Theoretical Elementary Particle Physics
Exercise 1

May 6, 2015

-to be handed in on Monday, 18th of May

1. **Color singlet hadrons (20 points)**

   Show that for quarks q_a $(a = 1, 2, 3 = r, g, b)$ in the fundamental representation of color $SU(3)_C$, the combinations $\bar{q}_a q_a$ (mesons) and $\varepsilon_{abc}\, q_a\, q_b\, q_c$ (baryons) are color singlets, i.e. invariant under $SU(3)_C$ transformations : $q_a \to U_{ab}\, q_b$, $\bar{q}_a \to \bar{q}_b U^\dagger_{ba}$.

2. **1-gluon exchange potential between different quark states (40 points)**

   We consider the interaction of 2 quarks due to the exchange of one gluon. In QCD, the 8 gluons form a color octet :

$$
\begin{aligned}
g_1 &= r\,\bar{g} \qquad g_2 = r\,\bar{b} \qquad g_3 = g\,\bar{r} \\
g_4 &= g\,\bar{b} \qquad g_5 = b\,\bar{r} \qquad g_6 = b\,\bar{g} \\
g_7 &= \frac{1}{\sqrt{2}}\,(r\,\bar{r} - g\bar{g}) \\
g_8 &= \frac{1}{\sqrt{6}}\left(r\,\bar{r} + g\bar{g} - 2\,b\bar{b}\right)
\end{aligned} \tag{1}
$$

   (a)(10 points) Show that the color factor due to 1-gluon exchange between 2 quarks of the same color , as shown in Fig. 1, is given by $+2/3$. Show this explicitly for a blue and a red quark.

   (b)(10 points) Calculate next the color factor arising from 1-gluon exchange between a quark and an anti-quark in a color singlet state :

$$
|q\bar{q}\rangle_{singlet} = \frac{1}{\sqrt{3}}\left(r\,\bar{r} + g\bar{g} + b\bar{b}\right) \tag{2}
$$

   as shown in Fig. 2. Show that this color factor is given by -8/3, yielding an attractive interaction. Note that an antiquark has opposite color charge as a quark.

   (c)(10 points) Likewise, show that the color factor arising from 1-gluon exchange between a quark and an anti-quark in a color octet state is given by +1/3, yielding a repulsive interaction. (choose one state, it is not necessary to show for all 8 of them!)

(d)(10 points) Show that the color factor due to 1-gluon exchange in the color singlet 3- quark state

$$|qqq\rangle_{singlet} = \frac{1}{\sqrt{6}}(r\,g\,b + g\,b\,r + b\,r\,g - r\,b\,g - g\,r\,b - b\,g\,r) \tag{3}$$

is given by -4, also giving rise to strong attraction. (**hint:** Consider one initial state, for instance $r\,g\,b$, and at the end multiply result by 6.)

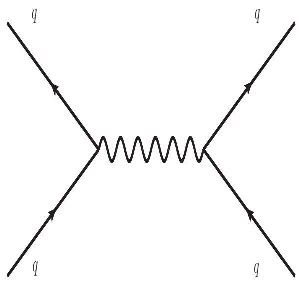

Figure 1: 1-gluon exchange potential between 2 quarks

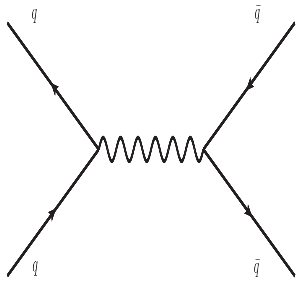

Figure 2: 1-gluon exchange potential between a quark and an anti-quark

3. **Path integrals in Quantum Mechanics (40 points)**

We derived the amplitude for the propagation of a particle from q' at t' to q'' at t'' to be

$$\langle q''(t'')|e^{-iHt}|q'(t')\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\frac{p_j^2}{2m}\delta t} \tag{4}$$

$$\tag{5}$$

where potential term $V(\bar{q}_j) = 1/2(q_j + q_{j+1})$ is now set to zero for simplicity. Useful formulae (see exercise) are :

$$\dot{q}_j = \frac{q_{j+1} - q_j}{\delta t} \qquad \delta t = \frac{t'' - t'}{N+1} \tag{6}$$

(a)(20 points) Consider general p_j integral. Complete the square in order to make it gaussian. When solving the integral treat it as a real integral, in particular use $\int_{-\infty}^{\infty} e^{-cx^2} dx = (\frac{\pi}{c})^{\frac{1}{2}}$ where c can consist of imaginary unit. (basically you assume that some other quantity in the exponential is imaginary in order to make the integral real). Using the result of the integration prove that $\int \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\frac{p_j^2}{2m}\delta t}$ can be written in the form

$$\left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \exp\left[\frac{im}{2\delta t}\sum_{j=1}^{N}(q_{j+1}-q_j)^2\right] \tag{7}$$

(b)(20 points) Now the total amplitude equals

$$\left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^{N} dq_k \exp\left[\frac{im}{2\delta t}\sum_{j=1}^{N}(q_{j+1}-q_j)^2\right] \tag{8}$$

Try to solve also the integrals over dq_k. First integrate over q_1, then q_2, etc. and try to look for a pattern. Again, as in a) part, complete the square,treat the integrals as real and apply the formula for gaussian integral. **Help** : The result is

$$\int \prod_{k=1}^{N} dq_k \exp\left[\frac{im}{2\delta t}\sum_{j=0}^{N}(q_{j+1}-q_j)^2\right] = \left[\frac{2i\pi\delta t}{m}\right]^{\frac{N}{2}} \sqrt{\frac{N!}{(N+1)!}} \exp\left[\frac{im(q_{N+1}-q_0)^2}{2(N+1)\delta t}\right] \tag{9}$$

Since $q_0 = q'$ and $q_{N+1} = q''$ it is now easy to express this result, together with a prefactor from (8) (and using eqs. (6)) as

$$\langle q''(t'')|e^{-iHt}|q'(t')\rangle = \sqrt{\frac{m}{2\pi i(t''-t')}} \exp\left[\frac{im(q''-q')^2}{2(t''-t')}\right] \tag{10}$$

which depends only on starting and final position and time and the mass of a particle.