PLANE-WAVE SOLUTIONS OF DIRAC EQUATION CONSTRUCTED FROM LORENTZ TRANSFORMATION

↳ FREE DIRAC PARTICLE <u>AT REST</u>

$$p^{\mu}_{(0)}\left(\frac{E}{c},\ \bar{p}\right) = \left(m_0 c,\ 0\right)$$

↑ IN REST FRAME

$$\psi_r(x) = w_r(0)\, e^{-\frac{i}{\hbar}\lambda_r (m_0 c^2) t}$$

$$\lambda_r = \begin{cases} +1 & , r = 1,2 \\ -1 & , r = 3,4 \end{cases}$$

$$w_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$w_3(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad w_4(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} \Sigma_3\, w_1 = +w_1 \\ \Sigma_3\, w_2 = -w_2 \\ \Sigma_3\, w_3 = +w_3 \\ \Sigma_3\, w_4 = -w_4 \end{cases}$$

w_r : EIGENFUNCTION OF z-COMP. OF "SPIN OPERATOR"

$$\bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$$

↳ TO DESCRIBE FREE PARTICLE WITH <u>FINITE</u> MOMENTUM $\bar{p}$

$$p^\mu \quad \left(\frac{E}{c}, \bar{p}\right) \qquad E^2 = c^2 \bar{p}^2 + m_0^2 c^4$$

PERFORM LORENTZ BOOST ON DIRAC SPINOR AT REST

$$w_r(\bar{p}) = S \, w_r(0)$$

$$\rightsquigarrow \quad S = \exp\left\{-\frac{i}{2}\,\omega\,\sigma_{01}\right\}$$

(FOR LORENTZ BOOST ALONG x-AXIS)

$$\downarrow \quad \sigma_{01} = -i\alpha_1$$

$$= \exp\left\{-\frac{\omega}{2}\,\alpha_1\right\}$$

$$= \cosh\frac{\omega}{2}\,\mathbb{1}_{4\times 4} - \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \sinh\frac{\omega}{2}$$

$$= \cosh\frac{\omega}{2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\tanh\frac{\omega}{2} \\ 0 & 1 & -\tanh\frac{\omega}{2} & 0 \\ \hline 0 & -\tanh\frac{\omega}{2} & 1 & 0 \\ -\tanh\frac{\omega}{2} & 0 & 0 & 1 \end{array}\right]$$

IN SYSTEM S : PARTICLE AT REST $\bar{p} = 0$

IN SYSTEM S' (MOVING WITH SPEED $-v$ ALONG x-AXIS)

↪ PARTICLE HAS MOMENTUM $\bar{p}$

$$\Big\} \quad -\frac{v}{c} = \tanh \omega$$

USE $\quad \tanh \omega = \dfrac{2 \tanh \omega/2}{1 + \tanh^2 \omega/2}$

$$1 + \sqrt{1 - \tanh^2 \omega} = \frac{2}{1 + \tanh^2 \omega/2}$$

$$\Downarrow$$

$$\tanh \frac{\omega}{2} = \frac{\tanh \omega}{1 + \sqrt{1 - \tanh^2 \omega}}$$

$$\hookrightarrow \quad -\tanh \frac{\omega}{2} = \frac{v/c}{1 + \sqrt{1-\beta^2}} = \frac{\gamma v/c}{1 + \gamma}$$

$$= \frac{\gamma m_0 v c}{m_0 c^2 + \gamma m_0 c^2}$$

$$\Big\} \quad p = \gamma m_0 v \qquad E = \gamma m_0 c^2$$

$$-\tanh \frac{\omega}{2} = \frac{p c}{m_0 c^2 + E}$$

$$\cosh \frac{\omega}{2} = \frac{1}{\sqrt{1 - \tanh^2 \frac{\omega}{2}}} = \sqrt{\frac{E + m_0 c^2}{2 m_0 c^2}}$$

$$S = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \left[\begin{array}{cc|cc} 1 & & 0 & \frac{cp}{E+m_0c^2} \\ & 1 & \frac{cp}{E+m_0c^2} & 0 \\ \hline 0 & \frac{cp}{E+m_0c^2} & 1 & \\ \frac{cp}{E+m_0c^2} & 0 & & 1 \end{array}\right]$$

$$= \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \left[\begin{array}{c|c} \mathbb{1} & \frac{c}{E+m_0c^2}\sigma_x p \\ \hline \frac{c}{E+m_0c^2}\sigma_x p & \mathbb{1} \end{array}\right]$$

$\rightsquigarrow$ FOR ARBITRARY LORENTZ BOOST ALONG DIRECTION $\bar{p}$

$$S = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \left[\begin{array}{c|c} \mathbb{1} & \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p} \\ \hline \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p} & \mathbb{1} \end{array}\right]$$

$$w_r(\bar{p}) = S\, w_r(0) \Rightarrow w_1(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} 1 \\ 0 \\ \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p}\begin{pmatrix}1\\0\end{pmatrix} \end{bmatrix}$$

$$w_2(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} 0 \\ 1 \\ \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p}\begin{pmatrix}0\\1\end{pmatrix} \end{bmatrix}$$

$$w_3(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} c\,\dfrac{\bar{\sigma}\cdot\bar{p}}{E+m_0c^2}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{bmatrix}$$

$$w_4(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} c\,\dfrac{\bar{\sigma}\cdot\bar{p}}{E+m_0c^2}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{bmatrix}$$

$\rightsquigarrow$ FREE PARTICLE SOLUTION

$$(m_0c^2)\,t = p_{(0)}^{\mu}\, x_{\mu}^{(0)}$$

(0) DENOTES REST FRAME

$$= p^{\mu} x_{\mu}$$ IN ARBITRARY FRAME (LORENTZ INVARIANT!)

$$\therefore \quad \boxed{\psi_r(x) = w_r(\bar{p})\, e^{-\frac{i}{\hbar}\lambda_r\, p_\mu x^\mu}}$$

⤳ DIRAC EQ.

$$\left(i\hbar\, \gamma^\mu \partial_\mu - m_0 c \right) \psi_r(x) = 0$$

$$\Downarrow$$

$$\left(\gamma^\mu p_\mu - \lambda_r m_0 c \right) w_r(\bar{p}) = 0$$

$$\Updownarrow$$

$$\underline{\underline{\left(\not{p} - \lambda_r m_0 c \right) w_r(\bar{p}) = 0}}$$

⤳ NORMALIZATION (BY DIRECT INSPECTION)
CHECK !

$$\bar{w}_r(\bar{p})\, w_{r'}(\bar{p})$$

$$= w_r^+(\bar{p})\, \gamma^0\, w_{r'}(\bar{p}) = \delta_{rr'}\, \lambda_r$$

⤳ COMPLETENESS (CHECK !)

$$\sum_{r=1}^{4} \lambda_r \left(w_r(\bar{p}) \right)_\alpha \left(\bar{w}_r(\bar{p}) \right)_\beta = \delta_{\alpha\beta}$$

$\alpha, \beta = 1 \ldots 4$

↳ POLARIZATION FOR DIRAC PARTICLE

⤳ REST FRAME

ASSUME PARTICLE SPIN ALONG Z-AXIS

$$w_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ SPIN PROJ $+\frac{\hbar}{2}$ ALONG Z-AXIS

$$w_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ SPIN PROJ $-\frac{\hbar}{2}$

IN REST FRAME $p^{\mu}_{(0)} = (m_0 c, 0)$

INTRODUCE POLARIZATION VECTOR $s^{\mu}_0 = (0, \bar{s})$

$\bar{s}^2 = 1$

$$\bar{\Sigma}.\bar{s}\, w_1(0) = + w_1(0)$$
$$\bar{\Sigma}.\bar{s}\, w_2(0) = - w_2(0)$$

$\bar{s}$ IS UNIT VECTOR GIVEN BY SPIN AXIS (e.g. IN ABOVE EXAMPLE $\bar{s} = \hat{e}_z$)

WE OBSERVE $s^{\mu}_0 s_{0\mu} = -\bar{s}^2 = -1.$

$$s^{\mu}_0 p_{0\mu} = 0$$

$\rightsquigarrow$ IN ARBITRARY FRAME

$$s^\mu = a^\mu{}_\nu \, (s_{(0)})^\nu$$

$\uparrow$ REST FRAME

$$\left.\begin{aligned} s^\mu s_\mu &= -1 \\ s^\mu p_\mu &= 0 \end{aligned}\right\} \quad \text{BECAUSE l.h.s IS LORENTZ INV.}$$

DEFINE SPINOR POLARIZATION IN REST FRAME

e.g. CHOOSE Z-AXIS AS SPIN AXIS

$$s = u_z$$

$$s_{(0)} = u_z^{(0)} = (0, \hat{e}_z).$$

NOTATION

	SPINOR WITH SPIN PROJ ALONG Z-AXIS IN REST FR. (Σ_3)
$u(p, u_z) \equiv w_1(\bar{p})$	$+1$
$u(p, -u_z) \equiv w_2(\bar{p})$	-1
$v(p, -u_z) \equiv w_3(\bar{p})$	$+1$
$v(p, u_z) \equiv w_4(\bar{p})$	-1

$$(\not{p} - m_0 c)\, u(p, \pm u_z) = 0 \qquad \text{POS. ENERGY SOLUTIONS}$$

$$(\not{p} + m_0 c)\, v(p, \pm u_z) = 0 \qquad \text{NEG. ENERGY SOLUTIONS.}$$

<u>NOTE</u> FOR NEGATIVE ENERGY SOLUTION

WE WILL INTERPRET ABSENCE OF DIRAC PARTICLE WITH $-p$ AND NEGATIVE SPIN PROJ.

AS AN ANTI-PARTICLE WITH $+p$ AND POSITIVE SPIN PROJ

$\Downarrow$

NOTATION $v(p, u_z)$

$\uparrow$ DENOTES SPIN PROJ OF ANTI-PARTICLE

- PROJECTION OPERATORS. FOR ENERGY & SPIN

↳ DEFINITIONS

$P_r(\bar{p})$ PROJECTS OUT SPINOR r

$$P_r(\bar{p})\, w_{r'}(\bar{p}) = \delta_{rr'}\, w_r(\bar{p})$$

$$P_r(\bar{p})\, P_{r'}(\bar{p}) = \delta_{rr'}\, P_r(\bar{p})$$

↳ ENERGY PROJECTORS

$$\boxed{\Lambda_r(\bar{p}) \equiv \frac{\lambda_r \not{p} + m_0 c}{2 m_0 c}}$$

<u>CHECK</u> $*$ $\Lambda_{1,2}(\bar{p}) = \dfrac{\not{p} + m_0 c}{2 m_0 c}$

$*$ $\Lambda_{3,4}(\bar{p}) = \dfrac{-\not{p} + m_0 c}{2 m_0 c}$

$$\left.\begin{matrix}\Lambda_{1,2}\\ \Lambda_{3,4}\end{matrix}\right\} \quad \Lambda_{\pm}(\bar{p}) \equiv \frac{\pm \not{p} + m_0 c}{2 m_0 c}$$

$$* \quad \Lambda_r(\bar{p})\, w_r(\bar{p}) = \frac{\lambda_r \not{p} + m_0 c}{2 m_0 c}\, w_r(\bar{p})$$

$$\downarrow \quad \not{p}\, w_r(\bar{p}) = \lambda_r\, m_0 c\, w_r(\bar{p})$$

$$= w_r(\bar{p})$$

$$* \quad \Lambda_r(\bar{p})\, \Lambda_{r'}(\bar{p}) = \frac{(\lambda_r \not{p} + m_0 c)(\lambda_{r'} \not{p} + m_0 c)}{4 m_0^2 c^2}$$

$$= \frac{1}{4 m_0^2 c^2} \left\{ \lambda_r \lambda_{r'} \not{p}\not{p} + (\lambda_r + \lambda_{r'})\, m_0 c \not{p} + m_0^2 c^2 \right\}$$

$$\downarrow \quad \not{p}\not{p} = \gamma_\mu \gamma_\nu\, p^\mu p^\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu)\, p^\mu p^\nu$$

$$= g_{\mu\nu}\, p^\mu p^\nu = p_\mu p^\mu$$

$$= \frac{E^2}{c^2} - \bar{p}^2 = m_0^2 c^2$$

$$= \frac{1}{4 m_0^2 c^2} \left\{ (\lambda_r \lambda_{r'} + 1)\, m_0^2 c^2 + (\lambda_r + \lambda_{r'})\, m_0 c \not{p} \right\}$$

$$= \frac{(1 + \lambda_r \lambda_{r'})}{2}\, \frac{(\lambda_r \not{p} + m_0 c)}{2 m_0 c}$$

$$= \left(\frac{1 + \lambda_r \lambda_{r'}}{2} \right) \Lambda_r(\bar{p})$$

$$\therefore \quad \Lambda_\pm^2(p) = \Lambda_\pm(p)$$

$$\Lambda_+ \Lambda_- = 0$$

$$\Lambda_+ + \Lambda_- = \mathbb{1}$$

SPIN PROJECTORS (FOR SPIN VECTOR s^μ)

$$\boxed{\Sigma(s) \equiv \frac{1 + \gamma_5 \not{s}}{2}}$$

IN REST FRAME $\quad \not{s} = \gamma_\mu s^\mu \quad$ (LORENTZ INV.)

$$= - \bar{\gamma} \cdot \bar{s}$$

* IF SPIN AXIS // Z-AXIS : $\bar{s} = \hat{e}_z$

$$\Sigma(u_z) = \frac{1 + \gamma_5 (-\gamma^3)}{2}$$

$$= \frac{1 + \Sigma^3 \gamma_0}{2} = \frac{1}{2} \left\{ 1\!\!1_{4\times 4} + \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right\}$$

$$\Sigma(u_z) \; u(p, u_z) = u(p, u_z)$$

$$\Sigma(u_z) \; v(p, u_z) = v(p, u_z)$$

BECAUSE

$$\frac{1 + \Sigma^3 \gamma_0}{2} \, w_{1,2}(0) = \frac{1 + \Sigma^3}{2} \, w_{1,2}(0) = \begin{cases} 1 \; , & w_1 \\ 0 \; , & w_2 \end{cases}$$

$$\frac{1 + \Sigma^3 \gamma_0}{2} \, w_{3,4}(0) = \frac{1 - \Sigma^3}{2} \, w_{3,4}(0) = \begin{cases} 0 \; , & w_3 \\ 1 \; , & w_4 \end{cases}$$

$$\Sigma(-u_z)\ u(p, u_z) = 0$$

$$\Sigma(-u_z)\ v(p, u_z) = 0$$

BECAUSE $\quad \dfrac{1 - \Sigma^3 \gamma_0}{2}\ w_{3,4}(0) = \dfrac{1 + \Sigma^3}{2}\ w_{3,4}(0)$

$$= \begin{cases} 1 & , w_3 \\ 0 & , w_4 \end{cases}$$

* FOR ARBITRARY SPIN AXIS

$$\Sigma(s) = \frac{1 + \gamma_5 \not{s}}{2}$$

$$\Sigma(s)\ u(p, s) = u(p, s)$$

$$\Sigma(s)\ v(p, s) = v(p, s)$$

$$\Sigma(-s)\ u(p, s) = \Sigma(-s)\ v(p, s) = 0.$$

↳ SIMULTANEOUS ENERGY & SPIN PROJECTORS

$$\mathcal{P}_1(\bar{p}) = \Lambda_+(\bar{p})\ \Sigma(u_z)$$

$$\mathcal{P}_2(\bar{p}) = \Lambda_+(\bar{p})\ \Sigma(-u_z)$$

$$\mathcal{P}_3(\bar{p}) = \Lambda_-(\bar{p})\ \Sigma(-u_z)$$

$$\mathcal{P}_4(\bar{p}) = \Lambda_-(\bar{p})\ \Sigma(+u_z)$$

3) DIRAC PARTICLE IN A CENTRAL POTENTIAL

- $$H_D = \bar{\alpha}.\hat{p}\, c + \beta m_0 c^2 + V(r)$$

$$H_D \psi = E \psi$$

$$\bar{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

↳ $\bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$, $\bar{S} = \frac{\hbar}{2} \bar{\Sigma}$ SPIN

↳ ORBITAL ANGULAR MOMENTUM $\bar{L} = \bar{r} \times \hat{p}$

↳ $$[H_D, S_i] = \frac{\hbar}{2} [\bar{\alpha}.\hat{p}\, c, \Sigma_i]$$

$$= \frac{\hbar}{2} c \left\{ \begin{pmatrix} 0 & \bar{\sigma}.\hat{p} \\ \bar{\sigma}.\hat{p} & 0 \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} - \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}.\hat{p} \\ \bar{\sigma}.\hat{p} & 0 \end{pmatrix} \right\}$$

$$= \frac{\hbar}{2} c \begin{pmatrix} 0 & [\bar{\sigma}.\hat{p}, \sigma_i] \\ [\bar{\sigma}.\hat{p}, \sigma_i] & 0 \end{pmatrix}$$

$$= i\hbar c\, \varepsilon_{jik}\, \hat{p}_j\, \alpha_k$$

$$= i\hbar c\, \varepsilon_{ijk}\, \hat{p}_k\, \alpha_j \neq 0$$

$\bar{S}$ DOES NOT COMMUTE WITH H_D

↳ $$[H_D, L_i] = \varepsilon_{ijk}\, [H_D, r_j \hat{p}_k]$$

$$= \varepsilon_{ijk}\, [c\, \alpha_\ell \hat{p}_\ell , r_j \hat{p}_k]$$

$$= \varepsilon_{ijk}\, c\, \alpha_\ell \underbrace{[\hat{p}_\ell , r_j]}_{-i\hbar\, \delta_{\ell j}} \hat{p}_k$$

$$= -i\hbar c\, \varepsilon_{ijk}\, \alpha_j \hat{p}_k \neq 0$$

↳ TOTAL ANGULAR MOMENTUM $\bar{J} \equiv \bar{L} + \bar{S}$

$$[H_D, J_i] = [H_D, S_i] + [H_D, L_i]$$

$$= 0$$

⇓

TOTAL ANGULAR MOMENTUM IS CONSERVED BY DIRAC EQ.

$$j = \ell \pm \frac{1}{2}$$

FOR GIVEN VALUE OF j, WE NEED ANOTHER CONSERVED QUANTITY TO DISTINGUISH BOTH CASES

↳ TRY $\underline{\underline{K \equiv \beta \left(\bar{\Sigma} . \bar{J} - \frac{\hbar}{2} \right)}}$

SHOW THAT $[H_D, K] = 0$

PROOF : $[H_D, \beta] = [\bar{\alpha}.\hat{\bar{p}}\, c, \beta]$ $\quad \alpha_i \beta = -\beta \alpha_i$

$$= -2c\, \bar{\gamma}.\hat{\bar{p}}$$

$$[H_D, \Sigma_i] = 2i\, c\, \varepsilon_{ijk}\, \alpha_j\, \hat{p}_k$$

$$[H_D, K] = -2c\, \bar{\gamma}.\hat{\bar{p}} \left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2} \right)$$

$$+ \beta \underbrace{[H_D, \Sigma_i]}_{2ic\, \varepsilon_{ijk}\, \alpha_j\, \hat{p}_k} J_i$$

$$= -2c\, \hat{p}_i \underbrace{\gamma_i\, \Sigma_j}\, J_j + \hbar c\, \bar{\gamma}.\hat{\bar{p}}$$

$$+ 2ic\, \varepsilon_{ijk}\, \gamma_j\, \hat{p}_k\, J_i$$

$$\gamma_i \Sigma_j = \begin{pmatrix} 0 & \sigma_i \sigma_j \\ -\sigma_i \sigma_j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \delta_{ij} + i\, \varepsilon_{ijk} \sigma_k \\ -\delta_{ij} - i\, \varepsilon_{ijk} \sigma_k & 0 \end{pmatrix}$$

$$= -2c \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \hat{\bar{p}}.\bar{J}$$

$$- 2ci\, \varepsilon_{ijk}\, \gamma_k\, \hat{p}_i\, J_j$$

$$+ \hbar c\, \bar{\gamma}.\hat{\bar{p}} + 2ic\, \varepsilon_{ijk}\, \gamma_j\, \hat{p}_k\, J_i$$

$$[H_D, K] = -2c \quad \beta\gamma_5 \; \hat{\bar{p}}.\bar{J}$$
$$- 2ci \; \cancel{\varepsilon_{ijk}\gamma_k \hat{P}_i} \; J_j$$
$$+ \hbar c \; \bar{\gamma}.\hat{\bar{p}}$$
$$+ 2ci \; \cancel{\varepsilon_{ijk} \; \gamma_k \; \hat{P}_i} \; J_j$$
$$= -2c \; \beta\gamma_5 \; \hat{\bar{p}}.\left(\bar{L} + \frac{\hbar}{2}\bar{\Sigma}\right) + \hbar c\,\bar{\gamma}.\hat{\bar{p}}$$
$$\downarrow \quad \hat{\bar{p}}.\bar{L} = 0$$
$$= -\hbar c \; \beta \; \bar{p}.\begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} + \hbar c \; \bar{\gamma}.\hat{\bar{p}}$$
$$= -\hbar c \; \bar{\gamma}.\hat{\bar{p}} + \hbar c \; \bar{\gamma}.\hat{\bar{p}}$$
$$\overset{!}{=} 0$$

$\hookrightarrow$ $$[K, \bar{J}] = 0$$

PROOF : $$[K, J_i] = \underbrace{[\beta, J_i]}\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right) + \beta\,[\bar{\Sigma}.\bar{J}, J_i]$$
$$= \frac{\hbar}{2}\underbrace{[\beta, \Sigma_i]}_{0}\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right) + \beta\,\Sigma_j \underbrace{[J_j, J_i]}_{i\hbar\,\varepsilon_{jik} J_k}$$
$$+ \beta\,\underbrace{[\Sigma_j, J_i]}_{2i\frac{\hbar}{2}\,\varepsilon_{jik}\Sigma_k} J_j$$
$$\overset{!}{=} 0$$

$\therefore$ H_D, J^2, J_z, K FORM A SET OF MUTUALLY COMMUTING OPERATORS.

- EIGENVALUES OF K

$\hookrightarrow$ $K = \beta\,(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2})$

$= \beta\,(\bar{\Sigma}.\bar{L} + \frac{\hbar}{2}\,\Sigma^2 - \frac{\hbar}{2})$

$\downarrow$ $\Sigma^2 = 3\,\mathbb{1}$

$$\underline{\underline{K = \beta\,(\bar{\Sigma}.\bar{L} + \hbar)}}$$

$K\psi = -\hbar\kappa\,\psi$ $\qquad$ $-\hbar\kappa$ IS EIGENVALUE

$\hookrightarrow$ COMPUTE K^2

$K^2 = \beta\,(\bar{\Sigma}.\bar{L} + \hbar)\,\beta\,(\bar{\Sigma}.\bar{L} + \hbar)$

$\downarrow$ $\Sigma_i\beta = \beta\Sigma_i$

$= (\bar{\Sigma}.\bar{L})^2 + 2\hbar\,(\bar{\Sigma}.\bar{L}) + \hbar^2$

$\downarrow$ $(\bar{\Sigma}.\bar{L})^2 = \Sigma_i\Sigma_j\,L_i L_j$

$= (\delta_{ij} + i\,\varepsilon_{ijk}\,\Sigma_k)\,L_i L_j$

$= L^2 + \frac{i}{2}\,\varepsilon_{ijk}\,\underbrace{[L_i, L_j]}_{i\hbar\varepsilon_{ij\ell}L_\ell}\,\Sigma_k + 4\,(\bar{S}.\bar{L}) + \hbar^2$

$\downarrow$ $\varepsilon_{ijk}\,\varepsilon_{ij\ell} = 2\,\delta_{k\ell}$

$= L^2 - \hbar\,(\bar{\Sigma}.\bar{L}) + 4\,(\bar{S}.\bar{L}) + \hbar^2$

$= L^2 + 2\,(\bar{S}.\bar{L}) + \hbar^2$

$= (\bar{L} + \bar{S})^2 - S^2 + \hbar^2$

$K^2 = J^2 - \underbrace{S^2}_{\frac{3\hbar^2}{4}\mathbb{1}} + \hbar^2 \Rightarrow$ $\qquad$ $\underline{\underline{K^2 = J^2 + \frac{\hbar^2}{4}}}$

EIGENVALUE $\quad \hbar^2 K^2 = \hbar^2 j(j+1) + \frac{\hbar^2}{4}$

$$K^2 = j^2 + j + \frac{1}{4}$$

$$= \left(j + \frac{1}{2}\right)^2$$

$$\underline{\underline{K = \pm \left(j + \frac{1}{2}\right)}}$$

$$\therefore \quad K\psi = \mp \hbar \left(j + \frac{1}{2}\right) \psi$$

$$\beta \begin{pmatrix} \bar{\sigma}.\bar{L} + \hbar & \\ & \bar{\sigma}.\bar{L} + \hbar \end{pmatrix} \psi = \mp \hbar \left(j + \frac{1}{2}\right) \psi$$

$$\Downarrow \qquad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

ψ_A, ψ_B ARE 2×1 SPINORS (PAULI)

$$\begin{pmatrix} \bar{\sigma}.\bar{L} + \hbar & \\ & -\bar{\sigma}.\bar{L} - \hbar \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \mp \hbar \left(j + \frac{1}{2}\right) \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\begin{pmatrix} \bar{\sigma}.\bar{L} & 0 \\ 0 & -\bar{\sigma}.\bar{L} \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar \begin{pmatrix} \mp \left(j + \frac{1}{2} \pm 1\right) \psi_A \\ \mp \left(j + \frac{1}{2} \mp 1\right) \psi_B \end{pmatrix}$$

ψ_A, ψ_B ARE EIGENSTATES OF J^2 AND $\bar{\sigma}.\bar{L}$

NOTE $\quad \bar{J} = \bar{L} + \frac{\hbar}{2} \bar{\sigma}$

$$\bar{J}^2 = \bar{L}^2 + \hbar \bar{\sigma}.\bar{L} + \frac{3}{4}\hbar^2$$

$$\hookrightarrow \quad L^2 = J^2 - \hbar\, \bar{\sigma}\cdot\bar{L} - \frac{3}{4}\hbar^2$$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 j(j+1) \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \frac{3}{4}\hbar^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$- \hbar^2 \begin{pmatrix} \mp\left(j + \frac{1}{2} \pm 1\right) \psi_A \\ \pm\left(j + \frac{1}{2} \mp 1\right) \psi_B \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} \left(j^2 + j \pm j \pm \frac{1}{2} + \frac{1}{4}\right) \psi_A \\ \left(j^2 + j \mp j \mp \frac{1}{2} + \frac{1}{4}\right) \psi_B \end{pmatrix}$$

$\ell = j \pm \frac{1}{2}$ ORBITAL ANGULAR MOMENTUM

DENOTE $\ell_\pm = j \pm \frac{1}{2}$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} \ell_\pm(\ell_\pm + 1)\, \psi_A \\ \ell_\mp(\ell_\mp \, 1)\, \psi_B \end{pmatrix}$$

$$\text{FOR} \begin{cases} K > 0 \Rightarrow \ell_A = \ell_+ \,, \quad \ell_B = \ell_- \\ K < 0 \Rightarrow \ell_A = \ell_- \,, \quad \ell_B = \ell_+ \end{cases}$$

SIGN OF K INDICATES WHETHER $\bar{L}$ AND $\bar{S}$ ARE PARALLEL OR ANTI-PARALLEL.

- SOLUTIONS OF DIRAC EQ. IN CENTRAL POTENTIAL

$\hookrightarrow \quad H_D \psi = E \psi$

$$\begin{pmatrix} 0 & c\,\bar{\sigma}\cdot\hat{\bar{p}} \\ c\,\bar{\sigma}\cdot\hat{\bar{p}} & 0 \end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = \begin{pmatrix} E - m_0c^2 - V & \\ & E + m_0c^2 - V\end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$$

$$\boxed{c\,\bar{\sigma}\cdot\hat{\bar{p}}\begin{pmatrix}\psi_B \\ \psi_A\end{pmatrix} = \begin{pmatrix} E - V - m_0c^2 & \\ & E - V + m_0c^2\end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}}$$

$$\hookrightarrow \quad \bar{\sigma}\cdot\hat{\bar{p}} = \underbrace{(\bar{\sigma}\cdot\bar{e}_r)(\bar{\sigma}\cdot\bar{e}_r)}_{1}\,(\bar{\sigma}\cdot\hat{\bar{p}})$$

$$(\bar{\sigma}\cdot\bar{e}_r)(\bar{\sigma}\cdot\hat{\bar{p}}) = \bar{e}_r\cdot\hat{\bar{p}} + i\,\bar{\sigma}\cdot(\bar{e}_r\times\hat{\bar{p}})$$

$$= (\bar{\sigma}\cdot\bar{e}_r)\left\{\bar{e}_r\cdot\hat{\bar{p}} + i\,\bar{\sigma}\cdot(\bar{e}_r\times\hat{\bar{p}})\right\}$$

$$\bar{e}_r\cdot\hat{\bar{p}} = -i\hbar\,\bar{e}_r\cdot\bar{\nabla} = -i\hbar\frac{\partial}{\partial r}$$

$$\bar{\sigma}\cdot\hat{\bar{p}} = \frac{1}{r}\,\frac{\bar{\sigma}\cdot\bar{r}}{r}\left\{-i\hbar\, r\frac{\partial}{\partial r} + i\,\bar{\sigma}\cdot\bar{L}\right\}$$

$\bar{\sigma}\cdot\bar{e}_r$ & $\bar{\sigma}\cdot\bar{L}$ ACT ONLY ON ANGULAR PARTS

↳ DENOTE

$$\psi_A \equiv g(r)\, \mathcal{Y}_{j l_A}^{m}$$

$$\psi_B \equiv i f(r)\, \mathcal{Y}_{j l_B}^{m}$$

$\Uparrow$

ANGULAR MOMENTUM EIGENSTATES

$$[\bar\sigma.\bar e_r \;, \bar J] = \frac{1}{r}\,[\bar\sigma.\bar r\,, \bar J]$$

$$= \frac{1}{r}\,[\bar\sigma.\bar r\,, \;\bar r \times \bar p + \frac{\hbar}{2}\bar\sigma]$$

$$[\bar\sigma.\bar e_r, J^i] = \frac{1}{r}\,\varepsilon_{ijk}\, r_j\, [\sigma_l r_l, p_k]$$

$$+ \frac{\hbar}{2r}\, r_l\, [\sigma_l, \sigma_i]$$

$$= \frac{1}{r}\, i\hbar \left(\varepsilon_{ijk}\, r_j\, \sigma_k + \varepsilon_{lik} r_l \sigma_k \right)$$

$$= 0$$

$(\bar\sigma.\bar e_r)\,\mathcal{Y}_{jl}^{m}$ ALSO EIGENSTATE OF J^2, J_z WITH j, m

↳ $(\bar\sigma.\bar e_r)$ IS PSEUDOSCALAR

CHANGES UNDER SPACE INVERSION $\mathcal{P}(\bar\sigma.\bar e_r)\mathcal{P} = -\bar\sigma.\bar e_r$

CHANGE OF PARITY OF STATE $(-1)^l \iff l$ CHANGES

$$(\bar\sigma.\bar e_r)\,\mathcal{Y}_{j l_A}^{m} = c\, \mathcal{Y}_{j l_B}^{m}$$

$$(\bar{\sigma}\cdot\bar{e}_r)^2 = 1 \implies C = \pm 1$$

OUR CONVENTIONS $\quad (\bar{\sigma}\cdot\bar{e}_r)\, \mathcal{Y}^{m}_{j\ell_A} = -\mathcal{Y}^{m}_{j\ell_B}$

$$K\psi = -\hbar K \psi$$

$$\Downarrow$$

$$\begin{cases} (\bar{\sigma}\cdot\bar{L})\,\psi_A = -\hbar(K+1)\,\psi_A \\ (\bar{\sigma}\cdot\bar{L})\,\psi_B = \hbar(K-1)\,\psi_B \end{cases}$$

$$\begin{cases} \bar{\sigma}\cdot\hat{\bar{p}}\;\psi_A = \frac{1}{r}(\bar{\sigma}\cdot\bar{e}_r)\left\{-i\hbar r\frac{\partial}{\partial r} - i\hbar(K+1)\right\}\psi_A \\ \bar{\sigma}\cdot\hat{\bar{p}}\;\psi_B = \frac{1}{r}(\bar{\sigma}\cdot\bar{e}_r)\left\{-i\hbar r\frac{\partial}{\partial r} + i\hbar(K-1)\right\}\psi_B \end{cases}$$

$$\therefore \begin{cases} (E - V - m_0c^2)\,\psi_A = \frac{c}{r}(\bar{\sigma}\cdot\bar{e}_r)\left\{-i\hbar r\frac{\partial}{\partial r} + i\hbar(K-1)\right\}\psi_B \\ (E - V + m_0c^2)\,\psi_B = \frac{c}{r}(\bar{\sigma}\cdot\bar{e}_r)\left\{-i\hbar r\frac{\partial}{\partial r} - i\hbar(K+1)\right\}\psi_A \end{cases}$$

$$\rightsquigarrow \left(E - V - m_0 c^2\right) g(r) \mathcal{Y}_{j l_A}^{m}$$

$$= \frac{c}{r} \left\{ -i\hbar r \frac{\partial}{\partial r} + i\hbar (K-1) \right\} i f(r) \underbrace{(\bar{\sigma} . \bar{e}_r) \mathcal{Y}_{j l_B}^{m}}_{- \mathcal{Y}_{j l_A}^{m}}$$

$$\rightsquigarrow (E - V + m_0 c^2) i f(r) \mathcal{Y}_{j l_B}^{m}$$

$$= \frac{c}{r} \left\{ -i\hbar r \frac{\partial}{\partial r} - i\hbar (K+1) \right\} g(r) \underbrace{(\bar{\sigma} . \bar{e}_r) \mathcal{Y}_{j l_A}^{m}}_{- \mathcal{Y}_{j l_B}^{m}}$$

$\therefore$ RADIAL EQUATIONS

$$\boxed{\begin{cases} \hbar c \left(- \dfrac{\partial f}{\partial r} + \dfrac{K-1}{r} f \right) = (E - V - m_0 c^2) g \\ \hbar c \left(\dfrac{\partial g}{\partial r} + \dfrac{K+1}{r} g \right) = (E - V + m_0 c^2) f \end{cases}}$$

SIMPLIFY USING $\quad F \equiv r f \quad , \quad G \equiv r g$

$$\begin{cases} \dfrac{\partial F}{\partial r} - K \dfrac{F}{r} = \dfrac{m_0 c^2 - E + V}{\hbar c} G \\ \dfrac{\partial G}{\partial r} + K \dfrac{G}{r} = \dfrac{m_0 c^2 + E - V}{\hbar c} F \end{cases}$$

4) QUANTIZATION OF DIRAC FIELD

- DIRAC LAGRANGIAN

↳ DIRAC EQ. $\underline{\underline{(i\hbar\, \gamma^\mu \partial_\mu - m_0 c)\psi = 0}}$

WITH $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

NOTE $\begin{cases} \gamma^{0\dagger} = \gamma^0 \\ \gamma^{i\dagger} = -\gamma^i \end{cases}$ $\quad \gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$ $\quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

↳ ADJOINT FIELD

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\,\gamma^0$$

TAKE † OF DIRAC EQ. :

$$-i\hbar\left(\partial_\mu \psi^\dagger\right)\gamma^{\mu\dagger} - \psi^\dagger m_0 c = 0$$

$$\Updownarrow$$

$$i\hbar\,(\partial_\mu\psi^\dagger)\,\gamma^0\gamma^\mu\gamma^0 + \psi^\dagger m_0 c = 0$$

$\Downarrow$ MULTIPLY BY γ^0 ON RIGHT

$$\underline{\underline{i\hbar\left(\partial_\mu\bar{\psi}\right)\gamma^\mu + \bar{\psi}\, m_0 c = 0}}$$

↳ LAGRANGIAN

TREAT ψ & $\bar{\psi}$ AS INDEPENDENT FIELDS (COMPLEX VALUED)

DIRAC EQS. FOR ψ & $\bar{\psi}$ CAN BE DERIVED FROM LAGRANGIAN

$$\boxed{\mathcal{L} = c\,\bar{\psi}\left[i\hbar\gamma^\mu\partial_\mu - m_0 c\right]\psi}$$

⇝ EULER-LAGRANGE EQ. FOR $\bar{\psi}$:

$$\frac{\partial\mathcal{L}}{\partial\psi} = c\,\bar{\psi}\,(-m_0 c)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = c\,\bar{\psi}\,(i\hbar\gamma^\mu)$$

$\Downarrow$

EL. EQ. $\quad \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = 0$

$\Downarrow$

$$c\,\bar{\psi}(-m_0 c) - c\,(\partial_\mu\bar{\psi})\,i\hbar\gamma^\mu = 0$$

$$\therefore \quad i\hbar\,(\partial_\mu\bar{\psi})\,\gamma^\mu + \bar{\psi}m_0 c = 0 \quad ✓$$

⇝ EULER-LAGRANGE EQ. FOR ψ:

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = c\,(i\hbar\gamma^\mu\partial_\mu - m_0 c)\,\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \qquad \therefore \quad i\hbar\gamma^\mu\partial_\mu\psi - m_0 c\,\psi = 0 \quad ✓$$

CONJUGATE MOMENTA:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\hbar \bar{\psi}\gamma^0 = i\hbar \psi^+$$

$$\bar{\pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

HAMILTONIAN

$$H = \int d^3\bar{x} \left(\pi \dot{\psi} + \dot{\bar{\psi}} \underbrace{\cancel{\bar{\pi}}}_{0} - \mathcal{L} \right)$$

$$= \int d^3\bar{x} \left(i\hbar \psi^+ \frac{\partial \psi}{\partial t} - c\, i\hbar \bar{\psi}\gamma^\mu \partial_\mu \psi + m_0 c^2 \bar{\psi}\psi \right)$$

$$= c \int d^3\bar{x}\ \bar{\psi} \left(i\hbar \gamma^0 \partial_0 - i\hbar \gamma^\mu \partial_\mu + m_0 c \right) \psi$$

$$= c \int d^3\bar{x}\ \bar{\psi} \left(-i\hbar \gamma^i \frac{\partial}{\partial x^i} + m_0 c \right) \psi$$

MOMENTUM

4 MOMENTUM $\quad cP^\nu \equiv \int d^3\bar{x} \left\{ c\, \pi\, \partial^\nu \psi - \mathcal{L} g^{0\nu} \right\}$

$\underline{\nu = 0}$ $\quad cP^i = c \int d^3\bar{x}\ \pi (\partial^i \psi)$

$$P^i = \int d^3\bar{x}\ \psi^+ (-i\hbar \nabla^i) \psi$$

NOTE $\quad \partial^i = \frac{\partial}{\partial x_i} = -\nabla^i = -\frac{\partial}{\partial x^i}$

↳ CHARGE

⇝ CONSIDER GLOBAL PHASE TRANSFORMATION

$$\psi \to e^{i\alpha}\psi$$

$$\bar{\psi} \to \bar{\psi} e^{-i\alpha}$$

α CONSTANT REAL NUMBER

⇝ FOR α INFINITESIMAL

$$\psi \to \psi + \underbrace{i(\delta\alpha)\psi}_{\delta\psi}$$

$$\bar{\psi} \to \bar{\psi} \underbrace{- i(\delta\alpha)\bar{\psi}}_{\delta\bar{\psi}}$$

⇝ $$\mathcal{L} = c\,\bar{\psi}\left[i\gamma^\mu \partial_\mu - m_0 c\right]\psi$$

$\mathcal{L}$ IS INVARIANT UNDER GLOBAL PHASE TF.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi)$$

$$+ \delta\bar{\psi}\underbrace{\cancel{\frac{\partial\mathcal{L}}{\partial\bar{\psi}}}}_{0}$$ (DIRAC EQ.)

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi\right)$$

$$= 0$$

$$\Updownarrow$$

CONSERVED CURRENT :

$$\partial_\mu \left(\bar{\psi} (i\hbar \gamma^\mu) \cdot (i(\delta\alpha)\psi) \right) = 0$$

$$\partial_\mu J^\mu = 0$$

$$J^\mu = \bar{\psi} \gamma^\mu \psi$$

$\rightsquigarrow$ CONSERVED CHARGE (q : ELECTRIC CHARGE OF FERMION)
SPIN 1/2

$$Q = q \int d^3\bar{x} \quad J^0(x)$$

$$= q \int d^3\bar{x} \quad \psi^+(x) \; \psi(x)$$

• SECOND QUANTIZATION OF DIRAC FIELD

↳ TREAT $\psi(x)$ AND $\bar{\psi}(x)$ AS FIELD OPERATORS WHICH CAN CREATE OR ANNIHILATE A DIRAC PARTICLE AT POSITION x .

⇝ WE LIKE TO MAKE A NORMAL MODE EXPANSION OF ψ, $\bar{\psi}$

EACH MODE CORRESPONDS WITH PARTICLE WITH MOMENTUM $\bar{p}$ & SPIN PROJECTION ALONG $\pm s$ (IN REST FRAME)

$$U(p,s) = A \begin{pmatrix} \chi_s \\ c\dfrac{\bar{\sigma}.\bar{p}}{E_p + m_0c^2}\chi_s \end{pmatrix} \rightsquigarrow (\not{p} - m_0 c)\, U(p,s) = 0$$

$$v(p,s) = A \begin{pmatrix} c\dfrac{\bar{\sigma}.\bar{p}}{E_p + m_0c^2}\chi_s' \\ \chi_s' \end{pmatrix} \rightsquigarrow (\not{p} + m_0 c)\, v(p,s) = 0$$

⇝ WE WILL SIMPLIFY NOTATION AND DENOTE SECOND ARGUMENT AS VALUE $(\pm \frac{1}{2})$ OF SPIN PROJECTION (s_z) ALONG AXIS z , i.e. $s_z = \pm \frac{1}{2}$ (IN UNITS $\hbar$)

$$U(p, s_z) \rightsquigarrow \chi_{s_z = +\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \chi_{s_z = -\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v(p, s_z) \rightsquigarrow \chi'_{s_z=+\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \chi'_{s_z=-\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

NOTE : OPPOSITE SPIN LABELING FOR v (ANTI-PARTICLE) AS FOR u (PARTICLE)

$\rightsquigarrow$ NORMALIZATION (A) OF SPINORS CHOSEN SUCH THAT

$$\bar{u}(p, s_z)\, u(p, s_z') = \delta_{s_z s_z'}$$

$$\bar{v}(p, s_z)\, v(p, s_z') = -\delta_{s_z s_z'}$$

CHECK : $A^2 \left(\chi^+_{s_z} \quad - \chi^+_{s_z} \dfrac{c\,\bar{\sigma}.\bar{p}}{E_p + m_0 c^2} \right) \begin{pmatrix} \chi_{s_z'} \\ \dfrac{c\,\bar{\sigma}.\bar{p}}{E_p + m_0 c^2}\chi_{s_z'} \end{pmatrix}$

$$= A^2 \chi^+_{s_z} \left(1 - \frac{c^2 (\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p})}{(E_p + m_0 c^2)^2} \right) \chi_{s_z'}$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \left(1 - \frac{c^2 \bar{p}^2}{(E_p + m_0 c^2)^2} \right)$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \left(1 - \frac{E_p^2 - m_0^2 c^4}{(E_p + m_0 c^2)^2} \right)$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \left(1 - \frac{E_p - m_0 c^2}{E_p + m_0 c^2} \right)$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \frac{2 m_0 c^2}{E_p + m_0 c^2}$$

$$\Downarrow$$

USING $\chi^{+}_{s_z} \chi_{s_z'} = \delta_{s_z s_z'}$

$$\bar{u}(p, s_z)\, u(p, s_z') = \delta_{s_z s_z'} \, A^2 \, \frac{2 m_0 c^2}{E_p + m_0 c^2}$$

$$= \delta_{s_z s_z'}$$

$$\Updownarrow$$

$$\boxed{A = \sqrt{\frac{E_p + m_0 c^2}{2 m_0 c^2}}}$$

⤳ CHECK THAT WITH THE ABOVE NORMALIZATION

$$u^{+}(p, s_z)\, u(p, s_z') = \frac{E_p}{m_0 c^2} \delta_{s_z s_z'}$$

$$v^{+}(p, s_z)\, v(p, s_z') = \frac{E_p}{m_0 c^2} \delta_{s_z s_z'}$$

⤳ NORMAL MODES ⤳ FREE PROPAGATING DIRAC PARTICLE

PLANE WAVE (IN VOLUME V) $\dfrac{e^{-\frac{i}{\hbar} p \cdot x}}{\sqrt{V}}$

NORMAL MODES $\psi^{+}(x) \sim \dfrac{e^{-\frac{i}{\hbar} p \cdot x}}{\sqrt{V}} \, u(p, s_z)$ POS. ENERGY SOLUTION

OR

$\psi^{-}(x) \sim \dfrac{e^{+\frac{i}{\hbar} p \cdot x}}{\sqrt{V}} \, v(p, s_z)$ NEG. ENERGY SOLUTION

$$\left(i\hbar\,\gamma^\mu \partial_\mu - m_0 c\right)\psi^+ = 0 \iff (\not{p} - m_0 c)\,u = 0$$

$$\left(i\hbar\,\gamma^\mu \partial_\mu - m_0 c\right)\psi^- = 0 \iff (\not{p} + m_0 c)\,v = 0$$

4. EXPANSIONS OF DIRAC FIELDS ψ & $\bar{\psi}$

$$\psi(x) = \sum_{\bar{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V}\right)^{1/2} \Big\{ b(\bar{p}, s_z)\, u(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} p \cdot x} + d^+(\bar{p}, s_z)\, v(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} p \cdot x} \Big\}$$

$$\bar{\psi}(x) = \sum_{\bar{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V}\right)^{1/2} \Big\{ b^+(\bar{p}, s_z)\, \bar{u}(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} p \cdot x} + d(\bar{p}, s_z)\, \bar{v}(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} p \cdot x} \Big\}$$

$b(\bar{p}, s_z)$ & $d^+(\bar{p}, s_z)$ ARE EXPANSION COEFFICIENTS WHICH WILL BECOME OPERATORS UPON SECOND QUANTIZATION

<u>NOTE</u> : NORMALIZATION FACTOR $\left(\dfrac{m_0 c^2}{E_p}\right)^{1/2}$

IS INTRODUCED TO GET SIMPLE ANTI-COMMUTATORS FOR b, d AFTER SECOND QUANTIZATION

SECOND QUANTIZATION

IMPOSE ANTI-COMMUTATION RELATIONS FOR EXPANSION COEFFICIENTS b & d

$$\{ b(\bar{p}, s_z), b^+(\bar{p}', s_z') \} = \delta_{\bar{p}\bar{p}'} \delta_{s_z s_z'}$$

$$\{ d(\bar{p}, s_z), d^+(\bar{p}', s_z') \} = \delta_{\bar{p}\bar{p}'} \delta_{s_z s_z'}$$

$$\{ b(\bar{p}, s_z), b(\bar{p}', s_z') \} = 0$$

$$\{ d(\bar{p}, s_z), d(\bar{p}', s_z') \} = 0$$

$$\{ b(\bar{p}, s_z), d(\bar{p}', s') \} = \{ b(\bar{p}, s_z), d^+(\bar{p}', s_z') \} = 0$$

$$\{ d(\bar{p}, s_z), b(\bar{p}', s_z') \} = \{ d(\bar{p}, s_z), b^+(\bar{p}', s_z') \} = 0$$

$\left\{ \begin{matrix} b(\bar{p}, s_z) \\ b^+(\bar{p}, s_z) \end{matrix} \right\}$ IS INTERPRETED AS $\left\{ \begin{matrix} \text{ANNIHILATION} \\ \text{CREATION} \end{matrix} \right\}$ OPERATOR

OF DIRAC <u>PARTICLE</u> WITH MOMENTUM $\bar{p}$ & SPIN PROJ. s_z

$\left\{ \begin{matrix} d(\bar{p}, s_z) \\ d^+(\bar{p}, s_z) \end{matrix} \right\}$ IS INTERPRETED AS $\left\{ \begin{matrix} \text{ANNIHILATION} \\ \text{CREATION} \end{matrix} \right\}$ OPERATOR

OF DIRAC <u>ANTI-PARTICLE</u> WITH MOMENTUM $\bar{p}$ & SPIN PROJ. s_z

$\rightsquigarrow$ VACUUM $|0\rangle$

$$b(\bar{p}, s_z)\,|0\rangle = 0$$

$$d(\bar{p}, s_z)\,|0\rangle = 0$$

$\rightsquigarrow$ NUMBER OPERATORS

$b^+(\bar{p}, s_z)\, b(\bar{p}, s_z)$: # PARTICLES WITH $\bar{p}, s_z$

$d^+(\bar{p}, s_z)\, d(\bar{p}, s_z)$: # ANTI-PARTICLES WITH $\bar{p}, s_z$

$\rightsquigarrow$ ANTI-COMMUTATION RELATION FOR FIELD OPERATORS

FROM ANTI-COMMUTATION RELATIONS FOR b, d

$$\left\{ \psi_\alpha(\bar{x}, t),\ \psi^+_\beta(\bar{x}', t) \right\}$$

AT EQUAL TIME t !

$$\left\{ \begin{array}{l} x^\mu = (ct, \bar{x}) \\ x'^\mu = (ct, \bar{x}') \end{array} \right.$$

$$= \sum_{\bar{p} s_z} \sum_{\bar{p}' s'_z} \frac{m_0 c^2}{V\,(E_p E_{p'})^{1/2}}$$

$$\cdot \left\{ b(\bar{p}, s_z)\, u_\alpha(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} p\cdot x} + d^+(\bar{p}, s_z)\, v_\alpha(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} p\cdot x}\right.,$$

$$\left. b^+(\bar{p}', s'_z)\, u^+_\beta(\bar{p}', s'_z)\, e^{+\frac{i}{\hbar} p'\cdot x'} + d(\bar{p}', s'_z)\, v^+_\beta(\bar{p}', s'_z)\, e^{-\frac{i}{\hbar} p'\cdot x'} \right\}$$

$$= \sum_{\bar{p}} \sum_{s_z} \frac{m_0 c^2}{E_p V} \left(u_\alpha(\bar{p}, s_z)\, u^+_\beta(\bar{p}, s_z)\ e^{+\frac{i}{\hbar} \bar{p}\cdot(\bar{x} - \bar{x}')} \right.$$

$$\left. + v_\alpha(\bar{p}, s_z)\, v^+_\beta(\bar{p}, s_z)\ e^{-\frac{i}{\hbar} \bar{p}\cdot(\bar{x} - \bar{x}')} \right)$$

$$\left\{\begin{array}{l} u^{+} = \bar{u}\,\gamma^{0} \\ v^{+} = \bar{v}\,\gamma^{0} \end{array}\right.$$

$$\sum_{s_z} u(\bar{p}, s_z)\, u^{+}(\bar{p}, s_z) = \underbrace{\sum_{s_z} u(\bar{p}, s_z)\, \bar{u}(\bar{p}, s_z)}_{\frac{(\not{p} + m_0 c)}{2 m_0 c}}\, \gamma^{0}$$

CHECK !

$$\sum_{s_z} v(\bar{p}, s_z)\, v^{+}(\bar{p}, s_z) = \underbrace{\sum_{s_z} v(\bar{p}, s_z)\, \bar{v}(\bar{p}, s_z)}_{-\frac{(-\not{p} + m_0 c)}{2 m_0 c}}\, \gamma^{0}$$

CHECK !

$$\therefore \quad \left\{ \psi_{\alpha}(\bar{x}, t) \,,\, \psi^{+}_{\beta}(\bar{x}', t) \right\}$$

$$= \sum_{\bar{p}} \frac{m_0 c^2}{E_p V} \left(\left[\frac{(\not{p} + m_0 c)}{2 m_0 c} \gamma^{0} \right]_{\alpha\beta} e^{\frac{i}{\hbar} \bar{p} \cdot (\bar{x} - \bar{x}')} + \left[\frac{(\not{p} - m_0 c)}{2 m_0 c} \gamma^{0} \right]_{\alpha\beta} e^{-\frac{i}{\hbar} \bar{p} \cdot (\bar{x} - \bar{x}')} \right)$$

$$\downarrow \qquad \sum_{\bar{p}} e^{\frac{i}{\hbar} \bar{p} \cdot (\bar{x} - \bar{x}')} = \frac{V}{(2\pi)^3} \int d^3\bar{p}\; e^{\frac{i}{\hbar} \bar{p} \cdot (\bar{x} - \bar{x}')}$$

$$= V\, \delta^{3}(\bar{x} - \bar{x}')$$

ONLY p^0 TERM SURVIVES $\qquad p^0 = E_p / c$

$$= \frac{m_0 c^2}{E_p}\, \delta^{3}(\bar{x} - \bar{x}')\, \frac{2 E_p / c}{2 m_0 c}\, \delta_{\alpha\beta}$$

$$\therefore \quad \left\{ \psi_{\alpha}(\bar{x}, t) \,,\, \psi^{+}_{\beta}(\bar{x}', t) \right\} = \delta_{\alpha\beta}\, \delta^{3}(\bar{x} - \bar{x}')$$

ANTI - COMMUTATION RELATIONS FOR FIELDS

ANALOGOUSLY

$$\left\{ \psi_\alpha(\bar{x},t)\,,\ \psi_\beta(\bar{x}',t) \right\} = 0$$

$$\left\{ \psi_\alpha^+(\bar{x},t)\,,\ \psi_\beta^+(\bar{x}',t) \right\} = 0$$