V

ELECTROMAGNETIC RADIATION

1) RADIATION BY A POINT CHARGE

2) RADIATION BY A SYSTEM OF CHARGES

   ⇒ RADIATION FIELDS

   ⇒ HALF - WAVE ANTENNA

   ⇒ MULTIPOLE EXPANSION

   ⇒ ELECTRIC DIPOLE RADIATION

   ⇒ MAGNETIC DIPOLE RADIATION

   ⇒ ELECTRIC QUADRUPOLE RADIATION

1) RADIATION BY A POINT CHARGE

⇒ • POINT CHARE AT REST OR IN UNIFORM MOTION

↓

SURROUNDED BY STATIC OR QUASISTATIC FIELDS

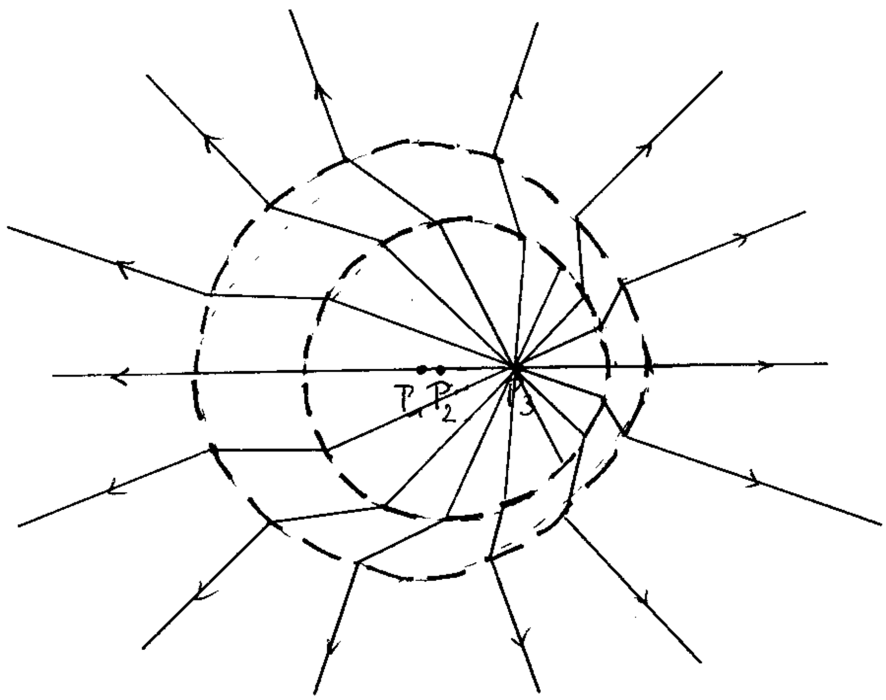
('RIGIDLY' ATTACHED TO CHARGE)

• POINT CHARGE WHICH ACCELERATES

$\bar{E}, \bar{B}$ FIELDS BREAK AWAY FROM POINT CHARGE

↓

PROPAGATE OUTWARD : RADIATION



$\bar{E}$ FIELD LINES

POINT CHARGE

$P_1 \rightarrow P_2$ SUDDEN ACCELERATION

$P_2 \rightarrow P_3$ UNIFORM MOTION

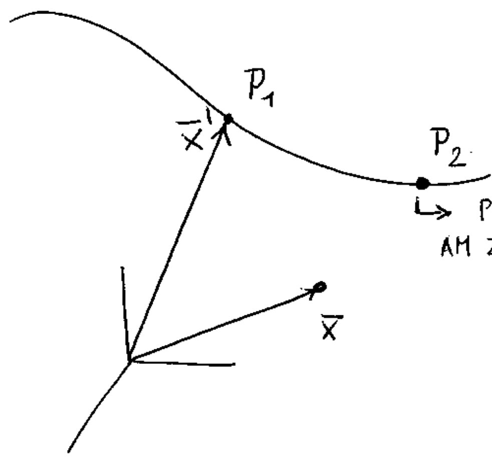
LONGITUDINAL (RADIAL) COMPONENT : COULOMB FIELD

TRANSVERSE COMPONENT : RADIATION FIELD

⇒ RETARDED FOUR - POTENTIAL

$$\partial_\mu \partial^\mu A^\nu = \underbrace{\frac{1}{c} J^\nu}_{\text{FOR GIVEN CHARGE DISTRIBUTION / CURRENT}} \qquad + \qquad \partial_\mu A^\mu = 0$$

LORENTZ GAUGE



↳ POSITION DER PUNKTLADUNG AM ZEITPUNKT t

P_2 : PRESENT POSITION OF POINT CHARGE AT TIME t

P_1 . EARLIER " " " "

↳ AT TIME $\underbrace{t - \frac{|\bar{x} - \bar{x}'|}{c}}_{\text{TIME IT TAKES FOR SIGNAL TO TRAVEL FROM } \bar{x}' \text{ TO } \bar{x}}$: RETARDED TIME

? WHAT IS POTENTIAL AT TIME t AT POSITION $\bar{x}$?

$$\boxed{A^\mu(t, \bar{x}) = \frac{1}{c} \int d^3\bar{x}' \, \frac{J^\mu\left(t - \frac{|\bar{x} - \bar{x}'|}{c}, \bar{x}'\right)}{4\pi \, |\bar{x} - \bar{x}'|}}$$

↑

RETARDED 4 - POTENTIAL

PROOF THAT RETARDED POTENTIAL IS A SOLUTION OF

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) A^\mu = \frac{1}{c} J^\mu$$

⇝ FOR TIME INDEPENDENT / STATIC CASE : OK

$$\nabla^2 A^\mu = - \frac{1}{c} J^\mu$$

$$\nabla^2 \frac{1}{4\pi |\bar{x} - \bar{x}'|} = - \delta^3(\bar{x} - \bar{x}')$$

⇝ FOR TIME DEPENDENT CASE

$$t' = t - \frac{|\bar{x} - \bar{x}'|}{c}$$ t' IS RETARDED TIME

$$\frac{\partial t'}{\partial t} = 1.$$

$$\leadsto \frac{\partial^2 A^\mu}{\partial t^2} = \frac{1}{c} \int d^3\bar{x}' \frac{1}{4\pi |\bar{x} - \bar{x}'|} \frac{\partial^2}{\partial t'^2} J^\mu(t', \bar{x}')$$

$$\leadsto \bar{\nabla} A^\mu = \frac{1}{4\pi c} \int d^3\bar{x}' \left[\frac{1}{|\bar{x} - \bar{x}'|} \cdot \frac{\partial J^\mu}{\partial t'} \cdot \nabla t' + J^\mu \bar{\nabla} \frac{1}{|\bar{x} - \bar{x}'|} \right]$$

$$\leadsto \nabla^2 A^\mu = \frac{1}{4\pi c} \int d^3\bar{x}' \left[\frac{1}{|\bar{x} - \bar{x}'|} \frac{\partial^2 J^\mu}{\partial t'^2} (\bar{\nabla} t')^2 + \frac{1}{|\bar{x} - \bar{x}'|} \frac{\partial J^\mu}{\partial t'} \nabla^2 t' + 2 \frac{\partial J^\mu}{\partial t'} \bar{\nabla} t' . \bar{\nabla} \frac{1}{|\bar{x} - \bar{x}'|} + J^\mu \nabla^2 \frac{1}{|\bar{x} - \bar{x}'|} \right]$$

$$\rightsquigarrow \quad \bar{\nabla} t' = -\frac{1}{c} \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|}$$

$$(\bar{\nabla} t')^2 = \frac{1}{c^2}$$

$$\rightsquigarrow \quad \nabla^2 t' = -\frac{1}{c} (\bar{x}-\bar{x}') \cdot \underbrace{\bar{\nabla} \frac{1}{|\bar{x}-\bar{x}'|}}_{-\frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}} - \frac{1}{c} \frac{1}{|\bar{x}-\bar{x}'|} \underbrace{\bar{\nabla} \cdot (\bar{x}-\bar{x}')}_{3}$$

$$= -\frac{2}{c} \frac{1}{|\bar{x}-\bar{x}'|}$$

$$\rightsquigarrow \quad \nabla^2 \frac{1}{4\pi |\bar{x}-\bar{x}'|} = -\delta^3(\bar{x}-\bar{x}')$$

$$\therefore \quad \left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right] A^\mu$$

$$= \frac{1}{4\pi c} \int d^3\bar{x}' \Big[\cancel{\frac{1}{|\bar{x}-\bar{x}'|} \frac{1}{c^2} \frac{\partial^2 J^\mu}{\partial t'^2}}$$

$$- \cancel{\frac{1}{|\bar{x}-\bar{x}'|} \frac{1}{c^2} \frac{\partial^2 J}{\partial t'^2}}$$

$$+ \frac{1}{|\bar{x}-\bar{x}'|^2} \cancel{\frac{2}{c}} \frac{\partial J^\mu}{\partial t'}$$

$$- 2 \frac{\partial J^\mu}{\partial t'} \frac{1}{c} \cancel{\frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|}} \cdot \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}$$

$$+ 4\pi \delta^3(\bar{x}-\bar{x}') \; J^\mu \Big]$$

$$\stackrel{!}{=} \frac{1}{c} J^\mu$$

⇒ LIÉNARD - WIECHERT POTENTIAL

↳ POTENTIAL FOR MOVING POINT CHARGE

$$A^0(t,\bar{x}) = \int d^3\bar{x}' \, \frac{\rho\left(t' = t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right)}{4\pi \, |\bar{x}-\bar{x}'|}$$

FOR POINT CHARGE $\frac{1}{|\bar{x}-\bar{x}'|}$ CAN BE TAKEN

OUT OF $\int d^3\bar{x}'$

$$A^0(t,\bar{x}) = \frac{1}{|\bar{x}-\bar{x}_p|} \int d^3\bar{x}' \, \rho\left(t' = t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right)$$

↑ POSITION POINT CHARGE AT $t' = t - \frac{|\bar{x}-\bar{x}_p|}{c}$

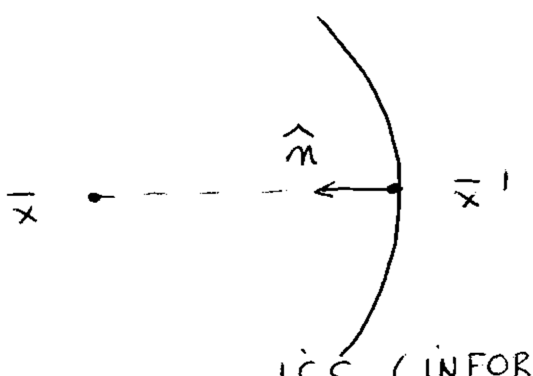
↳ FOR STATIC CASE : POINT CHARGE AT REST

$$\int d^3\bar{x}' \, \rho(\bar{x}') = q$$

↑ TOTAL CHARGE OF POINT CHARGE

↳ FOR MOVING CHARGE WITH VELOCITY $\bar{v}'$: CONSTANT OVER $d^3\bar{x}'$

↑ AT TIME t'



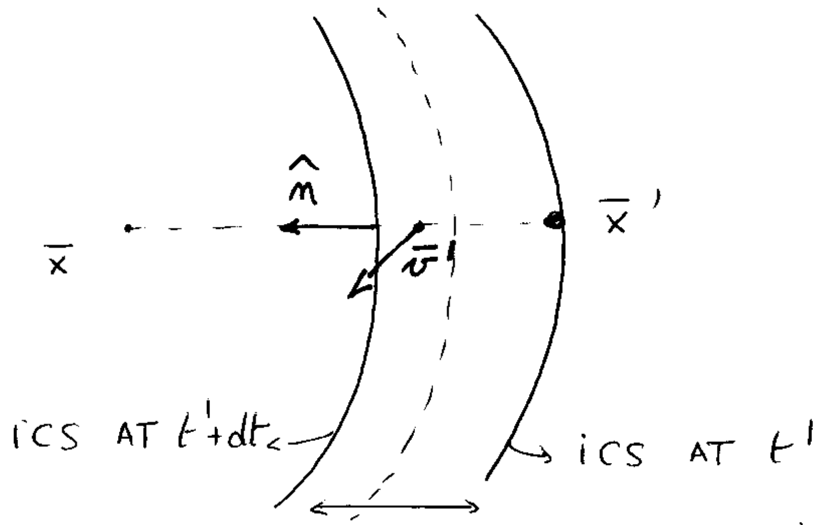
$\hat{n}$: UNIT VECTOR ALONG $\bar{x}-\bar{x}'$

ICS (INFORMATION - COLLECTOR SHELL) SHELL OF POINTS THAT ARE CAUSALY CONNECTED

MOVES ALONG $\hat{m}$ WITH SPEED c

dq : AMOUNT OF CHARGE INTERCEPTED BY ICS IN TIME dt

THIS IS CHARGE WHICH CONTRIBUTES TO POTENTIAL $A^0(t, \bar{x})$



$c\,dt$: DISTANCE ICS MOVED IN TIME dt

$c\,dt\left(\hat{n}\cdot\frac{\bar{v}'}{c}\right)$ FRACTION OF VOL. ELEMENT FROM WHICH CHARGE FLOWS OUT

$$dq \neq \rho\left(t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right) dV'$$

$$dq = \left(1 - \hat{n}\cdot\frac{\bar{v}'}{c}\right) \rho\left(t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right) dV'$$

$$\Downarrow$$

$$\int dV' \, \rho\left(t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right) = \frac{q}{1 - \hat{n}\cdot\frac{\bar{v}'}{c}}$$

NOTE: AT FIXED TIME t_0

$$\int dV' \, \rho(t_0, \bar{x}') = q$$

$$\boxed{A^0(t,\bar{x}) = \frac{q}{4\pi|\bar{x}-\bar{x}'|} \cdot \frac{1}{\left(1-\hat{n}\cdot\frac{\bar{v}'}{c}\right)}}$$

↳ ANALOGOUS FOR $\bar{A}$

$$\bar{A}(t,\bar{x}) = \frac{1}{c}\int d^3\bar{x}' \quad \frac{\bar{J}\left(t-\frac{|\bar{x}-\bar{x}'|}{c},\ \bar{x}'\right)}{4\pi|\bar{x}-\bar{x}'|}$$

FOR POINT PARTICLE

$$\bar{J}(t',\bar{x}') = \rho(t',\bar{x}')\,\bar{v}'$$

$$\boxed{\bar{A}(t,\bar{x}) = \frac{1}{c}\frac{q\bar{v}'}{4\pi|\bar{x}-\bar{x}'|} \cdot \frac{1}{\left(1-\hat{n}\cdot\frac{\bar{v}'}{c}\right)}}$$

↳ ABOVE EXPRESSIONS FOR A^0, $\bar{A}$ ARE CALLED : LIÉNARD - WIECHERT POTENTIAL

NOTE $\bar{A} = \frac{\bar{v}'}{c} A^0$ FOR POINT CHARGE

$\bar{x}'$, $\bar{v}'$ IN LW POTENTIALS ARE EVALUATED AT $t' = t - \frac{|\bar{x}-\bar{x}'|}{c}$

⇒ RADIATION FIELDS OF MOVING POINT CHARGE

↳

$$A^0(t,\bar{x}) = \frac{q}{4\pi|\bar{x}-\bar{x}'|} \cdot \frac{1}{\left(1-\hat{n}\cdot\frac{\bar{v}'}{c}\right)}$$

$$\bar{A}(t,\bar{x}) = \frac{\bar{v}'}{c} A^0(t,\bar{x})$$

* NOTE 1: IN THESE EQS $\left\| \begin{array}{l} \bar{x}' \equiv \bar{x}'(t') \\ t' = t - \frac{r}{c} \end{array}\right.$

$(\bar{r} \equiv \bar{x}-\bar{x}')$

$$\left\| \bar{v}' = \dot{\bar{x}}'(t') \equiv \frac{d\bar{x}'}{dt'}\right.$$

↓

IMPLICIT EQUATION $\underline{\underline{|\bar{x}-\bar{x}'(t')| = c\,(t-t')}}$

* NOTE 2 DIFFERENCE WITH GENERAL FORMULA

$$A^\mu(t,\bar{x}) = \frac{1}{c}\int d^3\bar{x}' \; \frac{J^\mu\left(t-\frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right)}{4\pi|\bar{x}-\bar{x}'|}$$

$\bar{x}'$ is FIXED POSITION COORDINATE IN A TIME-DEPENDENT CURRENT DISTR.

$\bar{x}'(t')$ is TIME-DEPENDENT POSITION (PATH) OF A POINT CHARGE

↳

$$\bar{E} = -\bar{\nabla}A^0 - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}$$

$$\bar{B} = \bar{\nabla}\times\bar{A}$$

DUE TO IMPLICIT EQ. CARE HAS TO BE TAKEN WITH TAKING DERIVATIVES!

↳ $\bar{\nabla} A^{0}$

$$\bar{\nabla} A^{0} = \frac{qc}{4\pi}\,\bar{\nabla}\left(\frac{1}{rc - \bar{r}\cdot\bar{v}'}\right) = -\frac{qc}{4\pi}\,\frac{\bar{\nabla}(rc - \bar{r}\cdot\bar{v}')}{(rc - \bar{r}\cdot\bar{v}')^{2}}$$

* IMPLICIT EQ. $\Rightarrow$ $\boxed{\bar{\nabla} r = -c\,\bar{\nabla} t'}$ (H1)

* $\bar{\nabla}(\bar{r}\cdot\bar{v}') = r_i(\bar{\nabla} v_i') + v_i'(\bar{\nabla} r_i)$

$$= r_i\,\frac{dv_i'}{dt'}\cdot\bar{\nabla} t' + v_i'\,\bar{\nabla}(x_i - x_i')$$

$$\downarrow \quad \bar{a}' \equiv \frac{d\bar{v}'}{dt'}$$

$$= (\bar{r}\cdot\bar{a})\,\bar{\nabla} t' + \bar{v}' - v_i'\,\underbrace{\bar{\nabla} x_i'}_{\frac{dx_i'}{dt'}\,\bar{\nabla} t'}$$

$$\boxed{\bar{\nabla}(\bar{r}\cdot\bar{v}') = \bar{v}' + (\bar{r}\cdot\bar{a} - v'^{2})\,\bar{\nabla} t'} \qquad \text{(H2)}$$

* $\bar{\nabla} r = \bar{\nabla}\sqrt{\bar{r}\cdot\bar{r}} = \dfrac{\bar{\nabla}(\bar{r}\cdot\bar{r})}{2\sqrt{\bar{r}\cdot\bar{r}}} = \dfrac{r_i(\bar{\nabla} r_i)}{r}$

$$= \frac{1}{r}\left\{r_i\,\bar{\nabla} x_i - r_i\,\bar{\nabla} x_i'\right\}$$

$$= \frac{1}{r}\left\{\bar{r} - r_i\,\frac{dx_i'}{dt'}\cdot\bar{\nabla} t'\right\} \Rightarrow \boxed{\bar{\nabla} r = \frac{\bar{r} - \bar{r}\cdot\bar{v}'(\bar{\nabla} t')}{r}}$$

(H3)

* COMBINING (H1) & (H3)

$$\frac{1}{r}\left(\bar{r} - \bar{r}.\bar{v}' \ \bar{\nabla}t'\right) = -c \ \bar{\nabla}t'$$

$$\downarrow$$

$$\bar{r} = \left(-rc + \bar{r}.\bar{v}'\right) \bar{\nabla}t'$$

$$\boxed{\bar{\nabla}t' = \frac{-\bar{r}}{rc - \bar{r}.\bar{v}'}} \qquad (H4)$$

* FINALLY

$$\bar{\nabla}A^0 = \frac{qc}{4\pi\,(rc - \bar{r}.\bar{v}')^2}\left\{ -\frac{c^2\bar{r}}{(rc-\bar{r}.\bar{v}')} + \bar{v}' - \frac{(\bar{r}.\bar{a} - \bar{v}'^2)\,\bar{r}}{(rc-\bar{r}.\bar{v}')}\right\}$$

$$\boxed{\bar{\nabla}A^0 = \frac{qc}{4\pi\,(rc - \bar{r}.\bar{v}')^3}\left\{ \bar{v}'\,(rc - \bar{r}.\bar{v}') - \bar{r}\,\left(c^2 - v'^2 + \bar{r}.\bar{a}\right)\right\}}$$

(H5)

$\hookrightarrow \quad \dfrac{\partial \bar{A}}{\partial t}$

$$* \quad \frac{\partial t'}{\partial t} = 1 - \frac{1}{c}\frac{\partial r}{\partial t}$$

$$= 1 - \frac{1}{cr}\,\bar{r}\cdot\frac{\partial \bar{r}}{\partial t}$$

$$= 1 + \frac{1}{cr}\,\bar{r}\cdot\underbrace{\frac{\partial \bar{x}'}{\partial t}}_{\frac{d\bar{x}'}{dt'}\cdot\frac{\partial t'}{\partial t}}$$

$$\left(1 - \frac{1}{cr}\,\bar{v}'\cdot\bar{r}\right)\frac{\partial t'}{\partial t} = 1$$

$$\boxed{\frac{\partial t'}{\partial t} = \frac{cr}{(cr - \bar{v}'\cdot\bar{r})}} \qquad (H6)$$

$$* \quad \frac{\partial \bar{A}}{\partial t} = \frac{q}{4\pi}\frac{\partial}{\partial t}\left(\frac{\bar{v}'}{rc - \bar{r}\cdot\bar{v}'}\right)$$

$$= \frac{q}{4\pi}\frac{\partial t'}{\partial t}\left\{\frac{\bar{a}'}{(rc - \bar{r}\cdot\bar{v}')} - \frac{\bar{v}'}{(rc - \bar{r}\cdot\bar{v}')^2}\frac{\partial}{\partial t'}\left(rc - \bar{r}\cdot\bar{v}'\right)\right\}$$

$$* \quad \frac{\partial r}{\partial t'} = \frac{1}{r}\,\bar{r}\cdot\frac{\partial \bar{r}}{\partial t'} = -\frac{1}{r}\,\bar{r}\cdot\frac{\partial \bar{x}'}{\partial t'}$$

$$\underline{\underline{\frac{\partial r}{\partial t'} = -\frac{1}{r}\,\bar{r}\cdot\bar{v}'}} \qquad (H7)$$

* $$\frac{\partial}{\partial t'}(\bar{r}.\bar{v}') = \bar{v}'.\frac{\partial \bar{r}}{\partial t'} + \bar{r}.\frac{\partial \bar{v}'}{\partial t'}$$

$$= -v'^2 + \bar{r}.\bar{a}' \qquad (H8)$$

* USING (H6), (H7) & (H8) WE OBTAIN

$$\frac{\partial \bar{A}}{\partial t} = \frac{q}{4\pi}\cdot\frac{cr}{(cr-\bar{v}'.\bar{r})^3}$$

$$\cdot\left\{\bar{a}'(cr-\bar{v}'.\bar{r}) + \frac{\bar{v}'}{r}c(\bar{r}.\bar{v}') + \bar{v}'(\bar{r}.\bar{a}'-v'^2)\right\}$$

$$\boxed{\begin{aligned}\frac{\partial \bar{A}}{\partial t} = \frac{qc^2}{4\pi}\cdot\frac{1}{(cr-\bar{v}'.\bar{r})^3} & \\ \cdot\Big\{(cr-\bar{v}'.\bar{r})\left(-\bar{v}'+\frac{r}{c}\bar{a}'\right) & \\ + \frac{r}{c}\left(c^2-v'^2+\bar{r}.\bar{a}'\right)\bar{v}'\Big\} &\end{aligned}}$$

(H9)

$\hookrightarrow$ $\underset{\sim}{\bar{E}}$

$$\bar{E} = -\bar{\nabla}A^0 - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

$\downarrow$ USING (H5) & (H9)

$$\bar{E} = -\frac{qc}{4\pi\,(cr - \bar{r}.\bar{v}')^3}\Big\{(rc - \bar{r}'.\bar{v}')\,\frac{r}{c}\,\bar{a}' + (c^2 - v'^2 + \bar{r}.\bar{a}')\Big(-\bar{r} + \frac{r}{c}\bar{v}'\Big)\Big\}$$

INTRODUCE $\boxed{\bar{U} \equiv \frac{c}{r}\,\bar{r} - \bar{v}'}$

$$\bar{E} = -\frac{q\,c}{4\pi\,(\bar{r}.\bar{U})^3}\Big\{(\bar{r}.\bar{U})\,\frac{r}{c}\,\bar{a}' - \frac{r}{c}\,(c^2 - v'^2 + \bar{r}.\bar{a}')\,\bar{U}\Big\}$$

$$\boxed{\bar{E} = \frac{q}{4\pi}\,\frac{r}{(\bar{r}.\bar{U})^3}\Big\{(c^2 - v'^2)\,\bar{U} + \bar{r}\times(\bar{U}\times\bar{a}')\Big\}}$$

* FOR $\bar{v}' = \bar{a}' = 0$ (STATIC CASE)

$$\bar{U} = \frac{c}{r}\,\bar{r} \qquad \bar{r}.\bar{U} = c\,\bar{r}$$

$$\bar{E} = \frac{q}{4\pi r^2}\,\hat{r}$$ STATIC COULOMB FIELD

* FOR PARTICLE IN UNIFORM MOTION $\bar{a}' = 0$

$$\boxed{\bar{E}_C = \frac{q}{4\pi} \frac{r}{(\bar{r}\cdot\bar{U})^3} (c^2 - v'^2)\, \bar{U}}$$

cf. RESULT FOR COULOMB FIELD

OBTAINED IN CHAPTER 1 USING LORENTZ TF.

(FIELD RELATIVE TO PRESENT POSITION OF CHARGE AS DERIVED IN CHAPTER 1)

THIS TERM $\sim \dfrac{1}{|\bar{x} - \bar{x}'|^2}$

$\bar{E}_C$ IS CALLED GENERALIZED COULOMB FIELD

* IN GENERAL $\bar{E} = \bar{E}_C + \bar{E}_R$

WITH $$\boxed{\bar{E}_R \equiv \frac{q}{4\pi} \frac{r}{(\bar{r}\cdot\bar{U})^3}\, \bar{r} \times (\bar{U} \times \bar{a}')}$$

↓

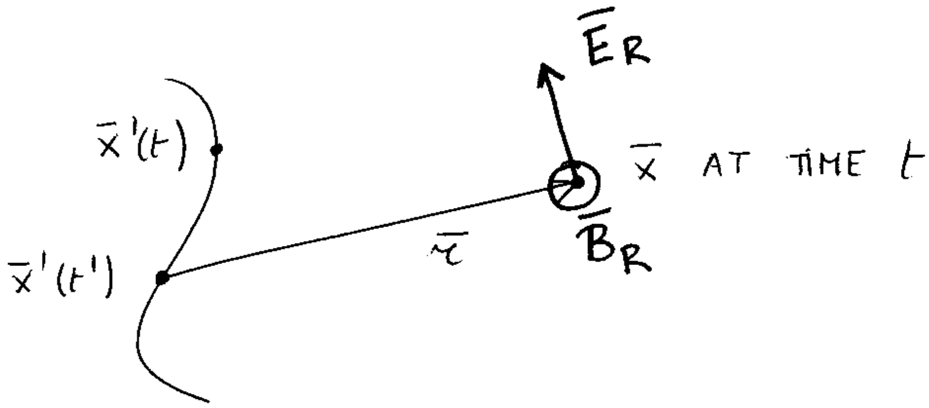
RADIATION FIELD

↳ PROPORTIONAL TO ACCELERATION $\bar{a}'$

↳ $\sim \dfrac{1}{|\bar{x} - \bar{x}'|}$ AT LARGE DISTANCES.

∴ $\bar{E}_R$ IS THE DOMINATING TERM AT LARGE DISTANCES !

$\bar{E}_R$ IS PERPENDICULAR TO $\bar{r}$



↳ $\bar{B}$

ANALOGOUS CALCULATION $\bar{B} = \bar{\nabla} \times \bar{A}$

(EXERCISE)

YIELDS $\boxed{\bar{B} = \hat{r} \times \bar{E}}$ $\bar{B} \perp \bar{E}$

$$\bar{B} = \underbrace{\hat{r} \times \bar{E}_C}_{\bar{B}_{BS}} + \underbrace{\hat{r} \times \bar{E}_R}_{\bar{B}_R}$$

$$* \quad \bar{B}_{BS} = -\frac{q}{4\pi} \frac{r}{(\bar{r}.\bar{u})^3} (c^2 - v'^2)(\hat{r} \times \bar{v}')$$

$$\boxed{\bar{B}_{BS} = \frac{q}{4\pi} \frac{r}{(\bar{r}.\bar{u})^3} (c^2 - v'^2)(\bar{v}' \times \hat{r})}$$

BIOT-SAVART LAW FOR POINT CHARGE

$$\bar{B}_{BS} \underset{v' \ll c}{\simeq} \frac{q}{4\pi c} \cdot \frac{1}{|\bar{x} - \bar{x}'|^2} \cdot (\bar{v}' \times \hat{r})$$

$*\quad \boxed{\bar{B}_{D_R} \equiv\sim \hat{r} \times \bar{E}_R}$

$\hookrightarrow \sim \dfrac{1}{|\bar{x} - \bar{x}'|}$ AT LARGE DISTANCES.

DOMINATING TERM AT LARGE DISTANCES

$\therefore \quad \bar{E}_R \perp \bar{B}_R$

$\bar{E}_R \perp \hat{r}$

$\bar{B}_R \perp \hat{r}$

⇒ POWER RADIATED BY AN ACCELERATED POINT CHARGE

↳ ENERGY FLUX

ENERGY FLUX ↷ POYNTING VECTOR

$$\bar{S} = c\,(\bar{E} \times \bar{B})$$

FOR RADIATION FIELDS

$$\bar{B}_R = \hat{r} \times \bar{E}_R$$

$$\underline{\underline{\bar{S} = c\,E_R^2\,\hat{r}}}$$

ENERGY FLUX FOR RADIATION FIELDS
↳ LEADING BEHAVIOR AT LARGE DISTANCE

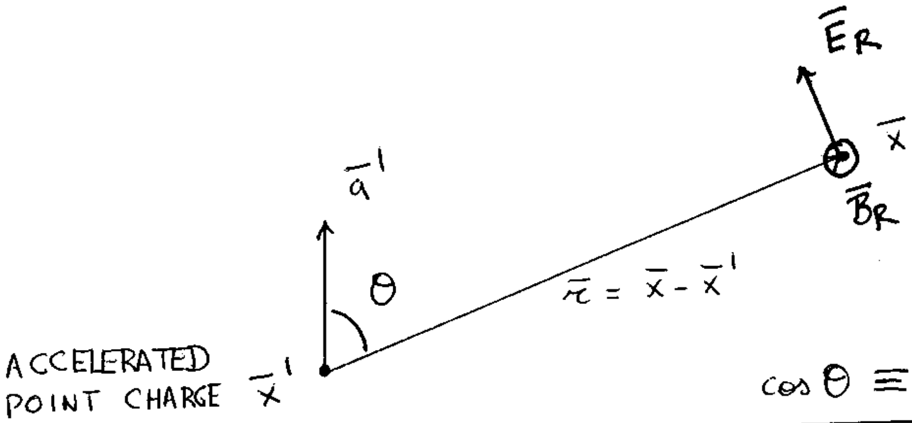
↳ RADIATION PATTERN: NON-RELATIVISTIC CASE

$$* \quad \bar{E}_R = \frac{q}{4\pi} \frac{r}{(\bar{r}\cdot\bar{u})^3}\, \bar{r} \times (\bar{u} \times \bar{a}')$$

NON-RELATIVISTIC LIMIT $v' \ll c$

$$\bar{u} \simeq c\,\hat{r}$$

$$\underline{\underline{\bar{E}_R \simeq \frac{q}{4\pi c^2} \frac{1}{|\bar{x}-\bar{x}'|} \left[\hat{r} \times (\hat{r} \times \bar{a}')\right]}}$$



$$\cos\theta \equiv \hat{r} \cdot \hat{a}'$$

$$\bar{E}_R = \frac{q}{4\pi c^2} \frac{1}{|\bar{x}-\bar{x}'|} \cdot \left[\cos\theta \, a' \hat{r} - \bar{a}' \right]$$

$$|\bar{E}_R| = \frac{q}{4\pi c^2} \frac{a' |\sin\theta|}{|\bar{x}-\bar{x}'|}$$

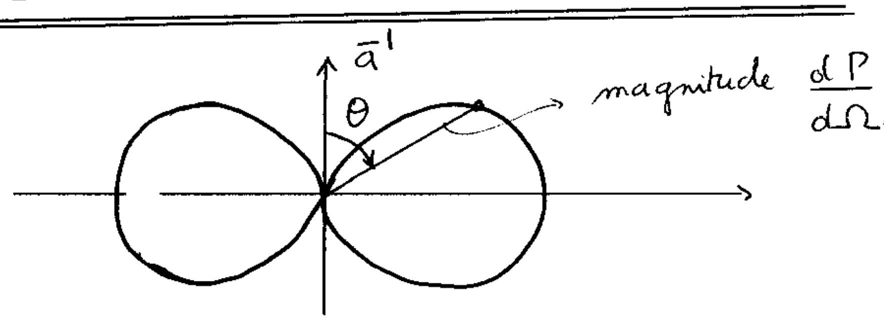
$$\boxed{\bar{S} = \frac{q^2}{(4\pi)^2 c^3} \frac{a'^2 \sin^2\theta}{|\bar{x}-\bar{x}'|^2} \hat{r}}$$

* POWER RADIATED IN SOLID ANGLE $d\Omega$

$$dP = \underset{\substack{\uparrow \\ \text{ENERGY / } \frac{\text{UNIT TIME}}{\text{UNIT SURFACE}}}}{|\bar{S}|} \cdot \underbrace{|\bar{x}-\bar{x}'|^2 d\Omega}_{\text{SURFACE CORR. WITH SOLID ANGLE } d\Omega}$$

$$\frac{dP}{d\Omega} = \frac{1}{4\pi}\left(\frac{q^2}{4\pi}\right) \cdot \frac{1}{c^3} \cdot a'^2 \sin^2\theta$$

DIPOLE
RADIATION
PATTERN



* TOTAL POWER RADIATED

$$P = \int d\Omega \; \frac{dP}{d\Omega}$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; \frac{dP}{d\Omega}$$

$x = \cos\theta$

$$= 2\pi \cdot \int_{-1}^{1} dx \; \frac{1}{4\pi} \cdot \frac{q^2}{4\pi} \frac{1}{c^3} \; a'^2 \underbrace{(1-x^2)}_{\sin^2\theta}$$

$$= 2\pi \cdot \frac{4}{3} \cdot \frac{1}{4\pi} \frac{q^2}{4\pi} \frac{a'^2}{c^3}$$

$$\boxed{P = \frac{2}{3} \cdot \frac{q^2}{4\pi} \cdot \frac{a'^2}{c^3}}$$

LARMOR FORMULA
FOR POWER RADIATED
BY AN ACCELERATED CHARGE

↳ RADIATION PATTERN : RELATIVISTIC CASE $v' \simeq c$

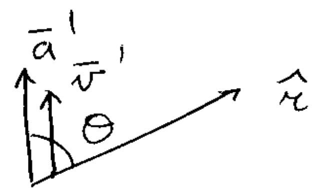
$$\hat{r} \cdot \bar{U} = c\left(1 - \hat{r} \cdot \frac{\bar{v}'}{c}\right)$$

$$\bar{E}_R = \frac{q}{4\pi c^2} \cdot \frac{1}{|\bar{x} - \bar{x}'|} \cdot \frac{1}{\left(1 - \hat{r} \cdot \frac{\bar{v}'}{c}\right)^3} \cdot \hat{r} \times \left(\left(\hat{r} - \frac{\bar{v}'}{c}\right) \times \bar{a}'\right)$$

* FOR $\bar{v}' \parallel \bar{a}'$ $\rightarrow$ $\bar{v}' \times \bar{a}' = 0$

$$\bar{E}_R = \frac{q}{4\pi c^2} \frac{1}{|\bar{x} - \bar{x}'|} \cdot \underbrace{\frac{1}{\left(1 - \hat{r} \cdot \frac{\bar{v}'}{c}\right)^3}}_{\text{EXTRA FACTOR COMPARED TO NON-RELATIVISTIC CASE}} \cdot \hat{r} \times (\hat{r} \times \bar{a}')$$

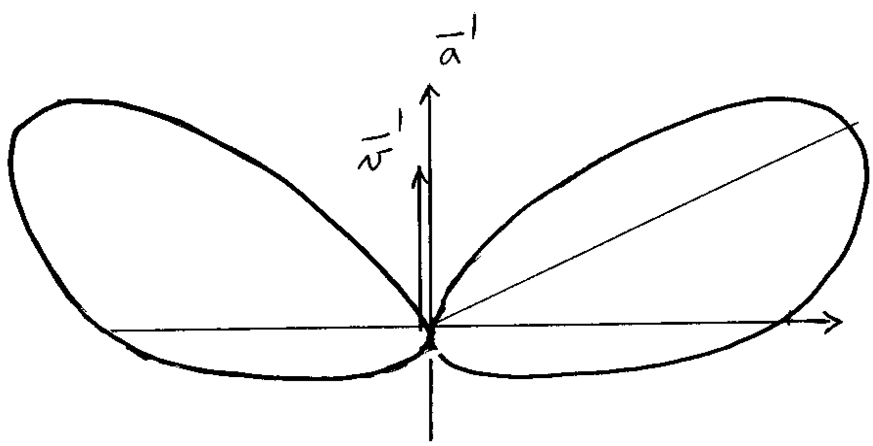
EXTRA FACTOR



$$\frac{1}{\left(1-\hat{r}\cdot\frac{\bar{v}'}{c}\right)^3} = \frac{1}{\left(1-\frac{v'}{c}\cos\theta\right)^3}$$

$$\leadsto \left.\frac{dP}{d\Omega}\right|_{\text{RECEIVER}} = \frac{q^2}{(4\pi)^2 c^3} \frac{a'^2 \sin^2\theta}{\left(1-\frac{v'}{c}\cos\theta\right)^6}$$

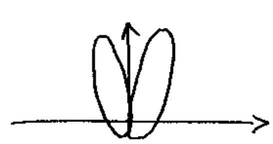
POWER RECEIVED AT FIXED LOCATION



DIPOLE PATTERN FOR ACCELERATED POINT CHARGE IN RELATIVISTIC MOTION

$v' \to c$ RADIATION MORE & MORE ENHANCED IN FORWARD DIRECTION

IN LIMIT $v' \to c$



RADIATION EMITTED AT VERY FORWARD ANGLES

$$\frac{dP}{d\Omega} \simeq \frac{1}{4\pi}\left(\frac{q^2}{4\pi}\right)\frac{1}{c^3} \frac{a'^2 \qquad \theta^2}{\left(1-\frac{v'}{c}+\underset{\substack{\uparrow \\ v'\simeq c}}{\frac{v'}{2c}}\theta^2\right)^6}$$

$\rightsquigarrow$ MAXIMUM OF $\frac{dP}{d\Omega}$

$$\frac{d}{d\theta}\left(\frac{dP}{d\Omega}\right) = 0$$

$$\Updownarrow$$

$$\frac{d}{d\theta}\left(\frac{\theta^2}{\left(1-\frac{v'}{c}+\frac{1}{2}\theta^2\right)^6}\right) = 0$$

$$\Updownarrow$$

$$\frac{2\theta}{(\quad)^6} - \frac{6\theta^2\theta}{(\quad)^7} = 0$$

$$\left(1-\frac{v'}{c}+\frac{1}{2}\theta^2\right) - 3\theta^2 = 0$$

$$\frac{5}{2}\theta^2 \simeq 1-\frac{v'}{c}$$

$$\theta_{MAX} \simeq \sqrt{\frac{2}{5}\left(1-\frac{v'}{c}\right)} \simeq \sqrt{\frac{1}{5}\left(1-\frac{v'^2}{c^2}\right)}$$

$$\simeq \frac{1}{\sqrt{5}\,\gamma}$$

$$\left.\frac{dP}{d\Omega}\right|_{\theta_{MAX}} \simeq \frac{1}{4\pi}\left(\frac{q^2}{4\pi}\right)\frac{1}{c^3}\cdot\frac{a'^2\,\frac{1}{5}\,\frac{1}{\gamma^2}}{\left(\frac{1}{2\gamma^2}+\frac{1}{10\gamma^2}\right)^6}$$

↑
PEAK IN RADIATED POWER

$$\left.\frac{dP}{d\Omega}\right|_{\theta_{MAX}} \simeq \frac{1}{4\pi}\,\frac{q^2}{4\pi}\cdot\frac{1}{c^3}\,\underbrace{\frac{5^5\cdot 2^6}{6^6}}_{\simeq 4.29}\,\left(a'^2\gamma^{10}\right)$$

↑ TENTH POWER OF PARTICLE ENERGY

$\rightsquigarrow$ ⚠ SUBTLETY : ABOVE POWER IS POWER RECEIVED AT FIXED LOCATION

$\neq$ POWER EMITTED SOURCE (IF $v \neq 0$)

$$\left.\frac{dP}{d\Omega}\right|_{\text{SOURCE}} = \underbrace{\left(1 - \hat{r}\cdot\frac{\bar{v}'}{c}\right)}_{\hat{r}\cdot\frac{\bar{U}}{c}} \left.\frac{dP}{d\Omega}\right|_{\text{RECEIVER}}$$

$$\left.\frac{dP}{d\Omega}\right|_{\text{SOURCE}} = \frac{q^2}{(4\pi)^2 c^3} \frac{a'^2 \sin^2\theta}{\left(1 - \frac{v'}{c}\cos\theta\right)^5}$$

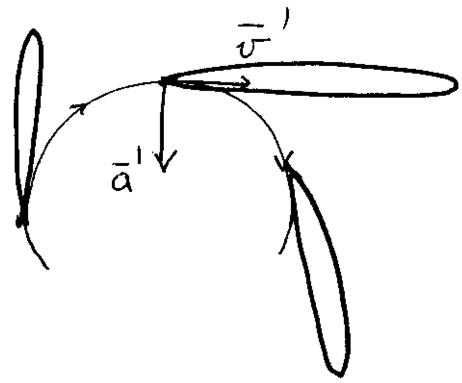
$\rightsquigarrow$ TOTAL POWER EMITTED BY SOURCE

$$P_{\text{SOURCE}} \simeq 2\pi \int_{-1}^{1} dx \; \frac{1}{4\pi} \cdot \frac{q^2}{4\pi} \frac{1}{c^3} \frac{a'^2(1-x^2)}{\left(1 - \frac{v'}{c}x\right)^5}$$

$$= \frac{1}{2} \frac{q^2}{4\pi} \frac{a'^2}{c^3} \underbrace{\int_{-1}^{1} dx \frac{(1-x^2)}{\left(1 - \frac{v'}{c}x\right)^5}}_{\frac{4}{3}\frac{1}{\left(1-\left(\frac{v'}{c}\right)^2\right)^3} = \frac{4}{3}\gamma^6}$$

$$\boxed{P_{\text{SOURCE}} = \frac{2}{3}\left(\frac{q^2}{4\pi}\right) \cdot \frac{a'^2}{c^3} \cdot \gamma^6}$$

* FOR $\underline{\underline{\bar{v}' \perp \bar{a}'}}$



SYNCHROTRON RADIATION (→ EXERCISE)

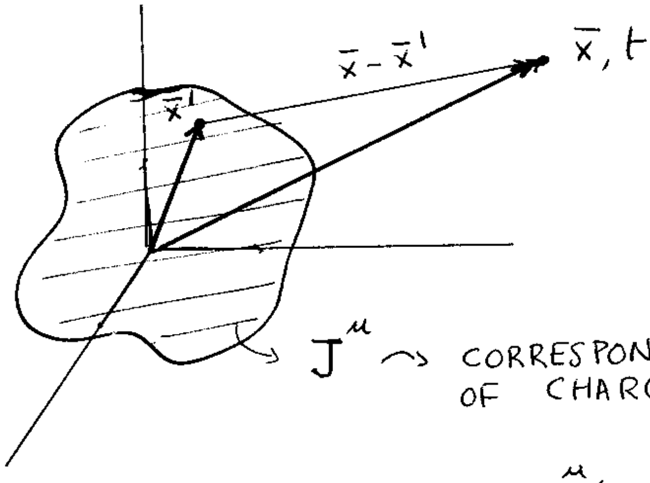
WHY CIRCULAR PARTICLE ACCELERATORS HAVE LARGE RADIUS

$$a_r = \frac{v^2}{R}$$

$$\underline{P} \sim \gamma^4 a^2$$

2) RADIATION BY A SYSTEM OF CHARGES

⇒ RADIATION FIELDS



J^μ ⤳ CORRESPONDS WITH SYSTEM OF CHARGES / CURRENTS

$$A^\mu(t,\bar{x}) = \frac{1}{c}\int d^3\bar{x}' \, \frac{J^\mu\left(t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right)}{4\pi\,|\bar{x}-\bar{x}'|}$$

WE ARE INTERESTED IN $\bar{E}, \bar{B}$ FIELDS FAR AWAY FROM SYSTEM OF CHARGES, i.e. $|\bar{x}| \gg L$ ↳ SIZE OF SYSTEM

$$\hookrightarrow |\bar{x}-\bar{x}'| = \left(\bar{x}^2 + \bar{x}'^2 - 2\,\bar{x}\cdot\bar{x}'\right)^{1/2}$$

$$\simeq |\bar{x}|\left(1 - \frac{\bar{x}\cdot\bar{x}'}{|\bar{x}|^2}\right)$$

$$= |\bar{x}| - \hat{x}\cdot\bar{x}'$$

$$\hookrightarrow \boxed{A^\mu(t,\bar{x}) \underset{|\bar{x}|\gg}{\simeq} \frac{1}{4\pi c\,|\bar{x}|}\int d^3\bar{x}'\; J^\mu\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}\cdot\bar{x}'}{c}, \bar{x}'\right)}$$

(FAR FIELD)

$$\hookrightarrow \quad \underline{\bar{B} = \bar{\nabla} \times \bar{A}}$$

$$\bar{B} = \frac{1}{4\pi c} \bar{\nabla}\left(\frac{1}{|\bar{x}|}\right) \times \int d^3\bar{x}' \; \bar{J}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}.\bar{x}'}{c}, \bar{x}'\right)$$

$$+ \frac{1}{4\pi c} \frac{1}{|\bar{x}|} \int d^3\bar{x}' \; \bar{\nabla} \times \bar{J}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}.\bar{x}'}{c}, \bar{x}'\right)$$

$$= - \frac{1}{4\pi c} \frac{1}{|\bar{x}|^2} \hat{x} \times \int d^3\bar{x}' \; \bar{J}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}.\bar{x}'}{c}, \bar{x}'\right)$$

$$+ \frac{1}{4\pi c} \frac{1}{|\bar{x}|} \int d^3\bar{x}' \; \bar{\nabla}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}.\bar{x}'}{c}\right) \times \frac{\partial \bar{J}}{\partial t}$$

$$\bar{\nabla}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}.\bar{x}'}{c}\right) = - \frac{\hat{x}}{c} + \frac{\bar{x}'}{c|\bar{x}|} - \frac{(\hat{x}.\bar{x}')\hat{x}}{c|\bar{x}|}$$

$$\downarrow$$

$$= - \frac{1}{4\pi c} \frac{1}{|\bar{x}|^2} \hat{x} \times \int d^3\bar{x}' \; \bar{J}$$

$$- \frac{1}{4\pi c^2} \frac{1}{|\bar{x}|} \hat{x} \times \int d^3\bar{x}' \; \frac{\partial \bar{J}}{\partial t}$$

$$+ \frac{1}{4\pi c^2} \frac{1}{|\bar{x}|^2} \int d^3\bar{x}' \left(\bar{x}' - (\hat{x}.\bar{x}')\hat{x}\right) \times \frac{\partial \bar{J}}{\partial t}$$

$\downarrow$ KEEP ONLY LEADING TERM $\sim \frac{1}{|\bar{x}|}$

$$\boxed{\begin{aligned} \bar{B} &\underset{|\bar{x}|\gg}{\simeq} - \frac{1}{4\pi c^2} \frac{1}{|\bar{x}|} \hat{x} \times \int d^3\bar{x}' \; \frac{\partial \bar{J}}{\partial t}\left(t - \frac{|\bar{x} - \bar{x}'|}{c}, \bar{x}'\right) \\ &= - \frac{1}{c} \hat{x} \times \frac{\partial \bar{A}}{\partial t} \end{aligned}}$$

$$\hookrightarrow \quad \bar{E} = -\bar{\nabla} A^0 - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

$$\rightsquigarrow \bar{\nabla} A^0 = -\frac{1}{4\pi}\frac{\hat{x}}{|\bar{x}|^2}\int d^3\bar{x}' \, \rho$$

$$+ \frac{1}{4\pi}\frac{1}{|\bar{x}|}\int d^3\bar{x}' \, \underbrace{\bar{\nabla}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}\cdot\bar{x}'}{c}\right)}_{-\frac{\hat{x}}{c} + O\left(\frac{1}{|\bar{x}|}\right)} \cdot \frac{\partial \rho}{\partial t}$$

$$\downarrow$$

$$\underset{|\bar{x}|>>}{\simeq} -\frac{1}{4\pi c}\frac{\hat{x}}{|\bar{x}|}\int d^3\bar{x}' \, \frac{\partial \rho}{\partial t}\left(t - \frac{|\bar{x}-\bar{x}'|}{c}, \bar{x}'\right)$$

$$= -\frac{\hat{x}}{c}\frac{\partial A^0}{\partial t}$$

$$\rightsquigarrow \bar{E} \underset{|\bar{x}|>>}{\simeq} +\frac{\hat{x}}{c}\frac{\partial A^0}{\partial t} - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

A^μ SATISFIES LORENTZ CONDITION

$$\partial_\mu A^\mu = 0 \quad \rightsquigarrow \quad \frac{1}{c}\frac{\partial A^0}{\partial t} + \bar{\nabla}\cdot\bar{A} = 0$$

$$\bar{E} \simeq -\hat{x}\left(\bar{\nabla}\cdot\bar{A}\right) - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

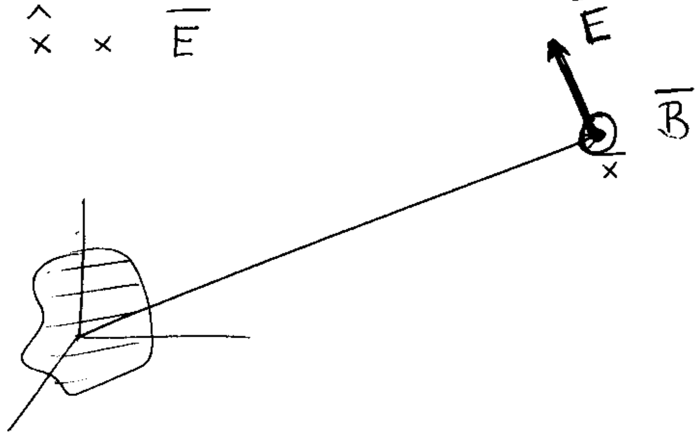
$$\bar{\nabla}\cdot\bar{A} \simeq -\frac{1}{c}\,\hat{x}\cdot\frac{\partial \bar{A}}{\partial t}$$

$$\boxed{\bar{E} \underset{|\bar{x}|>>}{\simeq} +\frac{1}{c}\left[\hat{x}\left(\hat{x}\cdot\frac{\partial \bar{A}}{\partial t}\right) - \frac{\partial \bar{A}}{\partial t}\right] = \frac{1}{c}\,\hat{x}\times\left(\hat{x}\times\frac{\partial \bar{A}}{\partial t}\right)}$$

$$\bar{E} = -\hat{x} \times \bar{B}$$

$$\Updownarrow$$

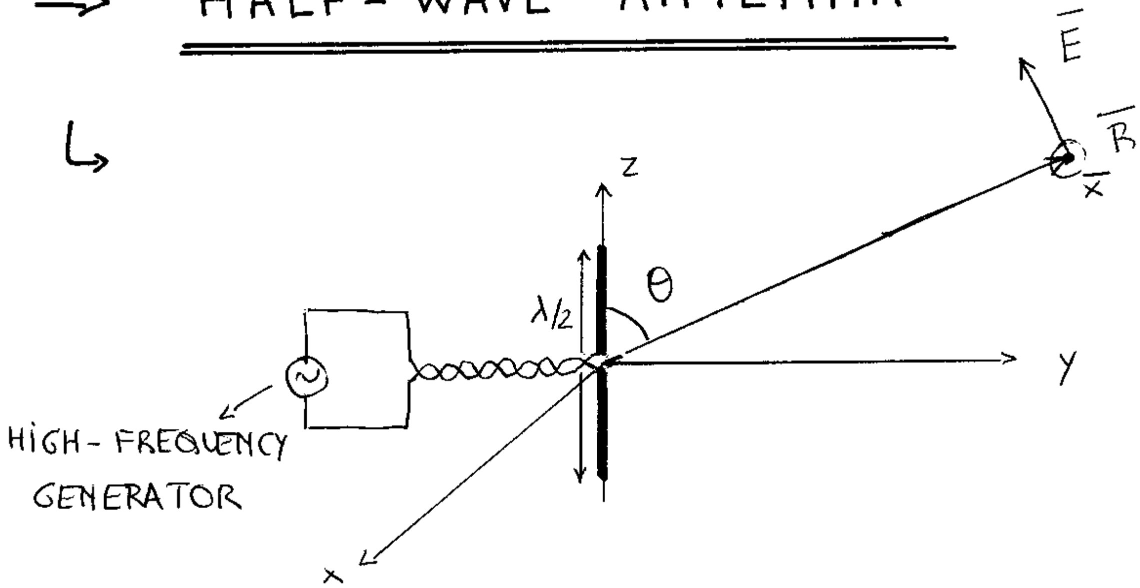
$$\bar{B} = \hat{x} \times \bar{E}$$



RADIATION FIELDS GIVE LEADING TERMS $\sim \dfrac{1}{|\bar{x}|}$

⇒ HALF-WAVE ANTENNA

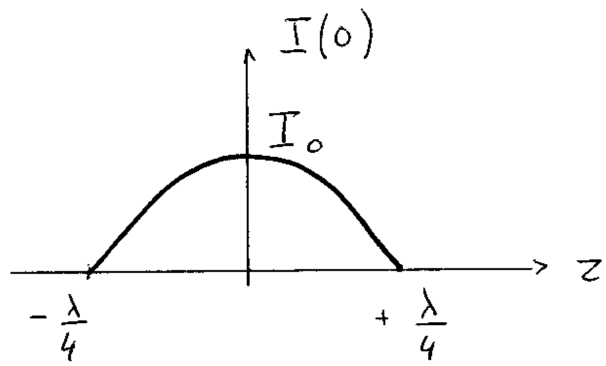
↳



ANTENNA: 2 PIECES OF WIRE OF LENGTH $\lambda/2$

CURRENT ALONG ANTENNA: (STANDING WAVE)

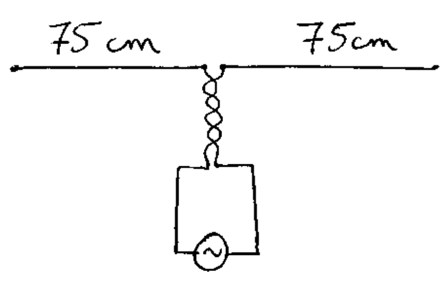
$$I(t) = I_0 \cos\left(\frac{2\pi z}{\lambda}\right) \cos \omega t$$



FREQUENCY $\omega = \dfrac{2\pi c}{\lambda}$

e.g. $\lambda = 3\ m \Rightarrow \nu = \dfrac{c}{\lambda} = 10^8\ Hz$

$= 100\ MHz$



$$\hookrightarrow \quad \bar{J}(t',\bar{x}') = \delta(x')\,\delta(y')\; I_0 \cos\left(\frac{2\pi}{\lambda} z'\right) \cos(\omega t')\, \bar{e}_z$$

$$\bar{A} = A_z\, \bar{e}_z$$

$$A_z(t,\bar{x}) \underset{|\bar{x}|\gg}{\simeq} \frac{1}{4\pi c|\bar{x}|} \int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} dz'\, I_0 \cos\left(\frac{2\pi}{\lambda} z'\right) \cos\left[\omega\left(t - \frac{|\bar{x}|}{c} + \frac{z'}{c}\cos\theta\right)\right]$$

$$= \frac{I_0}{8\pi c} \frac{1}{|\bar{x}|} \int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} dz' \left\{ \cos\left[\omega\left(t - \frac{|\bar{x}|}{c}\right) + z'\left(\frac{2\pi}{\lambda} + \frac{\omega}{c}\cos\theta\right)\right] + \cos\left[\omega\left(t - \frac{|\bar{x}|}{c}\right) - z'\left(\frac{2\pi}{\lambda} - \frac{\omega}{c}\cos\theta\right)\right]\right\}$$

$$\downarrow \qquad \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

$$= \frac{I_0}{8\pi c} \frac{1}{|\bar{x}|} \left\{ \frac{c}{\omega} \cdot \frac{1}{(1+\cos\theta)} \sin\left[z' \frac{\omega}{c}(1+\cos\theta) + \omega\left(t - \frac{|\bar{x}|}{c}\right)\right] + \frac{c}{\omega} \frac{1}{(1-\cos\theta)} \sin\left[z' \frac{\omega}{c}(1-\cos\theta) - \omega\left(t - \frac{|\bar{x}|}{c}\right)\right]\right\}\Bigg|_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}}$$

$$= \frac{I_0}{4\pi} \frac{1}{|\bar{x}|} \frac{1}{\omega} \left\{ \frac{1}{(1+\cos\theta)} \cos\left[\omega\left(t - \frac{|\bar{x}|}{c}\right)\right] \sin\left[\frac{\pi}{2}(1+\cos\theta)\right] + \frac{1}{(1-\cos\theta)} \cos\left[\omega\left(t - \frac{|\bar{x}|}{c}\right)\right] \sin\left[\frac{\pi}{2}(1-\cos\theta)\right]\right\}$$

$$\sin\left[\frac{\pi}{2} \pm \frac{\pi}{2}\cos\theta\right] = \cos\left(\frac{\pi}{2}\cos\theta\right)$$

$$\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta} = \frac{1}{2\cos^2\frac{\theta}{2}} + \frac{1}{2\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}$$

$$= \frac{2}{\sin^2\theta}$$

$$\boxed{A_z(t,\bar{x}) \underset{|\bar{x}|\gg}{\simeq} \frac{2 I_0}{4\pi |\bar{x}|\omega} \cos\omega\left(t - \frac{|\bar{x}|}{c}\right) \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}$$

$\hookrightarrow$

$$\bar{B} \underset{|\bar{x}|\gg}{\simeq} -\frac{1}{c}\,\hat{x} \times \frac{\partial \bar{A}}{\partial t}$$

$$\boxed{\bar{B} \simeq \frac{2 I_0}{4\pi c |\bar{x}|} \sin\omega\left(t - \frac{|\bar{x}|}{c}\right) \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} (\hat{x} \times \bar{e}_z)}$$

$\hookrightarrow$

$$\bar{S} = c\,(\bar{E} \times \bar{B})$$

$$= c\,B^2\,\hat{x}$$

NOTE $|\hat{x} \times \bar{e}_z| = \sin\theta$

$$|\bar{S}| = \left(\frac{I_0^2}{4\pi}\right) \frac{1}{c\pi|\bar{x}|^2} \sin^2\omega\left(t - \frac{|\bar{x}|}{c}\right) \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$\downarrow$

TIME - AVERAGE $\langle\ \rangle$

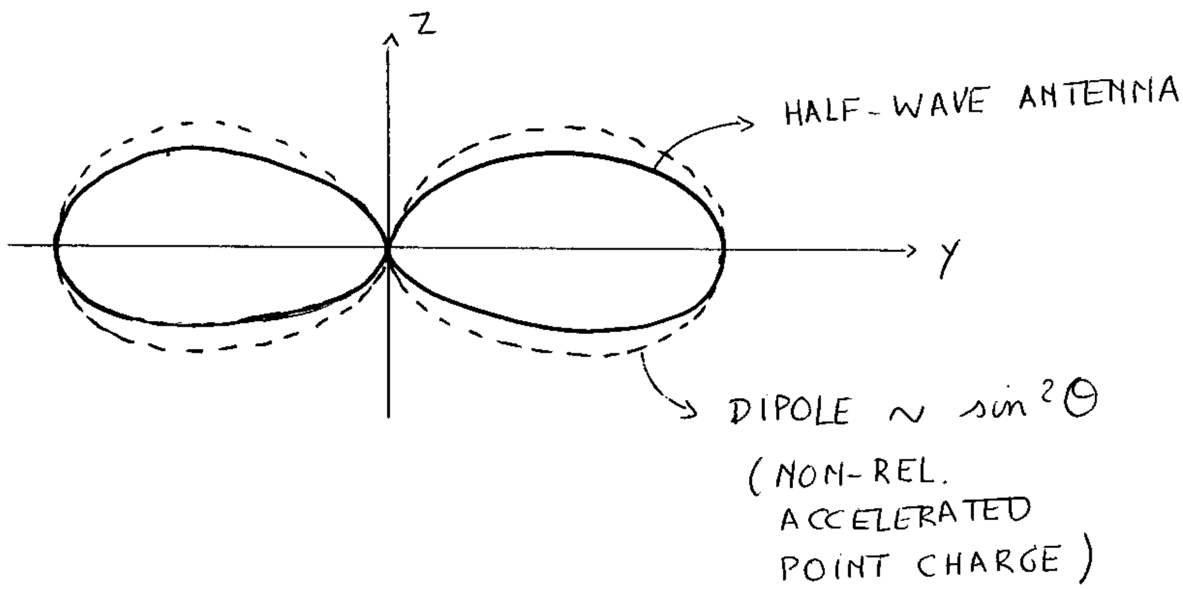
$$\langle |\bar{S}| \rangle = \left(\frac{I_0^2}{4\pi}\right) \frac{1}{2\pi c |\bar{x}|^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

↳ TIME-AVERAGED RADIATED POWER

$$\left\langle \frac{dP}{d\Omega} \right\rangle = |\bar{x}|^2 \left\langle |\bar{S}| \right\rangle$$

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{I_0^2}{4\pi} \cdot \frac{1}{2\pi c} \cdot \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}$$

RADIATION PATTERN



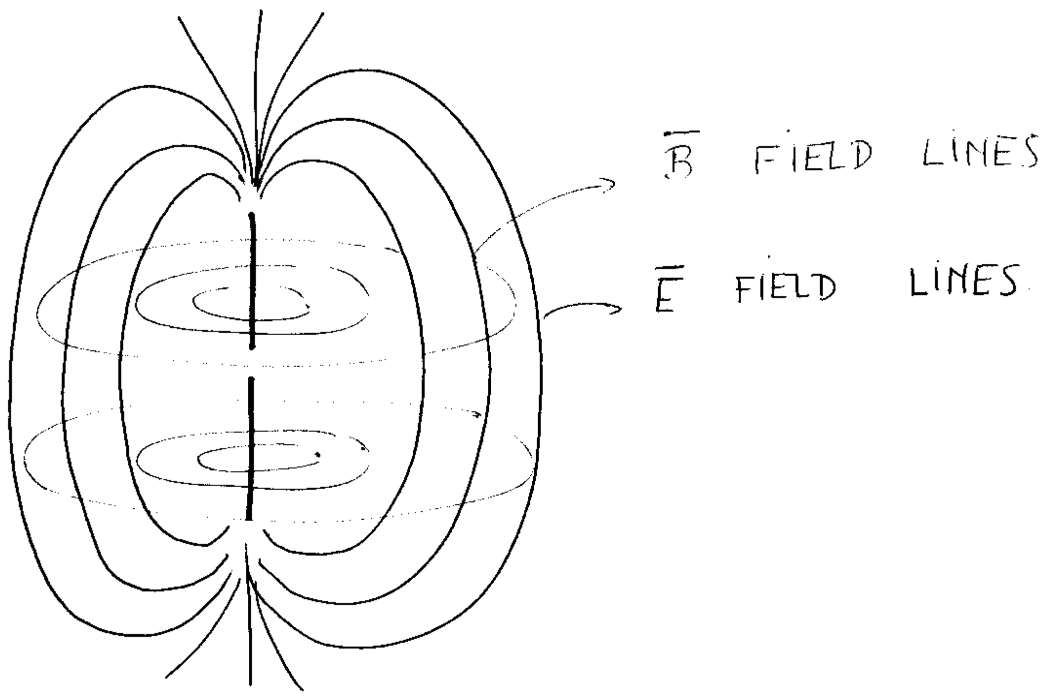
↳ TOTAL RADIATED POWER

$$\langle P \rangle = \int d\Omega \left\langle \frac{dP}{d\Omega} \right\rangle$$

$$= \frac{I_0^2}{4\pi} \cdot \frac{1}{2\pi c} \int_0^{2\pi} d\phi \underbrace{\int_0^{\pi} d\theta \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}}_{1.22}$$

$$\boxed{\langle P \rangle = \frac{I_0^2}{4\pi} \cdot \frac{(1.22)}{c}}$$

↳ FIELD PATTERNS



⇒ MULTIPOLE EXPANSION FOR RADIATION

↳ IF SIZE OF SYSTEM L << WAVELENGTH OF RADIATION λ

$$\underline{\underline{L \ll \lambda}}$$

ONE CAN MAKE POWER SERIES EXPANSION FOR CURRENT

(DOES NOT APPLY FOR CASES AS HALF-WAVE ANTENNA, WHERE $L = \lambda/2$)

$$\bar{A}(t,\bar{x}) \underset{|\bar{x}| \gg L}{\simeq} \frac{1}{4\pi c |\bar{x}|} \int d^3\bar{x}' \; \bar{J}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}\cdot\bar{x}'}{c}, \bar{x}'\right)$$

POWER SERIES EXPANSION FOR CURRENT

$$\bar{J}\left(t - \frac{|\bar{x}|}{c} + \frac{\hat{x}\cdot\bar{x}'}{c}, \bar{x}'\right)$$

$$= \bar{J}\left(t - \frac{|\bar{x}|}{c}, \bar{x}'\right) + \frac{\hat{x}\cdot\bar{x}'}{c} \frac{\partial \bar{J}}{\partial t}\left(t - \frac{|\bar{x}|}{c}, \bar{x}'\right) + \cdots$$

2° TERM OF ORDER $\frac{L}{c}\frac{1}{T}$ COMPARED TO FIRST

$$\rightsquigarrow \quad \frac{2^\circ \text{ TERM}}{1^\circ \text{ TERM}} \approx \frac{L}{cT} = \frac{L}{\lambda}$$

FOR $L \ll \lambda$ ⇒ SERIES CONVERGES RAPIDLY

$\frac{L}{T} \approx v$ TYPICAL VELOCITIES OF PARTICLES IN SYSTEM

$\frac{L}{\lambda} \ll 1 \iff \frac{v}{c} \ll 1$. <u>NON-RELATIVISTIC SYSTEMS.</u>

$$\boxed{\begin{aligned}\bar{A}(t,\bar{x}) \underset{\substack{|\bar{x}|\gg L\\ L\ll\lambda}}{\simeq} & \frac{1}{4\pi c|\bar{x}|}\int d^3\bar{x}'\ \bar{J}\left(t-\frac{|\bar{x}|}{c},\bar{x}'\right)\\ & + \frac{1}{4\pi c^2|\bar{x}|}\int d^3\bar{x}'\,(\hat{x}\cdot\bar{x}')\,\frac{\partial\bar{J}}{\partial t}\left(t-\frac{|\bar{x}|}{c},\bar{x}'\right)\\ & + \dots\end{aligned}}$$

↳ **MULTIPOLE EXPANSION FOR RADIATION**

2 APPROXIMATIONS :

$|\bar{x}| \gg L$: GIVES FAR-FIELDS , RADIATION FIELDS $\sim \frac{1}{|\bar{x}|}$

$\lambda \gg L$: FOR LONG-WAVELENGTHS

↳ POWER-SERIES EXPANSION FOR CURRENT

* <u>1° TERM</u> IN MULTIPOLE EXPANSION

$$\sim \int d^3\bar{x}'\ \bar{J} \quad \Rightarrow \quad \text{ELECTRIC DIPOLE RADIATION}$$

* <u>2° TERM</u> IN MULTIPOLE EXPANSION

$$\sim \int d^3\bar{x}'\ (\hat{x}\cdot\bar{x}')\ \frac{\partial\bar{J}}{\partial t} \quad \Rightarrow \quad \text{MAGNETIC DIPOLE RADIATION}$$

& ELECTRIC QUADRUPOLE RADIATION

* <u>HIGHER ORDER TERMS</u>

MAGNETIC QUADRUPOLE , ELECTRIC OCTUPOLE , ...

⇒ ELECTRIC DIPOLE RADIATION

↳ NOTATION : E1

FROM QUANTUM MECHANICS ⇒ EXPAND FIELD IN PARTIAL WAVES WITH GOOD ANGULAR MOMENTUM ℓ

⇒ FOR E.M. FIELD : 2 POLARIZATIONS

- ELECTRIC : $E\ell$
- MAGNETIC : $M\ell$

↳
$$\overline{A}_{E1}(t,\overline{x}) = \frac{1}{4\pi c|\overline{x}|}\int d^3\overline{x}'\ \overline{J}\left(t-\frac{|\overline{x}|}{c},\overline{x}'\right)$$

$$\rightsquigarrow \int d^3\overline{x}'\ \overline{J} = -\int d^3\overline{x}'\ \overline{x}'\left(\overline{\nabla}'.\overline{J}\right) + \cancel{\text{SURFACE TERM}} \to 0$$

$$\rightsquigarrow \overline{\nabla}'.\overline{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{(CURRENT CONSERVATION)}$$

$$\overline{A}_{E1}(t,\overline{x}) = \frac{1}{4\pi c|\overline{x}|}\frac{d}{dt}\int d^3\overline{x}'\ \overline{x}'\ \rho\left(t-\frac{|\overline{x}|}{c},\overline{x}'\right)$$

↳ ELECTRIC DIPOLE MOMENT

$$\boxed{\overline{P}(t) \equiv \int d^3\overline{x}'\ \overline{x}'\ \rho(t,\overline{x}')}$$

↳ $$\boxed{\overline{A}_{E1}(t,\overline{x}) = \frac{1}{4\pi c|\overline{x}|}\left[\frac{d\overline{p}}{dt}\right]_{t-\frac{|\overline{x}|}{c}}}$$

$$\overline{B}_{E1}(t,\overline{x}) = -\frac{1}{c}\,\hat{x}\times\frac{\partial\overline{A}}{\partial t}$$

$$= -\frac{1}{4\pi c^2|\overline{x}|}\,\hat{x}\times\left[\frac{d^2\overline{p}}{dt^2}\right]_{t-\frac{|\overline{x}|}{c}}$$

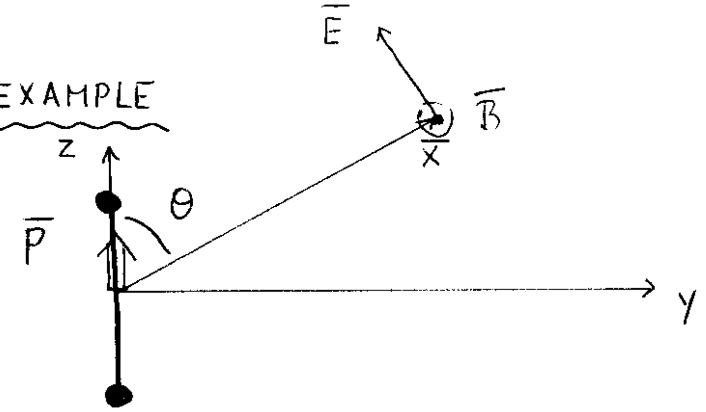
$$\overline{E}_{E1}(t,\overline{x}) = -\,\hat{x}\times\overline{B}_{E1}$$

∘∘ ELECTRIC DIPOLE RADIATION

$$\boxed{\begin{aligned}\overline{B}_{E1} &= -\frac{1}{4\pi c^2|\overline{x}|}\,\hat{x}\times[\ddot{\overline{p}}]\\ \overline{E}_{E1} &= -\,\hat{x}\times\overline{B}_{E1}\end{aligned}}$$

[ ] IS UNDERSTOOD TO BE EVALUATED AT $t-\frac{|\overline{x}|}{c}$

↳ EXAMPLE



$\overline{p}(t) = p_0\cos(\omega t)\,\overline{e}_z \Rightarrow$ HARMONIC MOTION OF CHARGE ALONG A STRAIGHT LINE

$$\overline{B}_{E1} = \frac{p_0\omega^2\cos\omega\left(t-\frac{|\overline{x}|}{c}\right)}{4\pi c^2|\overline{x}|}\sin\theta\left(\sin\phi\,\overline{e}_x - \cos\phi\,\overline{e}_y\right)$$

↳ POYNTING VECTOR

$$\bar{S} = c\,(\bar{E} \times \bar{B})$$

$$\bar{S}_{E1} = c\, B_{E1}^{2}\, \hat{x}$$

$$|\bar{B}_{E1}| = \frac{1}{4\pi c^2 |\bar{x}|} \sin\theta \,\left| [\ddot{\bar{p}}] \right|$$

WITH

$$\boxed{\bar{S}_{E1} = \frac{1}{(4\pi)^2 c^3} \frac{1}{|\bar{x}|^2} \sin^2\theta \,\left| [\ddot{\bar{p}}] \right|^2 \hat{x}}$$

↳ RADIATED POWER BY SOURCE

$$\rightsquigarrow \quad \frac{dP}{d\Omega} = |\bar{S}|\;|\bar{x}|^2$$

$$\boxed{\left.\frac{dP}{d\Omega}\right|_{E1} = \frac{1}{(4\pi)^2 c^3} \sin^2\theta \,\left| [\ddot{\bar{p}}] \right|^2}$$

⇝ TOTAL POWER

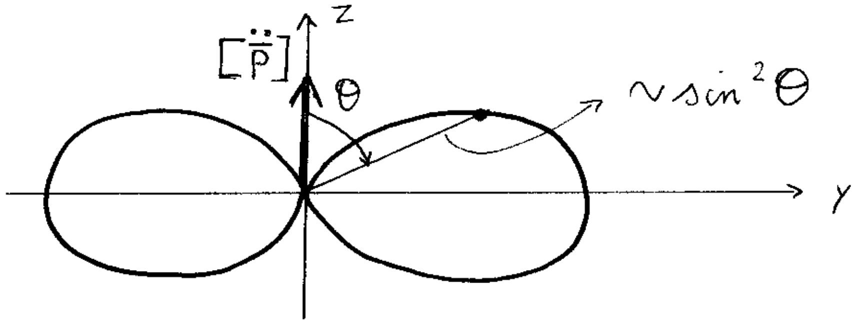
$$P = \int d\Omega \,\frac{dP}{d\Omega} \quad \Longrightarrow \quad \boxed{P_{E1} = \frac{2}{3} \cdot \frac{1}{4\pi c^3} \cdot \left| [\ddot{\bar{p}}] \right|^2}$$

↳ EXAMPLE : $\bar{p}(t) = p_0 \cos(\omega t)\ \bar{e}_z$

$$[\ddot{\bar{p}}] = -p_0 \omega^2 \cos\left[\omega\left(t - \frac{|\bar{x}|}{c}\right)\right] \bar{e}_z$$

$$\left.\frac{dP}{d\Omega}\right|_{E1} = \frac{1}{(4\pi)^2 c^3} p_0^2 \omega^4 \sin^2\theta \cos^2\left[\omega\left(t - \frac{|\bar{x}|}{c}\right)\right]$$

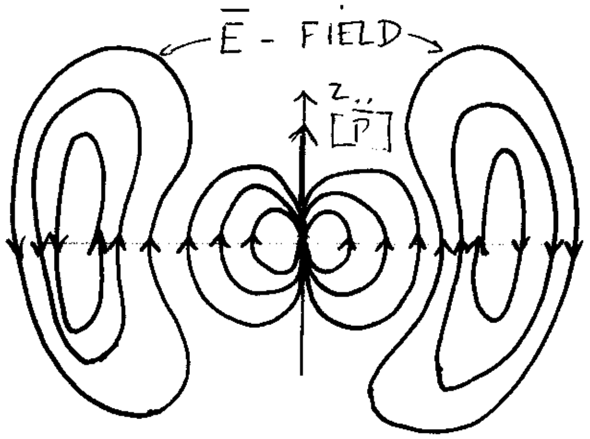
* RADIATION PATTERN (DONUT)



* $\bar{E}$-FIELD LINES ('HERTZ' DIPOLE)

$\boxed{t = 0}$

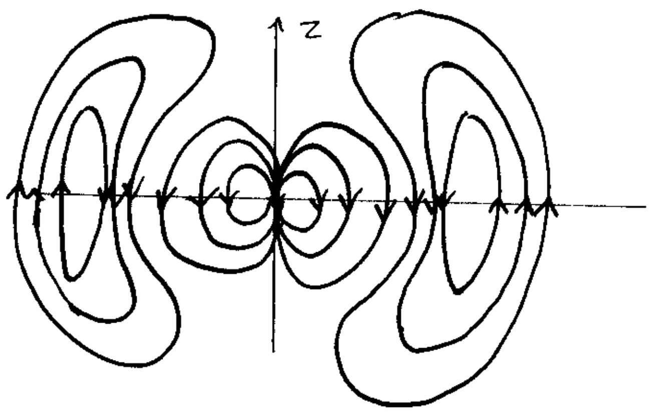
$$\bar{E}_{E1} \sim \cos\left(\omega \frac{|\bar{x}|}{c}\right)$$



$\boxed{t = T/2}$

$$\bar{E}_{E1} \sim \cos\left(\underbrace{\omega \frac{T}{2}}_{\pi} - \omega \frac{|\bar{x}|}{c}\right)$$

$$= -\cos\left(\omega \frac{|\bar{x}|}{c}\right)$$

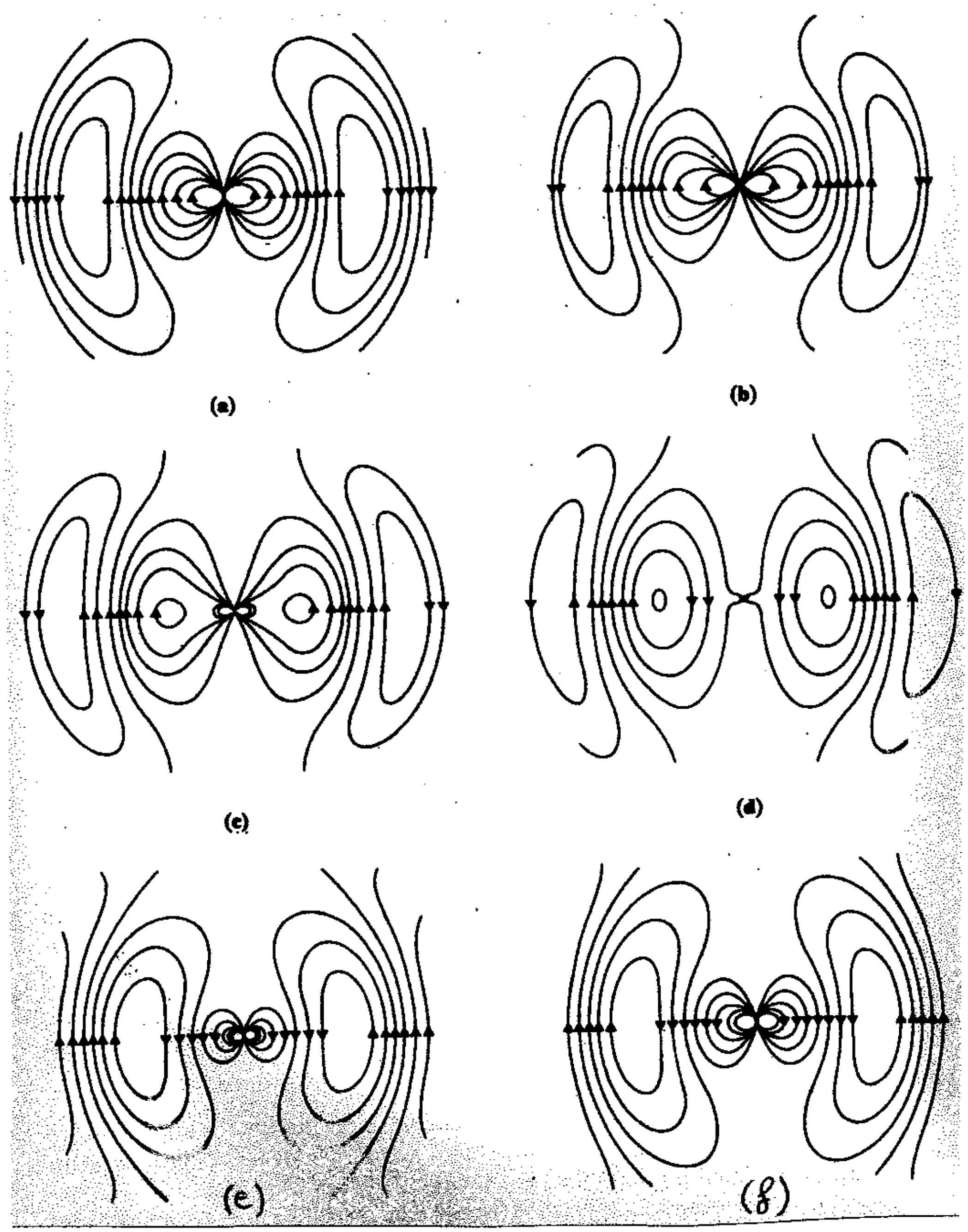


$\bar{E}$ - FIELD LINES : ELECTRIC DIPOLE RADIATION

FROM $t = 0 \longrightarrow$ TILL $t = \frac{T}{2}$

(HALF CYCLE)

Pattern of electric field lines surrounding a radiating electric dipole. The diagrams (*a*)–(*f*) span one half cycle, at intervals of $\frac{1}{12}$ cycle.



* EMITTED POWER

$$P_{E1} = \frac{2}{3} \frac{1}{4\pi c^3} p_0^2 \omega^4 \cos^2 \left[\omega \left(t - \frac{|\vec{x}|}{c} \right) \right]$$

OSCILLATES BETWEEN 0

AND $P_{E1}^{MAX} = \frac{2}{3} \frac{1}{4\pi c^3} p_0^2 \omega^4$

TIME-AVERAGED EMITTED POWER

$$\left\langle \cos^2 \omega \left(t - \frac{|\vec{x}|}{c} \right) \right\rangle = \frac{1}{2}$$

$$\boxed{\langle P_{E1} \rangle = \frac{1}{3} \frac{1}{4\pi c^3} p_0^2 \omega^4}$$

* WHY IS THE SKY BLUE ?

SUNLIGHT $\rightsquigarrow$ ABSORBED & RERADIATED BY ATMOSPHERIC DIPOLES (ATOMS IN ATMOSPHERE)

$\rightsquigarrow$ $P \sim \omega^4$

MORE ENERGY RADIATED FOR BLUE LIGHT THAN FOR RED LIGHT

⟹ MAGNETIC DIPOLE RADIATION

↳ NEXT TERM IN MULTIPOLE EXPANSION FOR RADIATION

$$\sim \int d^3\bar{x}' \, (\hat{x} \cdot \bar{x}') \, \frac{\partial \bar{J}}{\partial t}$$

$$\hat{x} \times (\bar{x}' \times \bar{J}) = (\hat{x} \cdot \bar{J})\, \bar{x}' - (\hat{x} \cdot \bar{x}')\, \bar{J}$$

$$\int d^3\bar{x}' \, (\hat{x} \cdot \bar{x}') \, \frac{\partial \bar{J}}{\partial t}$$

$$= -\frac{1}{2}\, \hat{x} \times \frac{d}{dt} \int d^3\bar{x}' \, (\bar{x}' \times \bar{J}) \quad \Longrightarrow \quad \text{M1}$$

$$+ \frac{1}{2} \int d^3\bar{x}' \left[(\hat{x} \cdot \bar{x}') \frac{\partial \bar{J}}{\partial t} + \left(\hat{x} \cdot \frac{\partial \bar{J}}{\partial t}\right) \bar{x}' \right] \Longrightarrow \text{E2}$$

* $\underline{1^\circ \text{ TERM}}$: MAGNETIC DIPOLE RADIATION (M1)

$$\boxed{\bar{A}_{M1} = -\frac{1}{4\pi c^2 |\bar{x}|} \, \frac{1}{2} \, \hat{x} \times \frac{d}{dt} \int d^3\bar{x}' \, (\bar{x}' \times \bar{J})}$$

* $\underline{2^\circ \text{ TERM}}$: ELECTRIC QUADRUPOLE RADIATION (E2)

$$\boxed{\bar{A}_{E2} = \frac{1}{4\pi c^2 |\bar{x}|} \, \frac{1}{2} \int d^3\bar{x}' \left[(\hat{x} \cdot \bar{x}') \frac{\partial \bar{J}}{\partial t} + \left(\hat{x} \cdot \frac{\partial \bar{J}}{\partial t}\right) \bar{x}' \right]}$$

↳ MAGNETIC DIPOLE MOMENT

$$\boxed{\bar{m}(t) \equiv \frac{1}{2c} \int d^3\bar{x}' \left[\bar{x}' \times \bar{J}(t, \bar{x}') \right]}$$

↳ $$\boxed{\bar{A}_{M1} = -\frac{1}{4\pi c\,|\bar{x}|} \hat{x} \times \left[\frac{d\bar{m}}{dt} \right]_{t - \frac{|\bar{x}|}{c}}}$$

SHORTHAND NOTATION $\bar{A}_{M1} = -\frac{1}{4\pi c\,|\bar{x}|} \hat{x} \times [\dot{\bar{m}}]$

$$\bar{B}_{M1} = -\frac{1}{c} \hat{x} \times \frac{\partial \bar{A}_{M1}}{\partial t}$$

$$\bar{E}_{M1} = -\hat{x} \times \bar{B}_{M1}$$

∴ MAGNETIC DIPOLE RADIATION

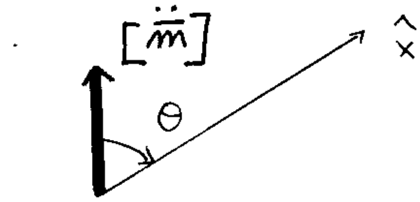
$$\boxed{\begin{aligned} \bar{B}_{M1} &= \frac{1}{4\pi c^2 |\bar{x}|} \hat{x} \times \left(\hat{x} \times [\ddot{\bar{m}}] \right) \\ \bar{E}_{M1} &= \frac{1}{4\pi c^2 |\bar{x}|} \left(\hat{x} \times [\ddot{\bar{m}}] \right) \end{aligned}}$$

[ ] EVALUATED AT $t - \frac{|\bar{x}|}{c}$

↳ RADIATION PATTERN

$$\rightsquigarrow \quad \bar{S}_{M1} = c\, E_{E1}^{2}\, \hat{x}$$

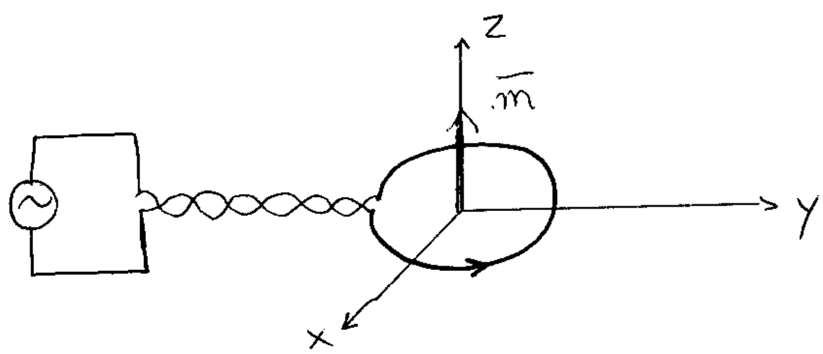
$$\boxed{\bar{S}_{M1} = \frac{1}{(4\pi)^2 c^3}\,\frac{1}{|\bar{x}|^2}\,\sin^2\theta\,\left|[\ddot{\bar{m}}]\right|^2\,\hat{x}}$$



$$\rightsquigarrow \quad \boxed{\left.\frac{dP}{d\Omega}\right|_{M1} = \frac{1}{(4\pi)^2 c^3}\,\sin^2\theta\,\left|[\ddot{\bar{m}}]\right|^2}$$

SAME RADIATION PATTERN AS FOR E1 RADIATION

↳ EXAMPLE : MAGNETIC DIPOLE ANTENNA



$$\bar{m}(t) = m_0 \cos(\omega t)\,\bar{e}_z$$

$$\rightsquigarrow \quad \left.\frac{dP}{d\Omega}\right|_{M1} = \frac{1}{(4\pi)^2 c^3}\, m_0^2\,\omega^4 \sin^2\theta\, \cos^2\left[\omega\left(t - \frac{|\bar{x}|}{c}\right)\right]$$

$\rightsquigarrow$ TIME-AVERAGED EMITTED POWER

$$\boxed{\langle P_{M1}\rangle = \frac{1}{3}\,\frac{1}{4\pi c^3}\, m_0^2\,\omega^4}$$

$\Rightarrow$ ELECTRIC QUADRUPOLE RADIATION

$$\hookrightarrow \quad \bar{A}_{E2} = \frac{1}{4\pi c^2 |\bar{x}|} \frac{1}{2} \int d^3\bar{x}' \left[(\hat{x} \cdot \bar{x}') \frac{\partial \bar{J}}{\partial t} + \left(\hat{x} \cdot \frac{\partial \bar{J}}{\partial t} \right) \bar{x}' \right]$$

$\rightsquigarrow$ USE $\int d^3\bar{x}' \left[(\hat{x} \cdot \bar{x}') \frac{\partial \bar{J}}{\partial t} + \left(\hat{x} \cdot \frac{\partial \bar{J}}{\partial t} \right) \bar{x}' \right]$

$$= - \int d^3\bar{x}' \quad \bar{x}' \, (\hat{x} \cdot \bar{x}') \left(\bar{\nabla}' \cdot \frac{\partial \bar{J}}{\partial t} \right)$$

INTEGRATION
BY PARTS

$\rightsquigarrow$ $\bar{\nabla}' \cdot \bar{J} = - \frac{\partial \rho}{\partial t}$

$\rightsquigarrow$ $\bar{A}_{E2} = \frac{1}{4\pi c^2 |\bar{x}|} \frac{1}{2} \frac{d^2}{dt^2} \int d^3\bar{x}' \; \bar{x}' (\hat{x} \cdot \bar{x}') \rho$

$\rightsquigarrow$ WE CAN MAKE A GAUGE TF.

ADD A TERM TO $\bar{A}$ WHICH DOES NOT
CONTRIBUTE TO PHYSICAL FIELDS $\quad \bar{B} = \bar{\nabla} \times \bar{A}$

$$\bar{B} = - \frac{1}{c} \hat{x} \times \frac{\partial \bar{A}}{\partial t}$$

TERM $\sim$ $\hat{x}$ IN $\bar{A}$ WILL NOT CONTRIBUTE TO $\bar{B}$

$$\therefore \quad \bar{A}_{E2} = \frac{1}{6} \frac{1}{4\pi c^2 |\bar{x}|} \frac{d^2}{dt^2} \int d^3\bar{x}' \left[3\bar{x}' (\hat{x} \cdot \bar{x}') - x'^2 \hat{x} \right] \cdot \rho \left(t - \frac{|x|}{c}, \bar{x}' \right)$$

↳ QUADRUPOLE MOMENT TENSOR

$$\boxed{Q_{ij}(t) \equiv \int d^3\bar{x}' \left[3\, x'_i\, x'_j - x'^2 \delta_{ij}\right] \rho(t, \bar{x}')}$$

INTRODUCE PROJECTION OF Q_{ij} ON VECTOR $\hat{x}$

$$\boxed{\bar{D}_i(t) \equiv Q_{ij}(t)\ \hat{x}_j}$$

$$= \int d^3\bar{x}' \left[3\, x'_i\, (\hat{x}.\bar{x}') - x'^2\, \hat{x}_j\right] \rho(t, \bar{x}')$$

↳ $$\boxed{\bar{A}_{E2}(t, \bar{x}) = \frac{1}{6} \frac{1}{4\pi c^2 |\bar{x}|} \left[\frac{d^2 \bar{D}}{dt^2}\right]_{t - \frac{|\bar{x}|}{c}}}$$

$$\bar{B}_{E2} = -\frac{1}{c}\ \hat{x} \times \frac{\partial \bar{A}_{E2}}{\partial t}$$

$$\boxed{\bar{B}_{E2} = -\frac{1}{6} \frac{1}{4\pi\, c^3 |\bar{x}|}\ \hat{x} \times \left[\dddot{\bar{D}}\right]}$$

↳ RADIATION PATTERN

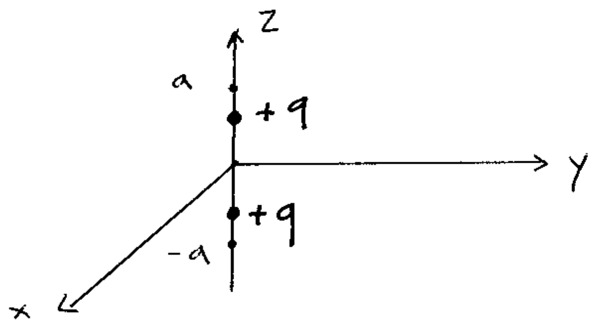
$$\boxed{\left.\frac{dP}{d\Omega}\right|_{E2} = \frac{1}{36} \frac{1}{(4\pi)^2 c^5} \left| \hat{x} \times \left[\dddot{\bar{D}}\right] \right|^2}$$

EXAMPLE : OSCILLATING QUADRUPOLE RADIATOR

2 EQUAL CHARGES OSCILLATING AT EQUAL & OPPOSITE DISTANCE FROM ORIGIN

$z(t) = \pm a \sin \omega t$

↪ POSITION OF CHARGES



$$\rightsquigarrow \quad \rho(t, \bar{x}') = q\, \delta(x')\, \delta(y') \left\{ \delta(z' - a \sin \omega t) + \delta(z' + a \sin \omega t) \right\}$$

$$\rightsquigarrow \quad Q_{11} = Q_{22} = - \int d^3\bar{x}'\, x'^2 \rho$$

$$= -2q\, a^2 \sin^2 \omega t$$

$$Q_{33} = \int d^3\bar{x}' (2z'^2 - x'^2 - y'^2)\, \rho$$

$$= 4q\, a^2 \sin^2 \omega t$$

$$Q_{12} = Q_{13} = Q_{23} = 0$$

$$\rightsquigarrow \quad \boxed{\bar{D}(t) = -2q\, a^2 \sin^2 \omega t \left(\hat{x}_1, \hat{x}_2, -2\hat{x}_3 \right)}$$

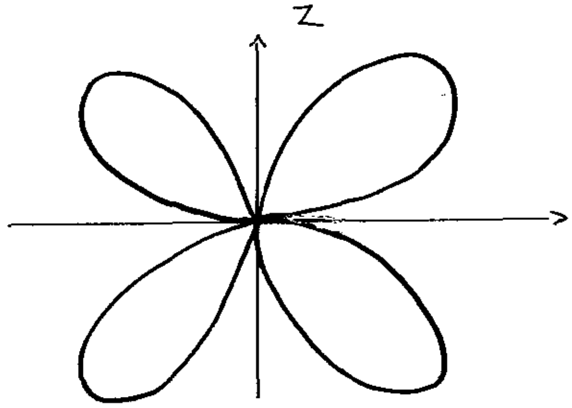
$\rightsquigarrow$ USING SPHERICAL COORDINATES

$$\hat{x}_1 = \sin\theta \cos\phi$$
$$\hat{x}_2 = \sin\theta \sin\phi$$
$$\hat{x}_3 = \cos\theta$$

$$\boxed{[\dddot{D}] = 8\, q\, a^2 \omega^3 \sin\left[2\omega\left(t - \frac{|\bar{x}|}{c}\right)\right] \cdot \left\{\sin\theta\cos\phi\, \bar{e}_x + \sin\theta\sin\phi\, \bar{e}_y - 2\cos\theta\, \bar{e}_z\right\}}$$

$$\hat{x} \times [\dddot{D}] = +\, 12\, q\, a^2 \omega^3 \sin^2\left[2\omega\left(t - \frac{|\bar{x}|}{c}\right)\right] \sin 2\theta \cdot \left(-\sin\phi\, \bar{e}_x + \cos\phi\, \bar{e}_y\right)$$

$$\boxed{\left.\frac{dP}{d\Omega}\right|_{E2} = \frac{q^2 a^4 \omega^6}{(4\pi)\,\pi c^5} \cdot \sin^2 2\theta \, \sin^2\left[2\omega\left(t - \frac{|\bar{x}|}{c}\right)\right]}$$



$$\rightsquigarrow \langle P_{E2} \rangle = \frac{q^2 a^4 \omega^6}{(4\pi)\,\pi c^5}\, 2\pi \underbrace{\int_0^{\pi} d\theta \sin\theta \sin^2 2\theta}_{16/15} \cdot \underbrace{\langle \sin^2[\cdots] \rangle}_{\frac{1}{2}}$$

$$\boxed{\langle P_{E2} \rangle = \frac{16}{15}\left(\frac{q^2}{4\pi}\right)\frac{a^4\omega^6}{c^5}}$$