$\underbracket{\psi_\alpha(x_1)\,\bar{\psi}}_\beta(x_2) = \langle 0 | T\{\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)\} | 0 \rangle$$
  $$= i\big(S_F(x_1 - x_2)\big)_{\alpha\beta}$$
  $$= -\,\underbracket{\bar{\psi}_\beta(x_2)\,\psi}_\alpha(x_1)$$

⤳ GENERALIZATION

$$T\{ABCD\ldots YZ\}$$

$$= N\{ABCD\ldots YZ\}$$

$$+ N\{\underbrace{AB}_{}CD\ldots YZ\}$$

$$+ N\{\underbrace{ABC}_{}D\ldots YZ\}$$

$$+ \ldots + N\{ABCD\ldots \underbrace{YZ}_{}\}$$

$$+ N\{\underbrace{AB}_{}\underbrace{CD}_{}\ldots YZ\}$$

$$+ \ldots$$

SUM OF N-ORDERED PRODUCTS WITH ALL POSSIBLE CONTRACTIONS $\underbrace{A(x_1)\,B(x_2)}_{}$ WITH $x_1^0 \neq x_2^0$

(AVOIDING EQUAL TIME CONTRACTIONS)

⤳ <u>NOTE</u> : PHASE FACTOR FOR FERMIONS

$$N\{\underbrace{A\,\underbrace{BC}_{}\,D\,E\,F\ldots J\,K}_{}\,L\,M\ldots\}$$

$$= (-1)^P\ \underbrace{AK}_{}\ \underbrace{BC}_{}\ \underbrace{EL}_{}\ N\{DF\ldots JM\ldots\}$$

P : # INTERCHANGES OF FERMION OPERATORS TO CHANGE ORDER