Exercise sheet 5
Theoretical physics 1 WS2015/2016
Lecturer: Prof. M. Vanderhaeghen
Assistant: Fabian Ewert

23.11.2015

Exercise 1 (30 points): Harmonic oscillator

The harmonic oscillator is probably the most important model system in all of physics. Throughout your studies you will encounter it in pretty much every lecture. A deep understanding is thus essential. Therefore, solve this exercise without looking things up in the lecture notes as far as possible.

As an example we look at a mass m attached to a spring with spring constant D. Its suspension is located at the origin and its length at rest in the homogeneous gravity field is L_0.

a) Introduce coordinates q, p, τ such that the energy of the oscillator becomes $E = \frac{1}{2}(p^2 + q^2)$ and $p = \frac{dq}{d\tau}$. What is the equation of motion for q in these coordinates?

b) The introduction of $\vec{\xi} = (q, p)^T$ allows us to reduce the problem to a linear system of differential equations of first order:

$$\dot{\vec{\xi}} = A\vec{\xi}, \qquad A \in \mathbb{R}^{2\times 2} \tag{1}$$

For general $A \in \mathbb{R}^{2\times 2}$, find the general solution of (1) in terms of the eigenvalues λ_i and the eigenvectors $\vec{a}_i$ of A. Distinguish between the cases $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2$.

c) Find the matrix A for the free, undamped oscillator, and for an oscillator damped by a force $F_D = -\alpha \frac{dz}{dt}$. With these matrices calculate the trajectories $z(t)$ corresponding to initial conditions $z(t=0) = z_0$, $\dot{z}(t=0) = \dot{z}_0$. Take into account the results of part b) when investigating different values of α. (There are three different regimes for the damped oscillator.) Does an appropriate choice of A lead to a solution of the driven oscillator?

d) Calculate the energy as a function of time $E(\tau) = \frac{1}{2}(p(\tau)^2 + q(\tau)^2)$ and use a PC to plot it for the three cases of part c). The choice of initial conditions that lead to descriptive plots is yours.

Exercise 2 (30 points): Phase space

In the lecture, phase space was introduced as the $q-p$-plane with trajectories drawn according to the solutions $(q(t), p(t))$. If multiple trajectories (corresponding to different initial conditions) are drawn in a way that shows the system's qualitative behavior for all initial conditions, it is called phase portrait. Sketch (without large calculations) the phase portrait of the following systems. For every system use at least three different initial conditions. Make special note of curves separating areas of qualitatively different trajectories (separatrices).

a) A free particle between two walls.

b) Vertical throw of a ball in the homogeneous gravity field.

c) Particle on a circular trajectory (the coordinate is the angle and the corresponding momentum is the angular momentum).

d) The plane, mathematical pendulum.

e) A particle in the double well potential $V(q) = \kappa(q-q_0)^2(q+q_0)^2$, $q_0 > 0$.

f) The radial coordinate r of an object in earth's gravity field (keyword: effective potential)

Exercise 3 (15 points): Driven oscillator: energy resonance

For a driven, damped, harmonic oscillator

$$\ddot{q}(t) + 2\beta\dot{q}(t) + \omega_0^2 q(t) = \frac{f}{m}\cos(\bar{\omega}t),$$

find the frequency at which the system absorbs the most energy per time. To this end, maximize the average power

$$\bar{P} = \frac{1}{T}\int_0^T dt\ f\cos(\bar{\omega}t)\dot{q}_0(t)$$

with respect to $\bar{\omega}$. $T = \frac{2\pi}{\bar{\omega}}$ is the period of the driving force and $q_0(t)$ is the solution of the inhomogeneous ODE known from the lecture. Sketch $\bar{P}(\bar{\omega})$.

Exercise 4 (15 points): Pendulum separatrix

The dimensionless energy of the plane mathematical pendulum is given by

$$\varepsilon = \frac{E}{mgl} = \frac{1}{2}\left(\frac{d\varphi}{d\tau}\right)^2 + (1 - \cos\varphi),$$

where $\tau = \omega t = \sqrt{\frac{g}{l}}t$ is the dimensionless time. The separatrix of the system is obtained for $\varepsilon = 2$. Find the trajectory of the pendulum for this energy if it starts at $\varphi(\tau = 0) = 0$.

Hint: Use appropriate substitutions and addition theorems to solve occurring integrals.