Theoretical Elementary Particle Physics
Exercise 2

May 22, 2015

-to be handed in on Friday, 5th of June

1. **Feynman parametrization (25 points)**

   (a)(10 points) Using standard integration techniques prove the identity :

$$\frac{1}{A\,B\,C} = 2\int_0^1 dx \int_0^{1-x} dz\, \frac{1}{\left[A + (B-A)\,x + (C-A)\,z\right]^3} \tag{1}$$

   (b)(15 points) Prove the following formula

$$\frac{1}{A_1\,A_2\,A_3\,\ldots\,A_n} = (n-1)!\int_0^1 \ldots \int_0^1 dx_1\,dx_2\,\ldots dx_n\,\frac{\delta\left(x_1 + x_2 + \ldots + x_n - 1\right)}{\left(x_1A_1 + x_2A_2 + x_3A_3 + \ldots + x_nA_n\right)^n} \tag{2}$$

   (**Hint:** Use mathematical induction. Useful relation is

$$\frac{1}{A_1A_2^n} = \int_0^1 dx_1 \int_0^1 dx_2\,\delta\left(x_1 + x_2 - 1\right)\frac{nx_2^{n-1}}{\left(x_1A_1 + x_2A_2\right)^{n+1}} \tag{3}$$

   If you use it prove it by performing several derivatives on $\left(A_1\,A_2\right)^{-1}$ with respect to A_2 and looking for a pattern or, alternatively, by means of the mathematical induction.)

2. **1-loop correction to the propagator in ϕ^4 theory (35 points)**

   (a)(30 points) Using Feynman rules for ϕ^4 theory (see exercise!) write the amplitude corresponding to the Feynman diagram in figure 1. The symmetry factor of this diagram is 2. Use dimensional regularization. The angular part of the d-dimensional integral equals $\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$. In the spatial part it is useful to make a substitution of the integration variable by introducing $x = \frac{\Delta}{l_E^2+\Delta}$. In this simplified example, which does not require usage of Feynman parametrization, $\Delta \equiv m^2$. Another useful identity is :

$$\int_0^1 dx\,x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4}$$

Take $d = 4 - \epsilon$ and expand the amplitude in powers of ϵ. Consider only ϵ^{-1} (divergent part) and ϵ^0 parts since other terms will vanish as you go to $\epsilon \to 0$ ($d \to 4$) limit. For the expansion of the Γ function near $z = -1$ pole use :

$$\Gamma(-1 + \frac{\epsilon}{2}) \approx -\frac{2}{\epsilon} + \gamma - 1 \tag{5}$$

where γ stands for the Euler-Mascheroni constant introduced in the exercises.
(b)(5 points) After extracting the ϵ^{-1} part in dimensional regularization, absorb this divergence into the renormalization constants δ_Z and δ_m by adding the amplitude of the diagram and the counterterm in figure 2 and requiring cancellation of infinite parts. Use the MS renormalization scheme, i.e. absorb only the divergent part in δ_m and δ_Z.
Solution : $\delta_Z = 0$, $\delta_m = \frac{\lambda m^2}{16\pi^2\epsilon}$

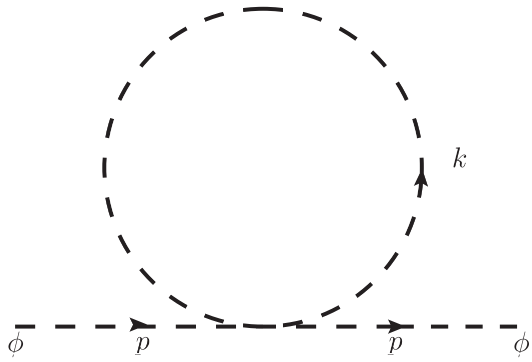

Figure 1: One loop correction to the propagator in ϕ^4 theory with 4-momenta labelled.

Figure 2: Counterterm that absorbs infinite contributions for diagram in figure 1. The Feynman rule for this counterterm is $i(p^2\delta_Z - \delta_m)$.

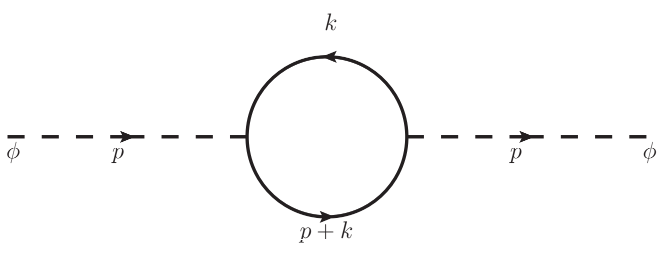

Figure 3: One loop correction to the propagator in Yukawa theory (interaction Lagrangian is $-g\,\overline{\psi}\psi\phi$) with 4-momenta labelled.

3. **1-loop correction to the propagator in Yukawa theory (40 points)**

One can study the interaction between a fermion field ψ and scalar field ϕ which is described by the Lagrangian term $-g\overline{\psi}\psi\phi$, g being the coupling constant. The 1-loop correction to the scalar propagator is presented in figure 3. The amplitude may be expressed in the following form

$$-g^2 \int \frac{d^d k}{(2\pi)^d} \mathrm{Tr}\left[\left(\not{k} + \not{p} + m\right)\left(\not{k} + m\right)\right] \frac{1}{\left((k+p)^2 - m^2\right)\left(k^2 - m^2\right)} \tag{6}$$

where m now represents the mass of the fermion propagating in the loop.

(a)(30 points) Starting from equation (6), perform dimensional regularization using the procedure described in the previous problem. Note that in this case Feynman parametrization should be employed. When shifting the integration 4-momentum variable k to $l = k + xp$ note that the terms in the numerator which are linear in l should be dropped because the integral of an asymmetrical function on a symmetrical interval vanishes. Show that the amplitude may be written in the form :

$$(d-1)\frac{4ig^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}} \tag{7}$$

where $\Delta = x(x-1)p^2 + m^2$ and x is a Feynman parameter.

(b)(10 points) Take $d = 4 - \epsilon$ and expand the factors containing ϵ. Use again equation (5) to expand Γ function near pole $z = -1$. After identifying the divergent part, using the counterterm from figure 2, obtain the expressions for renormalization constants δ_m and δ_Z. Use the MS renormalization scheme, i.e. absorb only the divergent part in δ_m and δ_Z. **Solution :** $\delta_Z = -4\frac{g^2}{(4\pi)^2\epsilon}$, $\delta_m = -\frac{24g^2m^2}{(4\pi)^2\epsilon}$