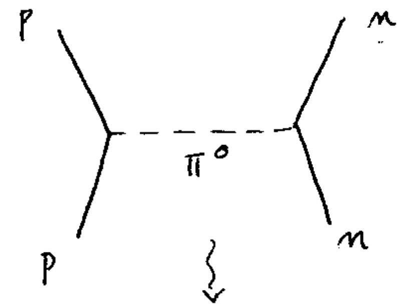
V

INTRODUCTION TO INTERACTING RELATIVISTIC QUANTUM FIELD THEORIES

V. INTERACTIONS & ELEMENTS OF FIELD THEORY

* QUANTUM EXCHANGE : YUKAWA THEORY

YUKAWA (1935) : SHORT RANGE FORCE BETWEEN PROTONS & NEUTRONS DUE TO EXCHANGE OF A PION



↓ MASS $m \simeq 135$ MeV (π^0)

EXCHANGED PARTICLE : VIRTUAL QUANTUM $\quad \Delta E \, \Delta t \sim \hbar$

⇓

SPIN 0 : DESCRIBED BY FIELD $\phi(t, \bar{x})$

SATISFIES KLEIN - GORDON EQ.

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi = 0$$

SOLUTIONS

- FOR FREE PROPAGATING PARTICLE $p^\mu(E, \bar{p})$

PLANE WAVE $\quad \phi(t, \bar{x}) = A \, e^{-i(Et - \bar{p}\cdot\bar{x})}$

$$E^2 = \bar{p}^2 + m^2$$

- FOR STATIC POTENTIAL (p OR n ACT AS SOURCE & SINK)

$$\phi \to U(\bar{x})$$

$$\left(- \nabla^2 + m^2 \right) U(\bar{x}) = 0$$

ASSUME SPHERICAL SYMMETRIC POTENTIAL

$$U(\bar{x}) = U(r) \qquad r = |\bar{x}|$$

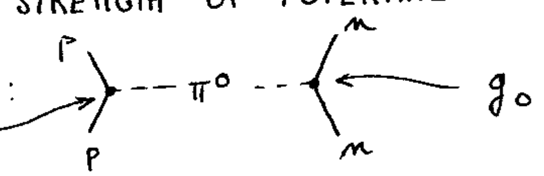
$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r} \right) = m^2 U$$

$\Downarrow$

SOLUTION $\quad \boxed{U(r) = \frac{g_0}{4\pi} \frac{1}{r} e^{-mr}}$

YUKAWA POTENTIAL ('SCREENED' $\frac{1}{r}$ POTENTIAL)

$\Rightarrow$ g_0 : STRENGTH OF POTENTIAL

g_0 :


$\Rightarrow$ R: RANGE OF POTENTIAL $\sim \frac{1}{m}$

$$R = \frac{(0.197 \text{ GeV fm})}{m} = \frac{0.197}{0.135} \text{ fm}$$

$$\simeq 1.5 \text{ fm}$$

SHORT RANGE OF STRONG INTERACTION $\Longleftrightarrow$ EXCHANGE OF MASSIVE PARTICLE

PION WAS OBSERVED IN 1947

* SPIN-0 PROPAGATOR

STATIC POTENTIAL $U(r)$

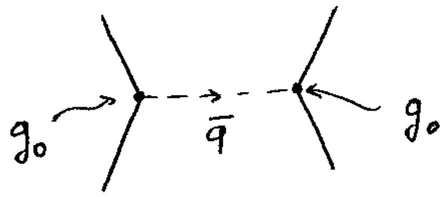
g_0 : COUPLING OF PARTICLE TO POTENTIAL

$\Downarrow$

FOURIER TF (TO MOMENTUM SPACE)

$$f(\bar{q}) = g_0 \int d^3\bar{r}\; U(r)\, e^{i\bar{q}\cdot\bar{r}}$$

↳ DESCRIBES EXCHANGE OF VIRTUAL PARTICLE WITH MOMENTUM $\bar{q}$



$$f(\bar{q}) = \frac{g_0^2}{4\pi} \int_0^\infty dr\; r^2\, \frac{e^{-mr}}{r} \cdot 2\pi \underbrace{\int_0^\pi d\cos\theta\; e^{i|\bar{q}|r\cos\theta}}_{\frac{2}{|\bar{q}|r}\sin(|\bar{q}|r)}$$

$$= g_0^2 \int_0^\infty dr\; \frac{1}{|\bar{q}|} \sin(|\bar{q}|r)\, e^{-mr}$$

$$= g_0^2\, \frac{1}{|\bar{q}|}\, \frac{1}{2i} \int_0^\infty dr\; e^{-mr}\left(e^{i|\bar{q}|r} - e^{-i|\bar{q}|r}\right)$$

$\downarrow$

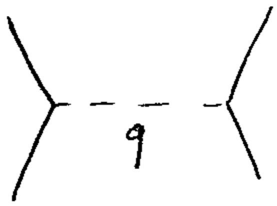
$$= g_0^2\, \frac{1}{2i|\bar{q}|} \left\{ -\frac{1}{i|\bar{q}|-m} - \frac{1}{i|\bar{q}|+m} \right\}$$

$$= g_0^2\, \frac{1}{|\bar{q}|^2 + m^2}$$

- FOR STATIC POTENTIAL (RANGE $\sim \frac{1}{m}$)

$$f(\bar{q}) = \frac{g_0^2}{|\bar{q}|^2 + m^2}$$

- FOR TIME-DEPENDENT EXCHANGE PROCESS



$q^\mu \; (q_0, \bar{q})$

REPLACE $\bar{q}^2 \to -q^2 = -q_0^2 + \bar{q}^2$

AMPLITUDE : $\dfrac{-g_0^2}{q^2 - m^2}$

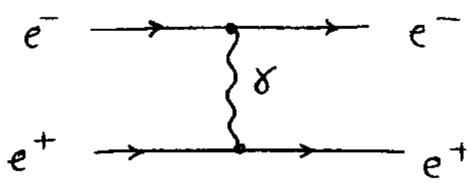
$\dfrac{1}{q^2 - m^2} \sim$ SPIN 0 PROPAGATOR

FROM AMPLITUDE FOR A PROCESS
WE CAN CALCULATE CROSS SECTIONS, DECAY RATES, ...

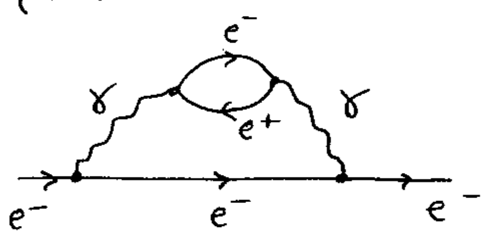
* FEYNMAN DIAGRAM

AMPLITUDE FOR PROCESS IS GRAPHICALLY DISPLAYED BY A FEYNMAN DIAGRAM (ARROWS INDICATE TIME SENSE)

e.g.

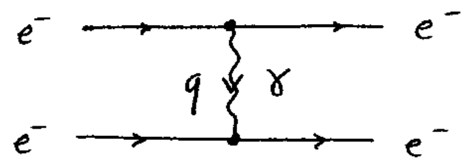


$\sim e^2$



$\sim e^4$

* ELECTROMAGNETIC INTERACTIONS

↳ 

q : 4-MOMENTUM OF γ

- STRENGTH $\sim \quad \alpha = \dfrac{e^2}{4\pi} = \dfrac{1}{137.036\ldots}$

$\Downarrow$

FINE STRUCTURE CONSTANT

- EXCHANGED PARTICLE : PHOTON (MASSLESS, SPIN 1)

$\Downarrow$

  - POTENTIAL : ∞ RANGE
  - PROPAGATOR $\sim \dfrac{1}{q^2}$

- CROSS SECTION (~ SCATTERING PROBABILITY)

$$d\sigma \sim \left| \frac{\alpha}{q^2} \right|^2 = \frac{\alpha^2}{q^4}$$

(RUTHERFORD CROSS SECTION)

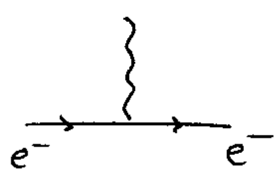
↳ EM COUPLING SMALL $\Rightarrow$ PERTURBATION SERIES IN α

e.g. e^- MAGNETIC MOMENT

$$\mu = g \cdot \underbrace{\frac{1}{2}}_{\text{SPIN}} \cdot \frac{e}{2m_e}$$

BOHR MAGNETON

- FOR DIRAC PARTICLE $g = 2$

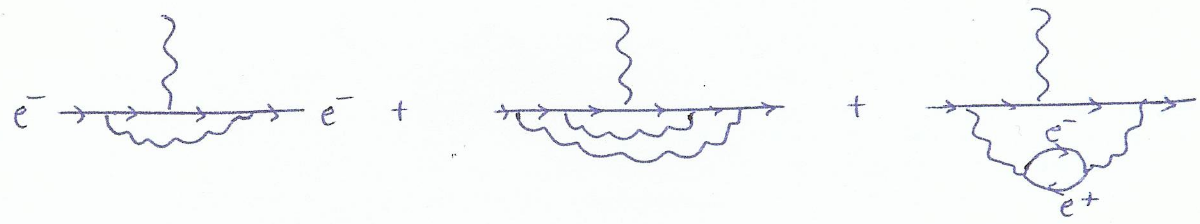


- DUE TO VIRTUAL γ PROCESSES (VACUUM POLARIZATION, ...)

$\Downarrow$ QUANTUM FIELD THEORY

$$g \neq 2$$

$\frac{g-2}{2}$: e^- ANOMALOUS MAGNETIC MOMENT



PERTURBATION SERIES

$$\frac{g-2}{2} = \underset{\substack{\uparrow \\ 0.00\underline{\underline{116}}}}{0.5 \left(\frac{\alpha}{\pi}\right)} - 0.32848 \left(\frac{\alpha}{\pi}\right)^2 + O\left(\frac{\alpha}{\pi}\right)^3$$

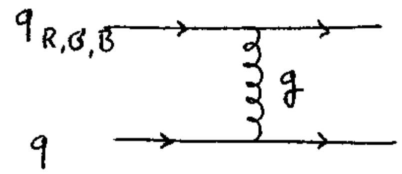
$$\left(\frac{g-2}{2}\right)^{\text{THEORY}} = (11\ 659\ 184.0 \pm 5.9) \times 10^{-10}$$

$$\left(\frac{g-2}{2}\right)^{\text{EXP}} = (11\ 659\ 208.9 \pm 6.3) \times 10^{-10}$$

THEORY $\leftrightarrow$ EXP

(2012) 3σ DEVIATION

* STRONG INTERACTIONS

↳ 

8 gluons ($R\bar{B}$, $R\bar{G}$, $B\bar{R}$, $B\bar{G}$, $G\bar{R}$, $G\bar{B}$, $R\bar{R} - G\bar{G}$, $R\bar{R} + G\bar{G} - 2B\bar{B}$

SU(3) GAUGE SYMMETRY : THEORY IS INVARIANT WHEN QUARK CHANGES ITS COLOR

MASSLESS $\longrightarrow$ PROPAGATOR $\sim \dfrac{1}{q^2}$

$\Downarrow$

$\dfrac{1}{r}$ POTENTIAL AT SHORT DISTANCES

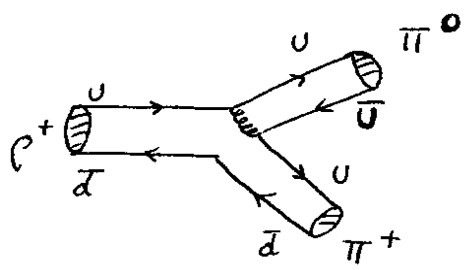
$$V(r) = \frac{a}{r} + \underbrace{b\,r}$$

1g EXCHANGE POTENTIAL

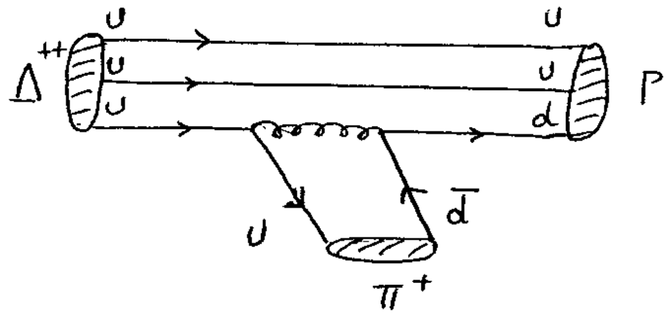
CONFINEMENT POTENTIAL AT LARGE DISTANCES

↳ STRONG DECAYS

$\rho^+ \rightarrow \pi^+ + \pi^0$



$\Delta^{++} \longrightarrow p + \pi^+$



* INTERACTION CROSS SECTION

STRENGTH OF SCATTERING PROCESS

e.g. $a + b \rightarrow c + d$ (e.g. $e^- + p \rightarrow e^- + p$)

a ↑ BEAM, b ↑ TARGET

ELASTIC e^-p SCATTERING

σ : REACTION CROSS SECTION (FOR EACH TARGET PARTICLE)

$$\equiv \frac{\text{TRANSITION PROBABILITY PER UNIT TIME} : W}{\text{INCIDENT FLUX} : \phi} \quad (\text{DIMENSION SURFACE}).$$

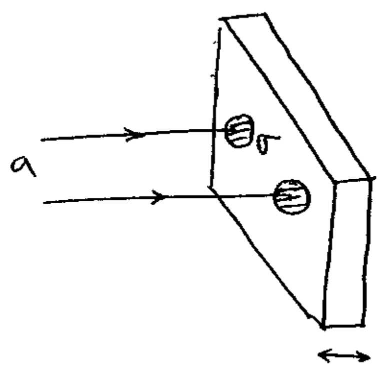
→ INCIDENT FLUX $\phi = n_a v_i$

n_a ↑ DENSITY OF PARTICLES IN BEAM

v_i ← RELATIVE VELOCITY OF BEAM (w.r.t TARGET)

→ # REACTIONS PER UNIT TIME & PER UNIT SURFACE

$$N = \phi \cdot (n_b \, \sigma \, dx)$$



dx (TARGET THICKNESS)

n_b : DENSITY OF TARGET

$n_b \, dx$: # TARGET PARTICLES / UNIT SURFACE

→ W: REACTION RATE (PER TARGET PARTICLE)

- NON-RELATIV. (FERMI'S GOLDEN RULE)

$$W = \frac{2\pi}{\hbar} \underbrace{|M_{fi}|^2}_{\substack{\text{MATRIX ELEMENT} \\ \langle f | \hat{O} | i \rangle}} \cdot \underbrace{\rho_f}_{\substack{\text{PHASE SPACE} \\ \text{OF FINAL STATE} \\ \text{(ENERGY DENSITY)}}}$$

- RELATIVISTICALLY

CALCULATE M_{fi} FROM FEYNMAN DIAGRAM

$$\rho_f = \frac{d^3\bar{p}_f}{(2\pi)^3\, 2E_f} \qquad \text{FOR 1 PARTICLE IN FINAL STATE}$$

$$\rho_f = \frac{d^3\bar{p}_c}{(2\pi)^3\, 2E_c} \frac{d^3\bar{p}_d}{(2\pi)^3\, 2E_d} \qquad \text{FOR 2 PARTICLE FINAL STA}$$

→ ENERGY-MOMENTUM IS CONSERVED

FACTOR $(2\pi)^4\, \delta^4(p_a + p_b - p_c - p_d)$

→ SPIN STATES

a: SPIN $s_a \Longrightarrow (2s_a+1)$ STATES

AVERAGE OVER INITIAL SPINS
& SUM OVER FINAL SPINS $\qquad \dfrac{1}{(2s_a+1)(2s_b+1)} \displaystyle\sum_{s_a, s_b, s_c, s_d}$

→ UNITS OF σ

barns	1 barn	$= 10^{-28}\ m^2$
millibarn	1 mb	$= 10^{-3}\ b$
microbarn	$1\ \mu b$	$= 10^{-6}\ b$
nanobarn	1 nb	$= 10^{-9}\ b$
picobarn	1 pb	$= 10^{-12}\ b$

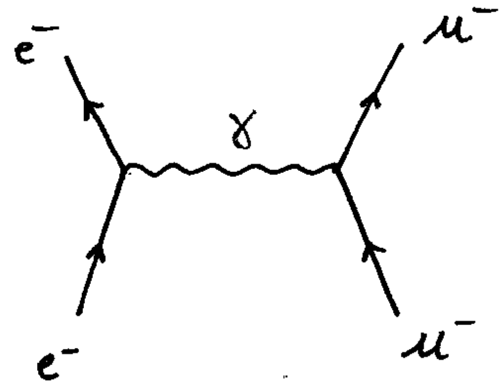
EXPRESS σ IN GeV^{-2}

& CONVERT BY $(\hbar c)^2 = (0.197\ GeV^2\ fm^2)$

+ express fm^2 in b

$1\ fm^2 = 10^{-2}\ b$

* EXAMPLE : " SPINLESS " ELECTRON - MUON SCATTERING



$\Rightarrow$ e^- & μ^- ARE SPIN 1/2 DIRAC PARTICLES

LET'S FIRST WORK IT OUT FOR THE CASE OF SPIN 0 PARTICLES (e.g. π^-, π^+)

$\Rightarrow$ SPIN - 0 PARTICLE SATISFIES KLEIN - GORDON EQ.

$$(\partial_\mu \partial^\mu + m^2)\, \phi = 0 \qquad \left(\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \right)$$

THIS FIELD EQ. FOLLOWS FROM LAGRANGIAN

$$L = \int d^3\bar{x}\, \mathcal{L} \quad , \quad \boxed{\mathcal{L}_{KG} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi}$$

$\Downarrow$

USE EULER - LAGRANGE EQUATIONS

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \quad \Rightarrow \quad (\partial_\mu \partial^\mu + m^2)\phi^\dagger = 0$$

& LIKEWISE FOR $\phi^\dagger$

⇒ INTERACTION OF CHARGED PARTICLE (CHARGE e) WITH ELECTROMAGNETIC FIELD

$$p^{\mu} \to p^{\mu} - e A^{\mu}$$

FOR $\quad \phi(x) = \phi_0 \, e^{-ip.x}$ $\quad$ PLANE WAVE

$$\boxed{\partial^{\mu} \longrightarrow \partial^{\mu} + ie A^{\mu}} \quad \text{MINIMAL SUBSTITUTION}$$

PERFORM THIS SUBSTITUTION IN $\mathcal{L}_{KG}$

$$\mathcal{L}_{KG} + \mathcal{L}_{INT} = \left[(\partial_{\mu} + ie A_{\mu}) \phi \right]^{+} (\partial^{\mu} + ie A^{\mu}) \phi - m^2 \phi^{+} \phi$$

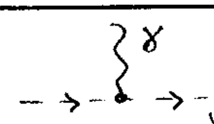
$$= (\partial_{\mu} \phi^{+})(\partial^{\mu} \phi) - m^2 \phi^{+} \phi$$

$$- ie A_{\mu} \phi^{+} \partial^{\mu} \phi + ie (\partial_{\mu} \phi)^{+} \phi A^{\mu}$$

$$+ e^2 \phi^{+} \phi A_{\mu} A^{\mu}$$

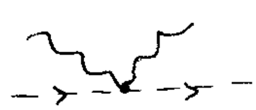
$$\boxed{\begin{aligned} \mathcal{L}_{INT} = &- e \left[i \phi^{+} (\partial^{\mu} \phi) - i (\partial^{\mu} \phi)^{+} \phi \right] A_{\mu} \\ &+ e^2 \phi^{+} \phi A_{\mu} A^{\mu} \end{aligned}}$$

TERM LINEAR IN A^{μ}



SPIN 0 PARTICLE

TERM QUADRATIC IN A^{μ}

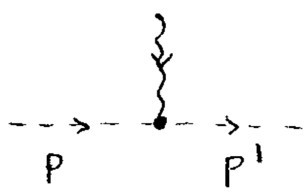


- TERM LINEAR IN A^μ

$$\mathcal{L}_{INT} = - e \, J^\mu A_\mu$$

J^μ : ELECTROMAGNETIC CURRENT

$$\boxed{J^\mu_{(x)} = i \, \phi^+ (\partial^\mu \phi) - i (\partial^\mu \phi^+) \phi}$$



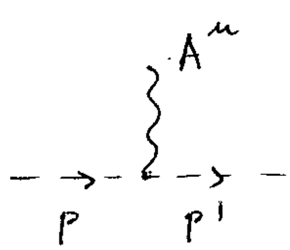
INITIAL $\phi_i \sim e^{-ipx}$

FINAL $\phi_f^+ \sim e^{+ip'x}$

IN MOMENTUM SPACE

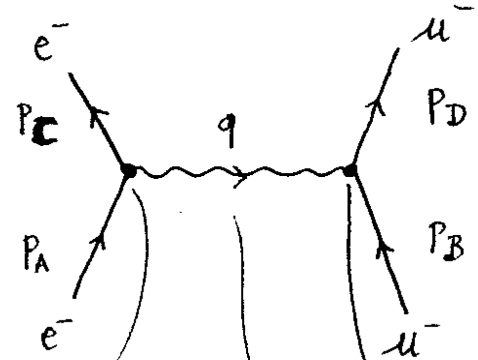
$$< p' \,|\, J^\mu(0) | \, p > = (p + p')^\mu$$

- FEYNMAN RULE (LOWEST ORDER : $i\mathcal{L}$)



$$\boxed{- ie \, (p + p')^\mu}$$

$\Rightarrow$ MATRIX ELEMENT FOR SPINLESS $e^- \mu^-$ SCATTERING



e^- HAS CHARGE $(-e)$
μ^-

$$\left[-i(-e)\,(p_B + p_D)^{\nu}\right]$$

$$\left[-i(-e)\,(p_A + p_C)^{\mu}\right]$$

PHOTON PROPAGATOR
(LORENTZ GAUGE)

$$-\frac{i\,g_{\mu\nu}}{q^2}$$

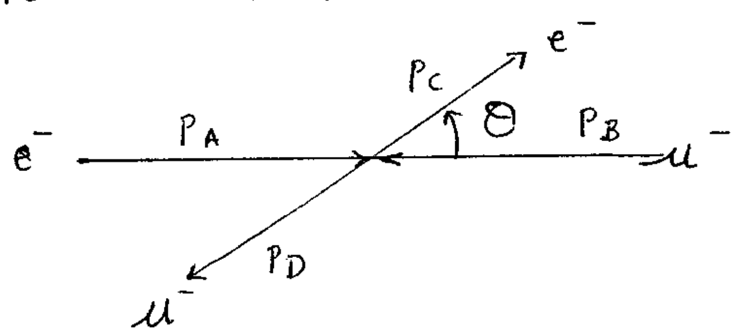
$$\mathcal{M}_{AB\to CD} = \left[+ie\,(p_A + p_C)^{\mu}\right]\left(\frac{-i\,g_{\mu\nu}}{q^2}\right)\left[ie\,(p_B + p_D)^{\nu}\right]$$

$\Uparrow$
MATRIX ELEMENT

$$\boxed{\mathcal{M}_{AB\to CD} = \frac{i\,e^2}{q^2}\,(p_A + p_C)^{\mu}\,(p_B + p_D)_{\mu}}$$

⇒ CROSS SECTION

- IN TOTAL C.M. (CENTER-OF-MASS) FRAME



$$P_A = (E_A, \bar{P}_A)$$

$$P_B = (E_B, \bar{P}_B) \qquad \text{IN C.M.} \quad \bar{P}_B = -\bar{P}_A, \quad E_B = E_A$$

$$P_C = (E_C, \bar{P}_C) \qquad \text{TOTAL ENERGY IS CONSERVED}$$

$$P_D = (E_C, -\bar{P}_D) \qquad E_C = E_A$$

TOTAL ENERGY INITIALLY $E_A + E_B = 2E_A$

INVARIANT $(P_A + P_B)^2 \equiv s = (2E_A)^2 - (\underbrace{\cancel{\bar{P}_A + \bar{P}_B}}_{0})^2$

$$\Downarrow$$

$$\boxed{E_A = \frac{\sqrt{s}}{2}}$$

s : MANDELSTAM INVARIANT

$$E_C = \sqrt{s}/2$$

LET'S CONSIDER HIGH BEAM ENERGIES

↳ $|\bar{P}_A| \gg m_e, m_\mu$ ⇒ WE CAN NEGLECT MASSES

$$E_A \simeq |\bar{P}_A|$$

$$P_A\left(\frac{\sqrt{s}}{2}, 0, 0, \frac{\sqrt{s}}{2}\right)$$

$$P_B\left(\frac{\sqrt{s}}{2}, 0, 0, -\frac{\sqrt{s}}{2}\right)$$

$$P_C\left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2}\sin\theta, 0, \frac{\sqrt{s}}{2}\cos\theta\right) \qquad P_D\left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s}}{2}\sin\theta, 0, -\frac{\sqrt{s}}{2}\cos\theta\right)$$

- $$\mathcal{M}_{AB \to CD} = \frac{ie^2}{q^2} \left(P_A + P_C \right)^\mu \left(P_B + P_D \right)_\mu$$

$$\Rightarrow \quad \left(P_A + P_C \right)^\mu \left(P_B + P_D \right)_\mu$$

$$= \left(\sqrt{s}\right) \cdot \left(\sqrt{s}\right)$$

$$- \left(\frac{\sqrt{s}}{2} \sin\theta \right) \cdot \left(- \frac{\sqrt{s}}{2} \sin\theta \right)$$

$$- \left(\frac{\sqrt{s}}{2} (1 + \cos\theta) \right) \cdot \left(- \frac{\sqrt{s}}{2} (1 + \cos\theta) \right)$$

$$= s \left(1 + \frac{1}{4} \sin^2\theta + \frac{1}{4} (1 + \cos\theta)^2 \right)$$

$$= \frac{s}{4} \left(6 + 2\cos\theta \right)$$

$$= \frac{s}{2} \left(3 + \cos\theta \right)$$

$$\Rightarrow \quad q^2 = \left(P_A - P_C \right)^2 = - \frac{s}{4} \left(1 - \cos\theta \right)^2 - \frac{s}{4} \sin^2\theta$$

$$= - \frac{s}{2} \sin^2\theta/2$$

$$\mathcal{M}_{AB \to CD} = - i e^2 \frac{(3 + \cos\theta)}{(1 - \cos\theta)} = - \frac{ie^2}{2} \frac{(3 + \cos\theta)}{\sin^2\frac{\theta}{2}}$$

- $$d\sigma = \underset{\substack{\uparrow \\ \text{INITIAL} \\ \text{FLUX}}}{\frac{1}{\Phi}} \cdot \underset{\substack{\uparrow \\ \text{TRANSITION} \\ \text{PROB / UNIT TIME}}}{W}$$

$$\Phi = \frac{|\bar{P}_A|}{E_A} + \frac{|\bar{P}_B|}{E_B}$$ TOTAL CURRENT (INITIAL) DENSITY

$$\simeq 2$$

$$W = \frac{1}{(2E_A)(2E_B)} \frac{d^3\bar{P}_C}{(2\pi)^3\, 2E_C} \cdot \frac{d^3\bar{P}_D}{(2\pi)^3\, 2E_D} \cdot (2\pi)^4 \delta^4(P_A + P_B - P_C - P_D)$$

$$\cdot\, |\mathcal{M}_{AB\to CD}|^2$$

$$d\sigma = \frac{1}{2s} \cdot \frac{1}{(2\pi)^2} \frac{d^3\bar{P}_C}{\underbrace{4E_C E_D}_{s}} \delta\Big(\underbrace{P_A^0 + P_B^0}_{\sqrt{s}} - \sqrt{\bar{P}_C^2 + \cancel{m_C^2}} - \sqrt{\bar{P}_C^2 + \cancel{m_\mu^2}}\Big)$$

$$\uparrow \quad |\bar{P}_D| = |\bar{P}_C|$$

$$\cdot\, |\mathcal{M}_{AB\to CD}|^2$$

$$= \frac{1}{2s^2(2\pi)^2} \cdot d\Omega \cdot d|\bar{P}_C|\, |\bar{P}_C|^2\, \underbrace{\delta\big(\sqrt{s} - 2|\bar{P}_C|\big)}_{\frac{1}{2}\delta\left(|\bar{P}_C| - \frac{\sqrt{s}}{2}\right)} |\mathcal{M}_{AB\to CD}|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{C.M.} = \frac{1}{2s^2(2\pi)^2} \cdot \left(\frac{\sqrt{s}}{2}\right)^2 \cdot \frac{1}{2} \cdot |\mathcal{M}_{AB\to CD}|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{C.M.} = \frac{1}{4s\,(4\pi)^2} \cdot e^4 \frac{(3+\cos\theta)^2}{(1-\cos\theta)^2}$$

$\downarrow$ INTRODUCE $\alpha \equiv \frac{e^2}{4\pi}$ FINE STRUCTURE CONST.

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)^{C.M} = \frac{\alpha^2}{4s} \cdot \frac{(3+\cos\theta)^2}{(1-\cos\theta)^2} = \frac{\alpha^2}{16s} \frac{(3+\cos\theta)^2}{\sin^4\theta/2}}$$

STRONG FORWARD PEAKING

$$\sim \frac{1}{\sin^4\theta/2}$$ RUTHERFORD FORMULA

NUMERICAL EXAMPLE : $\theta^{C.M.} = 90°$, $\sqrt{s} = 100$ GeV

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=90°} = \frac{9\alpha^2}{4s} = \frac{9}{4}\left(\frac{1}{137}\right)^2 \cdot \frac{1}{10^4}\ GeV^{-2}$$

$$= \frac{9}{4}\left(\frac{1}{137}\right)^2 \cdot \frac{(0.197)^2}{10^4}\ \underbrace{fm^2}_{10^{-2}\,b}$$

$$= \underbrace{\frac{9}{4}\left(\frac{197}{137}\right)^2}_{\simeq 4.65} \cdot \underbrace{10^{-12}\,b}_{pb}$$