QCD RUNNING COUPLING CONSTANT AND ASYMPTOTIC FREEDOM

$\Rightarrow$

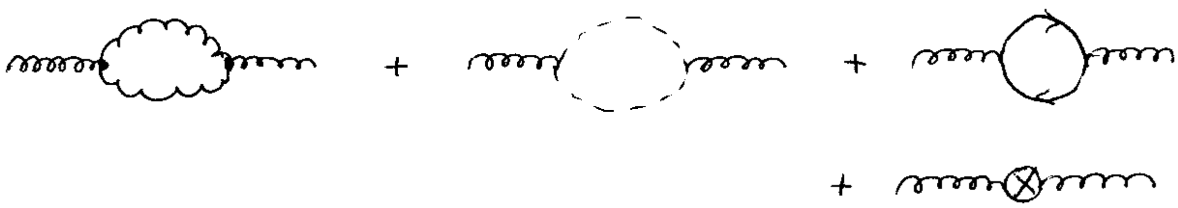
$$\hookrightarrow \boxed{Z_2 = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \frac{4}{3}\xi} + O(g^4)$$

$\Rightarrow$

$$Z_{1F} = Z_g \, Z_2 \, Z_3^{1/2}$$

$$\hookrightarrow \boxed{Z_{1F} = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \underbrace{\left[\frac{9}{4} + \frac{25}{12}\xi \right]}_{\frac{13}{3} \text{ FOR } \xi = 1}} + O(g^4)$$

$\Rightarrow$



$$\hookrightarrow \boxed{Z_3 = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left\{ \left(-\frac{13}{2} + \frac{2}{3} N_f \right) + \frac{3}{2}\xi \right\}} + O(g^4)$$

α_s^2

$$\Rightarrow \quad Z_g = Z_{1F}\, Z_2^{-1}\, Z_3^{-1/2}$$

$$Z_g = \left[1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon}\left(\frac{9}{4} + \frac{25}{12}\xi\right) + O(g^4)\right]$$

$$\cdot \left[1 + \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon}\left(\frac{4}{3}\xi\right) + O(g^4)\right]$$

$$\cdot \left[1 + \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon}\left(-\frac{13}{4} + \frac{1}{3}N_f + \frac{3}{4}\xi\right) + O(g^4)\right]$$

$$= 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon}\left(\frac{11}{2} - \frac{1}{3}N_f\right) + O(g^4)$$

$\Downarrow$

NOTE THAT THE ξ DEPENDENCE DROPS OUT !

CONSEQUENCE OF GAUGE INVARIANCE

$$\boxed{Z_g = 1 - \frac{g^2}{(4\pi)^2} \cdot \frac{1}{\varepsilon}\left(\frac{11}{2} - \frac{1}{3}N_f\right) + O(g^4)}$$

$$g_B = Z_g\, g$$

$$\Rightarrow \qquad g_B = Z_g \cdot g$$

IN D DIMENSIONS

$$[\mathcal{L}_{qqg}] = D$$

$\int d^D x \; \mathcal{L}$ IS DIMENSIONLESS

↳ MASS DIMENSION

$$\Downarrow$$

$$\left\{ \begin{aligned} [g_B \; \bar{q} \, \gamma^\mu A_{a\mu} \, q] &= D \\ [q] &= \frac{D-1}{2} \\ [A_a^\mu] &= \frac{D-2}{2} \end{aligned} \right.$$

$$\longrightarrow \quad [g_B] = 2 - \frac{D}{2} = \varepsilon$$

$\therefore$ IN D - DIMENSIONS

$$\boxed{g_B = \mu^\varepsilon \, Z_g \, g}$$

μ : MASS SCALE

g_B: BARE (UNPHYSICAL) COUPLING CONSTANT IS <u>IN</u>DEPENDENT OF μ

g: RENORMALIZED COUPLING CONSTANT
$\|$
PHYSICAL COUPLING CONSTANT AT MASS SCALE μ

⇒ β - FUNCTION

↳ DESCRIBES HOW THE PHYSICAL (RENORMALIZED) COUPLING CONSTANT VARIES WITH MASS SCALE

$$\boxed{\beta(g) \equiv + \mu \frac{\partial g}{\partial \mu} = \frac{\partial g}{\partial \ln \mu}}$$

$$g_B = \mu^{\varepsilon} \cdot Z_g \cdot g$$

↓

TAKE DERIVATIVE OF BOTH SIDES W.R.T $\ln \mu$

$$\mu \frac{\partial}{\partial \mu}$$

$$0 = \varepsilon \mu^{\varepsilon} Z_g g + \mu^{\varepsilon} \left(\mu \frac{\partial Z_g}{\partial \mu} \right) g + \mu^{\varepsilon} Z_g \left(\mu \frac{\partial g}{\partial \mu} \right)$$

USE $\quad Z_g = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{2\varepsilon} \beta_0$

WITH $\beta_0 \equiv 11 - \frac{2}{3} N_f$

↓

$$\mu \frac{\partial Z_g}{\partial \mu} = - \frac{1}{\varepsilon} \frac{\beta_0}{(4\pi)^2} g \cdot \underbrace{\left(\mu \frac{\partial g}{\partial \mu} \right)}_{\beta(g)}$$

$$\Downarrow$$

$$0 = \mu^{\varepsilon} Z_g \left\{ \varepsilon g + \frac{g}{Z_g} \mu \frac{\partial Z_g}{\partial \mu} + \beta(g) \right\}$$

$$\Downarrow$$

$$0 = \varepsilon g + \frac{g}{Z_g} \left(- \frac{1}{\varepsilon} \frac{\beta_0}{(4\pi)^2} g \, \beta(g) \right) + \beta(g)$$

$$-\varepsilon g = \beta(g) \left[1 - \frac{1}{\varepsilon} \frac{\beta_0}{(4\pi)^2} \frac{g^2}{Z_g} \right]$$

$\downarrow$ TO ORDER g^2

$$-\varepsilon g = \beta(g) \left[1 - \frac{1}{\varepsilon} \frac{\beta_0}{(4\pi)^2} g^2 + O(g^4) \right]$$

$$\Downarrow$$

$$\beta(g) = -\varepsilon g \left[1 + \frac{1}{\varepsilon} \frac{\beta_0}{(4\pi)^2} g^2 + O(g^4) \right]$$

$\Downarrow$ LIMIT $\varepsilon \to 0$

$$\boxed{\beta(g) = -\beta_0 \frac{1}{(4\pi)^2} g^3}$$

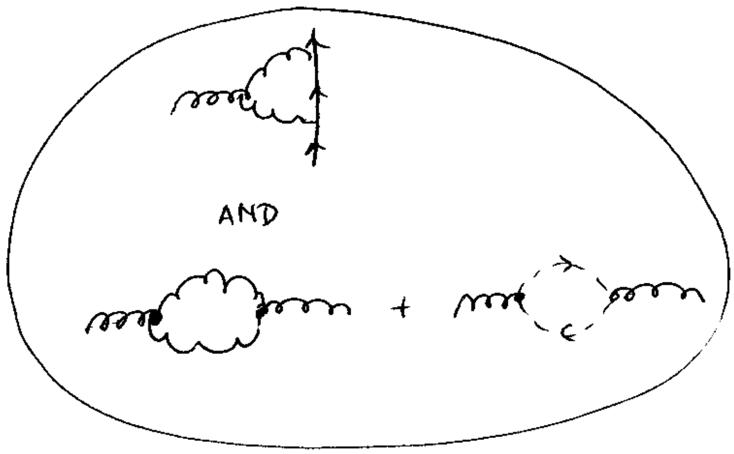
$\Rightarrow$ ASYMPTOTIC FREEDOM

$$\beta(g) \equiv \mu \frac{\partial g}{\partial \mu} = -\beta_0 \frac{1}{(4\pi)^2} \cdot g^3$$

$$\boxed{\beta_0 = 11 - \frac{2}{3} N_f}$$

FROM 

+ SIGN GIVEN BY



cfr. QED POLARIZATION OF VACUUM

DUE TO NON-ABELIAN NATURE OF THEORY

$\beta_0 > 0$ FOR $N_f < \frac{33}{2}$

IN NATURE $N_f = 6$ (u, d, s, c, b, t)

$\Downarrow$

$\beta_0 > 0$ (QCD)

COUPLING CONSTANT DECREASES AT $\mu \uparrow$, i.e. AT SHORT DISTANCES QUARKS BEHAVE AS FREE PARTICLES

ASYMPTOTIC FREEDOM (POLITZER, GROSS, WILCZEK) 1973

↳ IN **QED** WE HAVE AN OPPOSITE BEHAVIOR

N_f TERM DOMINATES

$\Downarrow$

$\beta_0 < 0$ (QED)

COUPLING CONSTANT INCREASES AT SHORT DISTANCES.

("INFRARED SLAVERY")

⟹ RUNNING COUPLING CONSTANT IN QCD

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = -\beta_0 \frac{1}{(4\pi)^2} g^3$$

$$\downarrow$$

$$\frac{dg}{g^3} = -\frac{\beta_0}{(4\pi)^2} \, d \ln \mu$$

$$d\left(-\frac{1}{2}\frac{1}{g^2}\right) = -\frac{\beta_0}{(4\pi)^2} \, d \ln \mu$$

INTEGRATE BETWEEN MASS SCALE μ_1 & MASS SCALE μ_2

$$-\frac{1}{2}\frac{1}{g^2(\mu_2)} + \frac{1}{2}\frac{1}{g^2(\mu_1)} = -\frac{\beta_0}{(4\pi)^2} \ln \frac{\mu_2}{\mu_1}$$

$$\underbrace{\frac{1}{g^2(\mu_1)} - \frac{\beta_0}{(4\pi)^2} \ln \mu_1^2}_{\text{DEPENDS ONLY ON } \mu_1} = \underbrace{\frac{1}{g^2(\mu_2)} - \frac{\beta_0}{(4\pi)^2} \ln \mu_2^2}_{\text{DEPENDS ONLY ON } \mu_2}$$

$\downarrow$ HENCE BOTH SIDES HAVE TO BE EQUAL TO A CONSTANT

$$\equiv -\frac{\beta_0}{(4\pi)^2} \ln \Lambda_{QCD}^2$$

|| Λ_{QCD} IS FUNDAMENTAL PARAMETER (MASS SCALE) IN QCD

$\therefore$ AT SOME MASS SCALE Q

$$\frac{1}{g^2(Q)} - \frac{\beta_0}{(4\pi)^2} \ln Q^2 = -\frac{\beta_0}{(4\pi)^2} \ln \Lambda_{QCD}^2$$

$$\downarrow$$

$$\frac{1}{g^2(Q)} = +\frac{\beta_0}{(4\pi)^2} \ln\left(\frac{Q^2}{\Lambda_{QCD}^2}\right)$$

DEFINE

$$\boxed{\alpha_s(Q) \equiv \frac{g^2(Q^2)}{4\pi} = \frac{4\pi}{\beta_0 \ln\left(Q^2/\Lambda_{QCD}^2\right)}}$$

QCD RUNNING COUPLING CONSTANT

EXPERIMENT $\Lambda_{QCD} \simeq 0.2$ GeV

$\rightarrow$ $Q \uparrow \Rightarrow \alpha_s \downarrow$ AT LARGE SCALES $\downarrow$ PERTURBATION THEORY

$\rightarrow$ AT SMALL SCALES (LARGE DISTANCES) $\alpha_s \uparrow$

WHEN $Q = \Lambda_{QCD}$ $\rightarrow$ FORMULA HAS POLE (LANDAU POLE)

α_s DIVERGES

⇒ RUNNING COUPLING CONSTANT IN QED

- **β - FUNCTION IN QED**

$$e_B = \mu^{\varepsilon} Z_e \, e$$

VACUUM POLARIZATION

$$\beta(e) \equiv \mu \frac{\partial e}{\partial \mu} \qquad Z_e = 1 + \frac{e^2}{(4\pi)^2} \frac{1}{2\varepsilon} \cdot \frac{4}{3}$$

$$\boxed{\beta(e) = \frac{e^3}{(4\pi)^2} \cdot \frac{4}{3}} > 0$$

INCREASES FOR SHORTER DISTANCES

$$\mu \frac{\partial e}{\partial \mu} = \frac{e^3}{(4\pi)^2} \cdot \frac{4}{3}$$

$$\downarrow$$

$$-\frac{1}{2} d\left(\frac{1}{e^2}\right) = d\left(\frac{4}{3} \cdot \frac{1}{(4\pi)^2} \ln \mu\right)$$

$\downarrow$ INTEGRATE BETWEEN LOW MASS SCALE m (MASS OF e^-) AND LARGE MASS SCALE Q

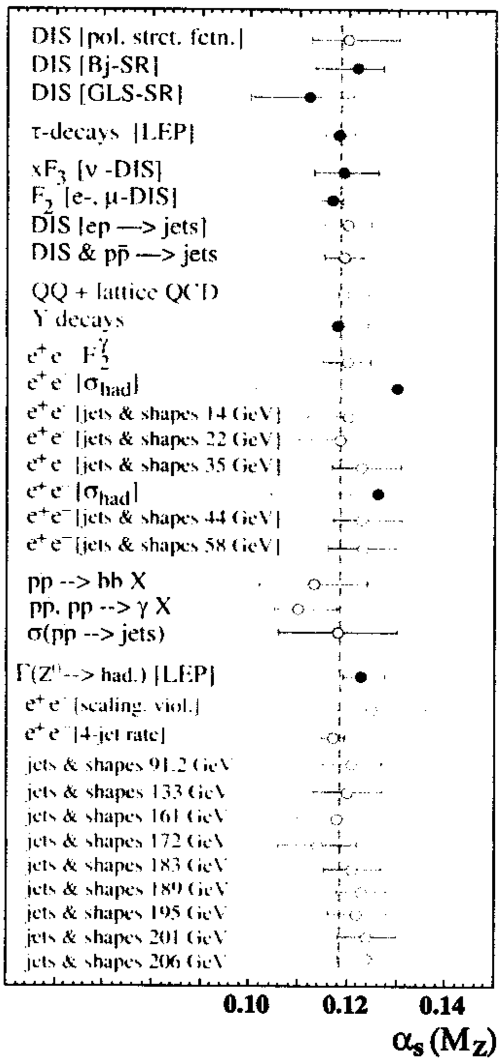
$$\frac{1}{e^2(Q^2)} - \frac{1}{e^2(m^2)} = -\frac{4}{3} \frac{1}{(4\pi)^2} \ln \frac{Q^2}{m^2}$$

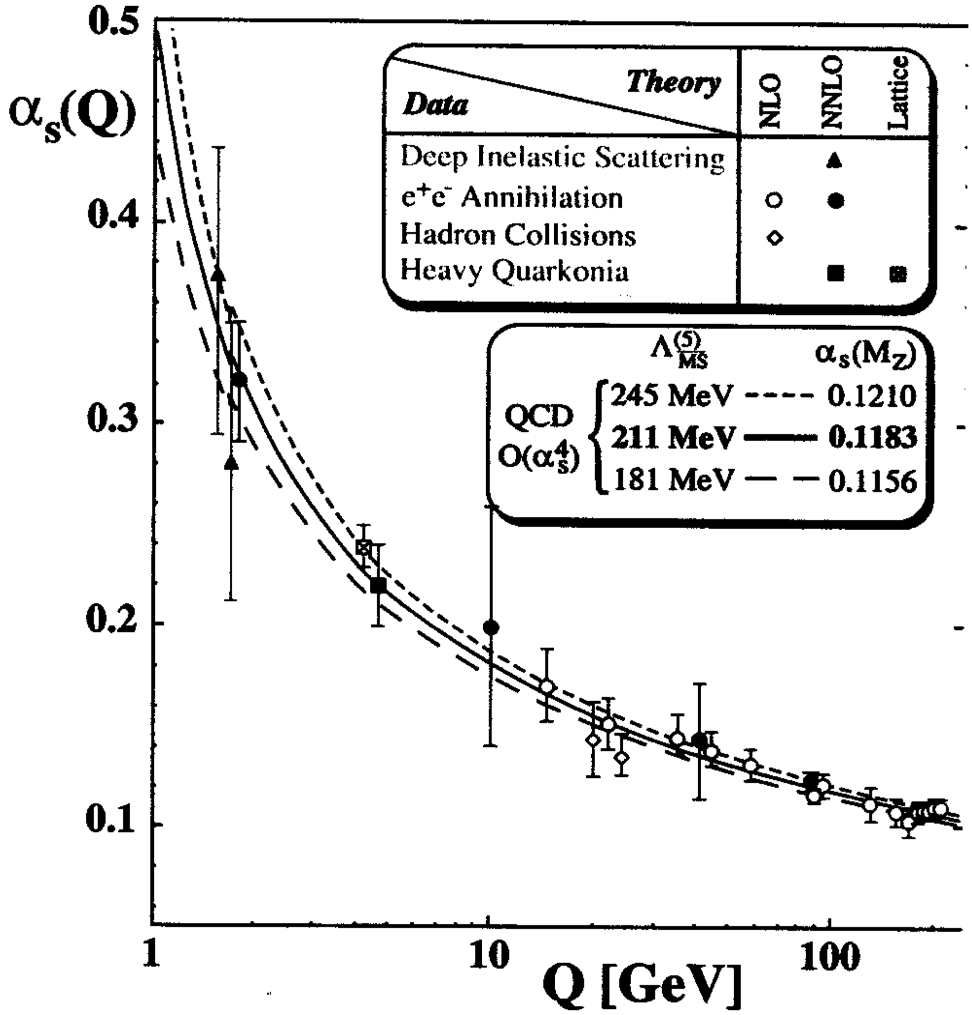
$$\boxed{\alpha_{em}(Q^2) \equiv \frac{e^2(Q^2)}{4\pi} = \alpha_{em}(m^2)\left[1 + \frac{4}{3} \frac{\alpha_{em}}{4\pi} \ln \frac{Q^2}{m^2}\right]}$$

↳ FINE STRUCTURE 'CONSTANT'

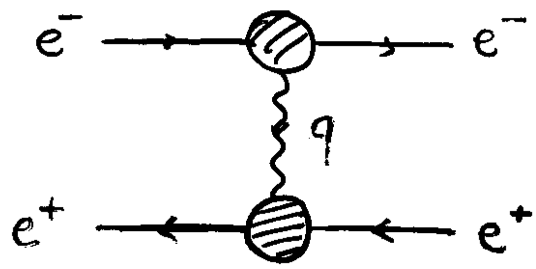
$$\alpha_{em}(m^2) \approx \frac{1}{137}$$

QCD : running coupling constant





QED COUPLING CONSTANT



$q^2 < 0$

$q^2 \equiv -Q^2$

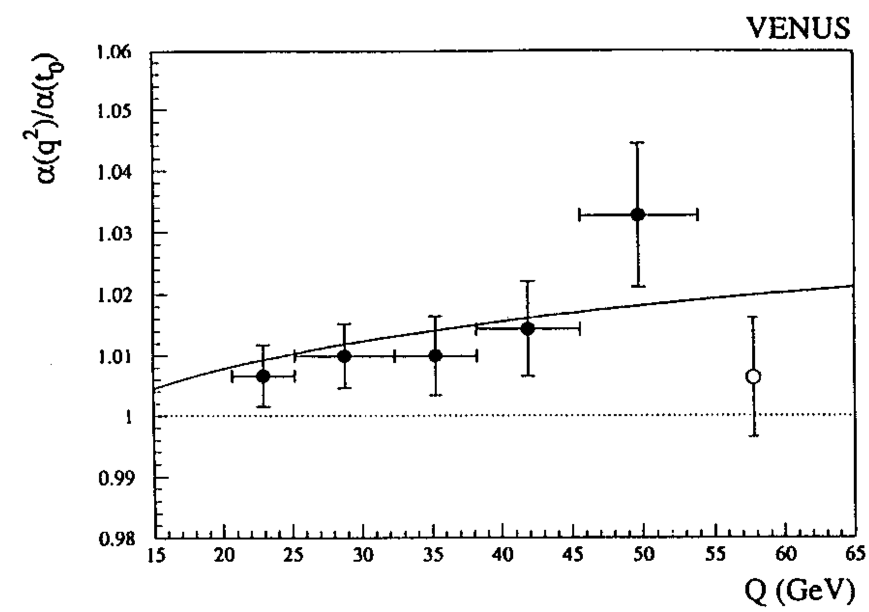
BHABHA SCATTERING

$\Downarrow$

$$\alpha_{em}(Q^2) = \frac{e^2(Q^2)}{4\pi}$$

TRISTAN

e^+e^- COLLIDER (KEK)



t_0 : LOW SCALE

$$\alpha(0) \simeq \frac{1}{137}$$