IV

ELECTROMAGNETIC WAVES

1) ELECTROMAGNETIC WAVES IN VACUUM
2) ELECTROMAGNETIC WAVES IN MATTER (NON- CONDUCTING MEDIA)
3) ELECTROMAGNETIC WAVES IN CONDUCTORS
4) WAVE GUIDES & RESONANT CAVITIES

1) ELECTROMAGNETIC WAVES IN VACUUM

⇒ WAVE EQUATION

↳ IN ELECTROSTATICS, MAGNETOSTATICS $\dot{\bar{E}} = 0$, $\dot{\bar{B}} = 0$

NOW

$$\begin{aligned} \bar{\nabla} \cdot \bar{E} &= 0 \\ \bar{\nabla} \cdot \bar{B} &= 0 \\ \bar{\nabla} \times \bar{E} &= -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \\ \bar{\nabla} \times \bar{B} &= \frac{1}{c}\frac{\partial \bar{E}}{\partial t} \end{aligned}$$

↳ $$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$$

$$\Downarrow$$

$$\bar{\nabla}\,(\underbrace{\cancel{\bar{\nabla} \cdot \bar{E}}}_{0}) - \nabla^2 \bar{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{E}$$

$$\frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} - \nabla^2 \bar{E} = 0$$

$$\Box \bar{E} = 0$$

↳ d'ALEMBERT OPERATOR

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

↳ $$\bar{\nabla} \times (\bar{\nabla} \times \bar{B}) = \frac{1}{c}\frac{\partial}{\partial t}\bar{\nabla} \times \bar{E}$$

$$\Downarrow$$

$$\bar{\nabla}(\underbrace{\cancel{\bar{\nabla} \cdot \bar{B}}}_{0}) - \nabla^2 \bar{B} = -\frac{1}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2}$$

$$\frac{1}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2} - \nabla^2 \bar{B} = 0$$

↳ FOR POTENTIAL

$$\partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = 0$$

$$\Box A^\nu - \partial^\nu \left(\partial_\mu A^\mu \right) = 0$$

⇝ GAUGE CONDITION (COULOMB GAUGE) $\begin{cases} \phi = 0 \\ \bar{\nabla} . \bar{A} = 0 \end{cases}$

THIS IS SPECIAL CASE OF LORENTZ GAUGE

$$\partial_\mu A^\mu = 0$$

⇝ $\underline{\underline{\Box A^\nu = 0}} \iff \underline{\underline{\frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} = 0}}$

⇝ $\bar{E}$ & $\bar{B}$ FIELDS FROM

$$\bar{E} = - \frac{1}{c} \frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

⇒ MONOCHROMATIC PLANE WAVES

↳ GENERAL SOLUTION

SPECIAL CASE : ONE SPATIAL COORDINATE Z

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial z^2} = 0$$

f STANDS FOR COMPONENT OF $\bar{E}, \bar{B}$, OR $\bar{A}$

$$\Longleftrightarrow \quad \left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) f = 0$$

⇝
$$\begin{cases} \eta = z - ct \\ \xi = z + ct \end{cases}$$

$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)$$

⇝ EQUATION FOR f

$$\frac{\partial^2}{\partial \eta\, \partial \xi} f = 0$$

⇝ SOLUTIONS
$$f = f_1(\eta) + f_2(\xi)$$
$$= f_1(z - ct) + f_2(z + ct)$$

ARBITRARY FUNCTIONS

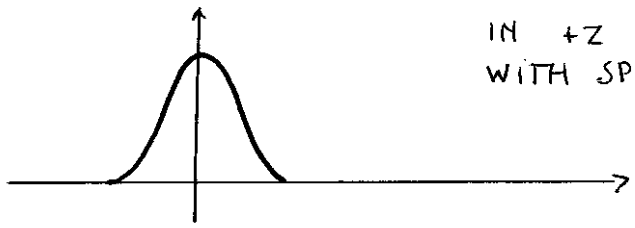
↳ INTERPRETATION

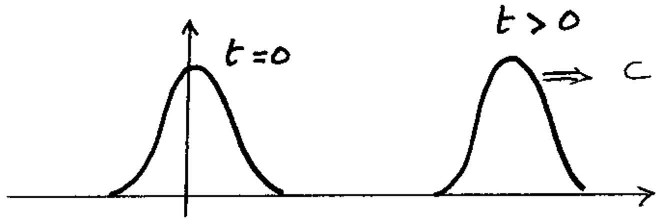
⇝ e.g. $f_1(z-ct)$

SOLUTION HAS SAME VALUE FOR $z = ct +$ CONSTANT

⇓

PROPAGATING WAVES
IN +z DIRECTION
WITH SPEED c (SPEED OF LIGHT)





⇝ ANALOGOUS $f_2(z+ct)$

PROPAGATING WAVE IN −z DIRECTION
WITH SPEED c

↳ MONOCHROMATIC PLANE WAVES

⇝ $f(z,t) = A\cos(kz-\omega t) + B\cos(kz+\omega t)$

k : WAVE NUMBER

ω : ANGULAR FREQUENCY ⇝ WAVES OF 1 FREQUENCY: MONOCHROMATIC

WAVE EQ ⇒ SPEED OF WAVE $c = \dfrac{\omega}{k}$

⇝ INSTEAD OF WORKING WITH cos, sin

WE CAN WORK WITH $e^{i\alpha} = \cos\alpha + i \sin\alpha$

$$f(z,t) = C\, e^{i(kz - \omega t)} + D\, e^{-i(kz + \omega t)}$$

AT END WE CAN TAKE REAL PART

⇝ WAVE TRAVELING ALONG A DIRECTION $\bar{n}$

$$\underline{\underline{\bar{k} \equiv \frac{\omega}{c}\, \bar{n}}}$$

$\bar{k}$: WAVE VECTOR

$$f(\bar{x},t) = C'\, e^{i(\bar{k}.\bar{x} - \omega t)}$$

⇝ FOR $\bar{A}$, $\bar{E}$, $\bar{B}$

$$\bar{A} = \mathrm{Re}\, \bar{A}_0\, e^{i(\bar{k}.\bar{x} - \omega t)}$$

$$\bar{E} = \mathrm{Re}\, \bar{E}_0\, e^{i(\bar{k}.\bar{x} - \omega t)}$$

$$\bar{B} = \mathrm{Re}\, \bar{B}_0\, e^{i(\bar{k}.\bar{x} - \omega t)}$$

$\bar{A}_0$, $\bar{E}_0$, $\bar{B}_0$ ARE CONSTANT VECTORS.

WILL NOT WRITE Re (REAL PART) EXPLICITELY

IN FOLLOWING

⇒ POLARIZATION

$$\bar{A} = \bar{A}_0 \, e^{i(\bar{k}\cdot\bar{x} - \omega t)}$$

↳ $$\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} = \frac{i\omega}{c}\bar{A} = i k \bar{A}$$

$$\bar{B} = \bar{\nabla} \times \bar{A} = + i \bar{k} \times \bar{A}$$

⇝ $$\underline{\underline{\bar{B} = \bar{n} \times \bar{E}}} \qquad (\bar{k} = k\bar{n})$$

↳ $\bar{\nabla}.\bar{E} = 0 \implies \bar{n}\cdot\bar{A} = 0$

OR

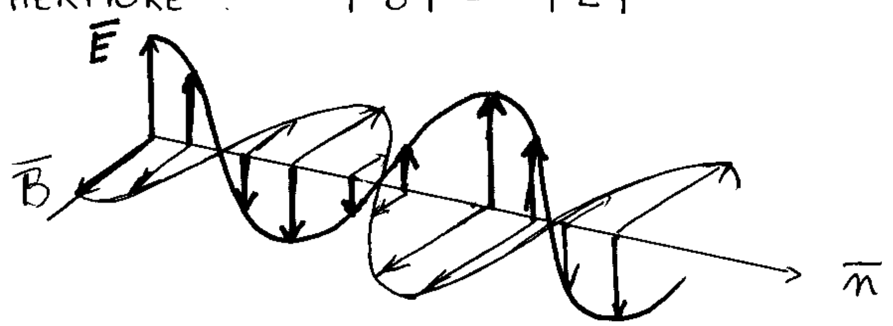
$$\underline{\underline{\bar{n}.\bar{E} = 0}}$$

↳ BOTH $\bar{E}$ & $\bar{B}$ ARE PERPENDICULAR TO DIRECTION OF PROPAGATION $\bar{n}$

<u>AND</u> ARE PERPENDICULAR TO EACH OTHER $(\bar{E} \perp \bar{B})$

$\bar{E}$ & $\bar{B}$ ARE <u>TRANSVERSE WAVES</u>

FURTHERMORE : $|\bar{B}| = |\bar{E}|$



↳ LINEAR VS ELLIPTICAL POLARIZATION

$\bar{E}$ & $\bar{B}$ HAVE 2 POSSIBLE DIRECTIONS $\perp$ TO $\bar{n}$

e.g. $\bar{E} = \bar{E}_0 \, e^{i(\bar{k}\cdot\bar{x} - \omega t)}$

$$\bar{E}_0 = E_1 \hat{e}_1 + E_2 \hat{e}_2$$

⇝ $\hat{e}_1, \hat{e}_2$: 2 UNIT VECTORS $\perp$ TO $\hat{n}$

e.g. FOR $\hat{n} = \hat{e}_z$ (WAVE ALONG z-DIRECTION)

$$\hat{e}_1 = \hat{e}_x$$
$$\hat{e}_2 = \hat{e}_y$$

⇝ E_1, E_2 ARE IN GENERAL COMPLEX NUMBERS.

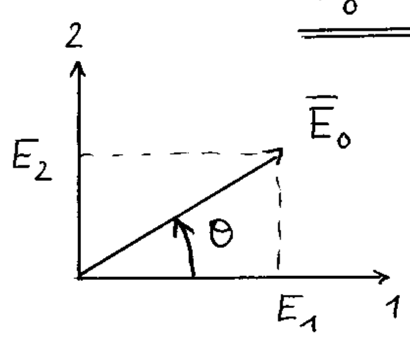
1) LINEAR POLARIZATION

⇝ $E_1 = 0$ ↷ POLARIZATION ALONG $\hat{e}_2$
(i.e. $\bar{E}_0 = E_2 \hat{e}_2$)

⇝ $E_2 = 0$ ↷ POLARIZATION ALONG $\hat{e}_1$
(i.e. $\bar{E}_0 = E_1 \hat{e}_1$)

⇝ E_1 & E_2 HAVE SAME PHASE

$$\underline{\bar{E}_0 = E_0 \left(\cos\theta \, \hat{e}_1 + \sin\theta \, \hat{e}_2 \right)}$$



$$\begin{cases} E_1 = E_0 \cos\theta \\ E_2 = E_0 \sin\theta \end{cases}$$

$$E_0 = \sqrt{E_1^2 + E_2^2}$$

2) ELLIPTICAL POLARIZATION

E_1 & E_2 HAVE DIFFERENT PHASES

⇝ CIRCULAR POLARIZATION (SPECIAL CASE)

$$\bar{E}_0 = E_0 \left(\hat{e}_1 \pm i \hat{e}_2 \right)$$

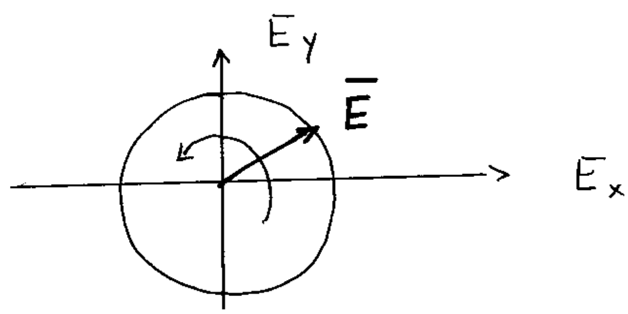
↓

$$\bar{E} = E_0 \left(\hat{e}_1 \pm i \hat{e}_2 \right) e^{i(\bar{k}\cdot\bar{x} - \omega t)}$$

(E_0 REAL)

FOR $\bar{n} = \hat{e}_z$ COMPONENTS OF $\bar{E}$ (TAKE REAL PART)

$$\begin{cases} E_x = E_0 \cos (kz - \omega t) \\ E_y = \mp E_0 \sin (kz - \omega t) \end{cases}$$



VECTOR $\bar{E}$ SWEEPS AROUND CIRCLE WITH ANGULAR FREQUENCY ω

$$\bar{\mathcal{E}}_\pm \equiv \frac{1}{\sqrt{2}} \left(\hat{e}_1 \pm i \hat{e}_2 \right)$$

$\bar{\mathcal{E}}_+$: COUNTERCLOCKWISE ROTATION : RIGHT CIRCULAR POL.

$\bar{\mathcal{E}}_-$: CLOCKWISE ROTATION : LEFT CIRCULAR POL.

~> **GENERAL CASE** OF ELLIPTICAL POLARIZATION

$$\overline{E}_0 = (\overline{a}_1 \pm i\,\overline{a}_2)\, e^{-i\alpha}$$

$$\overline{a} \equiv \overline{a}_1 + i\,\overline{a}_2 \qquad \overline{a}_1,\ \overline{a}_2 \ \text{REAL}$$

$\overline{a}^2$ REAL

$$a^2 = a_1^2 \pm 2i\,\overline{a}_1 \cdot \overline{a}_2 - a_2^2$$

$$\overline{a}_1 \cdot \overline{a}_2 = 0 \implies \overline{a}_1 \perp \overline{a}_2$$

FOR $\overline{n} = \hat{e}_z$

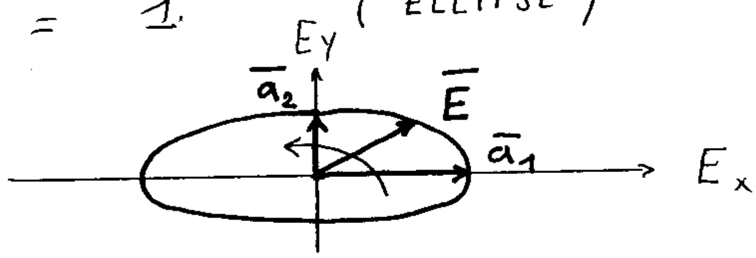
$\overline{a}_1 \parallel \hat{e}_x$

$\overline{a}_2 \parallel \hat{e}_y$

$$\overline{E} = \mathrm{Re}\left(\overline{a}\, e^{i(kz - \omega t - \alpha)}\right)$$

$$\begin{cases} E_x = a_1 \cos(kz - \omega t - \alpha) \\ E_y = \mp a_2 \sin(kz - \omega t - \alpha) \\ E_z = 0 \end{cases}$$

$$\frac{E_x^2}{a_1^2} + \frac{E_y^2}{a_2^2} = 1 \qquad \text{(ELLIPSE)}$$



$\overline{E}$ - FIELD ROTATES (WITH ω) ON ELLIPSE

⇒ ENERGY - MOMENTUM OF PLANE WAVES

↳ POINTING VECTOR

$$\bar{S} = c\,(\bar{E}\times\bar{B})$$

$$\left(\frac{1}{c} S^i = \Theta^{0i} \right)$$

$$= c\ \bar{E}\times(\bar{n}\times\bar{E})$$

$$= c\left(E^2\,\bar{n} - \underbrace{(\bar{E}\cdot\bar{n})}_{0}\,\bar{E}\right)$$

ENERGY IS TRANSPORTED IN DIRECTION $\bar{n}$

↳ ENERGY DENSITY

$$U = \frac{1}{2}\left(\bar{E}^2 + \bar{B}^2\right)$$

$|\bar{E}| = |\bar{B}|$ FOR PLANE WAVES

$$= \bar{E}^2$$

↳ MOMENTUM DENSITY

$$\bar{P} = \frac{1}{c}\left(\bar{E}\times\bar{B}\right) = \frac{1}{c^2}\,\bar{S} = \frac{1}{c}U\,\bar{n}$$

↳ ENERGY - MOMENTUM RELATION FOR MASSLESS PARTICLES

$$c\,|\bar{P}| = E$$

IN QUANTUM THEORY : PHOTONS (MASSLESS) ARE QUANTA OF E.M. FIELD

↳ $k^\mu = \left(\frac{\omega}{c}, \bar{k}\right),\ k^2 = 0$ ‖ $E = \hbar\omega \qquad \bar{P} = \hbar\bar{k} \qquad p^\mu = \hbar k^\mu$

2) ELECTROMAGNETIC WAVES IN MATTER

⇒ PROPAGATION IN LINEAR MEDIA

↳ IN MATTER, IN REGIONS WHERE $\rho = 0$, $\bar{J} = 0$

(NO FREE CHARGES, NO FREE CURRENTS)

$$\boxed{\begin{array}{ll} \bar{\nabla}\cdot\bar{D} = 0 & \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} \\ \bar{\nabla}\cdot\bar{B} = 0 & \bar{\nabla}\times\bar{H} = \frac{1}{c}\frac{\partial\bar{D}}{\partial t} \end{array}}$$

↳ NOTE 1) $\bar{\nabla}\cdot\bar{E} = -\underset{\substack{\Uparrow \\ \text{POLARIZATION CHARGE}}}{\bar{\nabla}\cdot\bar{P}}$ $\Rightarrow$ $\bar{D} = \bar{E} + \bar{P}$

2) $\bar{\nabla}\times\bar{B} = \frac{1}{c}\frac{\partial\bar{E}}{\partial t} + \frac{1}{c}\underbrace{\left(c\,\bar{\nabla}\times\bar{M}\right)}_{\substack{\Updownarrow \\ \text{MAGNETIZATION CURRENT}}} + \frac{1}{c}\underset{\substack{\Uparrow \\ \text{POLARIZATION CURRENT}}}{\frac{\partial\bar{P}}{\partial t}}$

$$\bar{H} = \bar{B} - \bar{M}$$

THIS EQ. GIVES $\bar{\nabla}\times\bar{H} = \frac{1}{c}\frac{\partial\bar{D}}{\partial t}$

↳ IN LINEAR MEDIA

$$\begin{array}{l} \bar{D} = \varepsilon\bar{E} \\ \bar{B} = \mu\bar{H} \end{array} \Rightarrow \left\| \begin{array}{ll} \bar{\nabla}\cdot\bar{E} = 0\,, & \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} \\ \bar{\nabla}\cdot\bar{B} = 0\,, & \bar{\nabla}\times\bar{B} = \mu\varepsilon\frac{1}{c}\frac{\partial\bar{E}}{\partial t} \end{array} \right.$$

↳ INDEX OF REFRACTION

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\frac{1}{c} \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$$

$$\bar{\nabla} (\underbrace{\cancel{\bar{\nabla}} \cdot \bar{E}}_{0}) - \nabla^2 \bar{E} = -\frac{\mu \varepsilon}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\boxed{\frac{\mu \varepsilon}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} - \nabla^2 \bar{E} = 0}$$

SOLUTIONS $\bar{E} = \bar{E}_0 \cos(\omega t - \bar{k} . \bar{x})$

WITH $-\frac{\mu \varepsilon}{c^2} \omega^2 + \bar{k}^2 = 0$

SPEED OF WAVE $v = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon \mu}}$

INDEX OF REFRACTION $n \equiv \sqrt{\varepsilon \mu}$ $\Rightarrow$ $v = \frac{c}{n}$

FOR MOST MATERIALS : $\mu \simeq 1.$

$n \simeq \sqrt{\varepsilon} > 1$

$v < c$

LIGHT PROPAGATES SLOWER THROUGH MATTER THAN THROUGH VACUUM

↳ SOLUTIONS

$$\bar{E} = \bar{E}_0 \, e^{-i(\omega t - \bar{k}\cdot\bar{x})}$$

↳ CAN HAVE LINEAR, CIRCULAR, OR ELLIPTICAL POLARIZATION

$$\bar{B} = \bar{B}_0 \, e^{-i(\omega t - \bar{k}\cdot\bar{x})}$$

$$\rightsquigarrow \quad \bar{\nabla} \times \bar{B} = i\left(\bar{k} \times \bar{B}_0\right) e^{-i(\omega t - \bar{k}\cdot\bar{x})}$$

$$\rightsquigarrow \quad \frac{1}{c}\frac{\partial \bar{E}}{\partial t} = -i \underbrace{\left(\frac{\omega}{c}\right)}_{\frac{k}{n}} \bar{E}_0 \, e^{-i(\omega t - \bar{k}\cdot\bar{x})}$$

$$\bar{\nabla} \times \bar{B} = n^2 \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$$

$$\Downarrow$$

$$\bar{B}_0 = n\left(\hat{k} \times \bar{E}_0\right)$$

$$\bar{E}_0 \cdot \hat{k} = 0$$
$$\bar{B}_0 \cdot \hat{k} = 0$$

TRANSVERSE WAVES

MAGNITUDES OF $\bar{E}$ & $\bar{B}$ FIELDS DIFFER BY FACTOR n

↳ ENERGY DENSITY / POINTING VECTOR

$$\Rightarrow \quad U = \frac{1}{2}\left(\bar{E}.\bar{D} + \bar{H}.\bar{B}\right)$$

$$= \frac{1}{2}\left(\varepsilon \bar{E}^2 + \frac{1}{\mu}\bar{B}^2\right)$$

$\Rightarrow$ POINTING VECTOR : ENERGY FLUX IN MEDIUM

$$\frac{d}{dt}\int d^3\bar{x}\; U = \frac{d}{dt}\frac{1}{2}\int d^3\bar{x}\left(\varepsilon \bar{E}^2 + \frac{1}{\mu}\bar{B}^2\right)$$

$$= \int d^3\bar{x}\left(\varepsilon \bar{E}.\frac{\partial \bar{E}}{\partial t} + \frac{1}{\mu}\bar{B}.\frac{\partial \bar{B}}{\partial t}\right)$$

$$= c\int d^3\bar{x}\left\{\bar{E}.(\bar{\nabla}\times\bar{H}) - \bar{H}.(\bar{\nabla}\times\bar{E})\right\}$$

NOTE
$$\bar{\nabla}.(\bar{E}\times\bar{H}) = \varepsilon_{ijk}\nabla^i(E^j H^k)$$

$$= \varepsilon_{ijk} E^j \nabla^i H^k + \varepsilon_{ijk} H^k \nabla^i E^j$$

$$= -\bar{E}.(\bar{\nabla}\times\bar{H}) + \bar{H}.(\bar{\nabla}\times\bar{E})$$

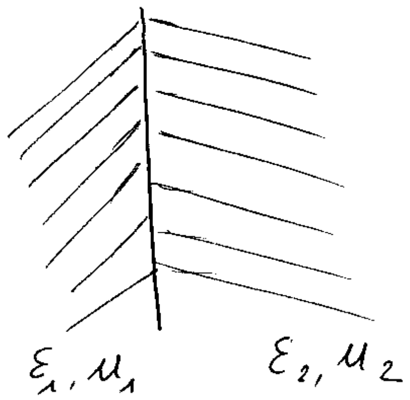
$$\frac{d}{dt}\int d^3\bar{x}\; U = -c\int d^3\bar{x}\;\bar{\nabla}.(\bar{E}\times\bar{H})$$

$$\int d^3\bar{x}\left[\frac{\partial U}{\partial t} + c\bar{\nabla}.(\bar{E}\times\bar{H})\right] = 0$$

POINTING VECTOR IN MEDIUM
$$\underline{\underline{\bar{S} = c\,(\bar{E}\times\bar{H})}}$$

$$= \frac{c}{\mu}(\bar{E}\times\bar{B})$$

⇒ BOUNDARY CONDITIONS IN MATTER



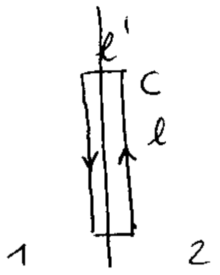
WAVE MOVES FROM ONE TRANSPARENT MEDIUM (1) TO ANOTHER TRANSPARENT MEDIUM (2)

BOUNDARY CONDITIONS ?

↳ $\bar{\nabla} . \bar{D} = 0 \quad \Rightarrow \quad D_{\perp 1} = D_{\perp 2}$ (COMPONENTS PERPENDICULAR TO SURFACE)

$$\underline{\underline{\varepsilon_1 E_{\perp 1} = \varepsilon_2 E_{\perp 2}}}$$

↳ $\bar{\nabla} \times \bar{E} = - \frac{1}{c} \frac{\partial \bar{B}}{\partial t}$



$$\oint_C d\bar{\ell} . \bar{E} = - \frac{1}{c} \frac{\partial}{\partial t} \int d\bar{S} . \bar{B}$$

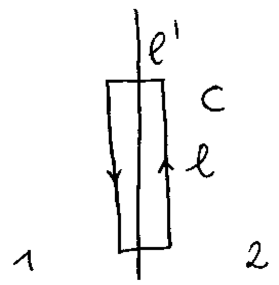
$$\ell . \left(E_{\parallel 2} - E_{\parallel 1} \right) = - \frac{1}{c} \frac{\partial}{\partial t} \underset{\substack{\downarrow \\ 0}}{(\text{FLUX THROUGH LOOP})}$$

IN LIMIT $\ell' \to 0$

$$\underline{\underline{E_{\parallel 1} = E_{\parallel 2}}}$$

$\hookrightarrow \quad \bar{\nabla} \cdot \bar{B} = 0 \quad \Rightarrow \quad \underline{\underline{B_{\perp 1} = B_{\perp 2}}}$

$\hookrightarrow \quad \bar{\nabla} \times \bar{H} = \dfrac{1}{c} \dfrac{\partial \bar{D}}{\partial t}$



$$\oint_C d\bar{l} \cdot \bar{H} = \frac{1}{c} \frac{\partial}{\partial t} \int d\bar{S} \cdot \bar{D}$$

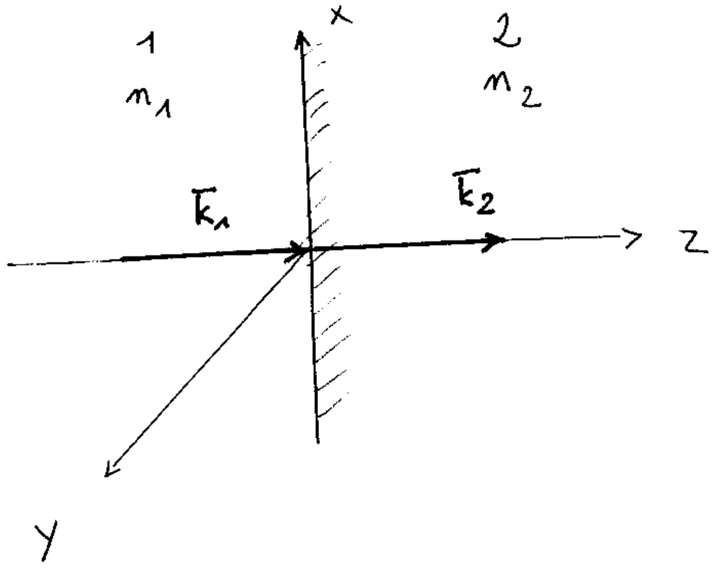
$$\downarrow \quad l' \to 0$$

$$H_{\parallel 2} = H_{\parallel 1}$$

$$\underline{\underline{\frac{1}{\mu_1} B_{\parallel 1} = \frac{1}{\mu_2} B_{\parallel 2}}}$$

$\Rightarrow$ REFLECTION & TRANSMISSION :

NORMAL INCIDENCE



WAVE TRAVELING ALONG z FROM 1 TO 2

PERPENDICULAR TO BOUNDARY BETWEEN 2 MEDIA

INCIDENT WAVE GIVES RISE TO REFLECTED & TRANSMITTED WAVES

↳ INCIDENT WAVE I IN 1

$$\bar{E}_I = E_{0I} \, e^{i(k_1 z - \omega t)} \, \bar{e}_x$$

↓

LIN. POL. ALONG x- AXIS.

$$\bar{B}_I = n_1 \left(\bar{e}_z \times \bar{E}_I \right)$$

$$= \frac{c}{v_1} E_{0I} \, e^{i(k_1 z - \omega t)} \, \bar{e}_y$$

↳ REFLECTED WAVE R IN 1 : TRAVELING IN -Z DIRECTION

$$\bar{E}_R = E_{0R} \, e^{i(-k_1 z - \omega t)} \, \bar{e}_x$$

$$\bar{B}_R = n_1 \left((-\bar{e}_z) \times \bar{E}_R \right)$$

$$= - \frac{c}{v_1} E_{0R} \, e^{i(-k_1 z - \omega t)} \, \bar{e}_y$$

↳ TRANSMITTED WAVE T IN 2 : TRAVELING IN +Z DIRECTION

$$\bar{E}_T = E_{0T} \, e^{i(k_2 z - \omega t)} \, \bar{e}_x$$

$$\bar{B}_T = \frac{c}{v_2} E_{0T} \, e^{i(k_2 z - \omega t)} \, \bar{e}_y$$

BOUNDARY CONDITIONS AT $z = 0$

ON LEFT $\bar{E}_I + \bar{E}_R \quad , \quad \bar{B}_I + \bar{B}_R$

ON RIGHT $\bar{E}_T \quad , \quad \bar{B}_T$

$$\rightsquigarrow \quad \varepsilon_1 E_{\perp 1} = \varepsilon_2 E_{\perp 2} = 0 \quad \text{TRIVIAL}$$

$$B_{\perp 1} = B_{\perp 2} = 0 \quad \text{TRIVIAL}$$

$$\rightsquigarrow \quad E_{\parallel 1} = E_{\parallel 2}$$

$$\Downarrow$$

$$E_{0I} + E_{0R} = E_{0T}$$

$$\rightsquigarrow \quad \frac{1}{\mu_1} B_{\parallel 1} = \frac{1}{\mu_2} B_{\parallel 2}$$

$$\Downarrow$$

$$\frac{1}{\mu_1} \frac{c}{v_1} \left(E_{0I} - E_{0R} \right) = \frac{1}{\mu_2} \frac{c}{v_2} E_{0T}$$

$$\downarrow$$

$$E_{0I} - E_{0R} = \frac{\mu_1 n_2}{\mu_2 n_1} E_{0T}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$\therefore \quad E_{0I} - E_{0R} = \beta E_{0T}$$

$$\begin{cases} E_{0I} + E_{0R} = E_{0T} \\ E_{0I} - E_{0R} = \beta E_{0T} \end{cases}$$

$$\Downarrow$$

$$E_{0R} = \left(\frac{1-\beta}{1+\beta} \right) E_{0I}$$

$$E_{0T} = \left(\frac{2}{1+\beta} \right) E_{0I}$$

$\rightsquigarrow$ FOR SPECIAL CASE $\mu_1 = \mu_2 = 1$ $\Rightarrow$ $\beta = \frac{v_1}{v_2}$
(GOOD APPROXIMATION IN MANY CASES)

$$E_{0R} = \left(\frac{v_2 - v_1}{v_2 + v_1} \right) E_{0I}$$

$$E_{0T} = \left(\frac{2 v_2}{v_2 + v_1} \right) E_{0I}$$

<u>REAL</u> AMPLITUDES IN TERMS OF INDICES OF REFRACTION

$$E_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0I}$$

$$E_{0T} = \left(\frac{2 n_1}{n_1 + n_2} \right) E_{0I}$$

⇝ INTENSITY OF WAVE

$I \equiv \langle S \rangle$ AVERAGE POWER TRANSPORTED PER UNIT OF SURFACE

$\uparrow$ POINTING VECTOR

$$\bar{S} = \frac{c}{\mu} \left(\bar{E} \times \bar{B} \right)$$

$$B_0 = n E_0$$

$$\langle \cos^2 \rangle = \frac{1}{2}$$

$$I = \frac{1}{2} \frac{c}{\mu} n E_0^2$$

$\downarrow$ $c = n v$, $n^2 = \varepsilon \mu$

$$\underline{\underline{I = \frac{1}{2} \varepsilon v E_0^2}}$$

⇝ REFLECTION & TRANSMISSION COEFF.

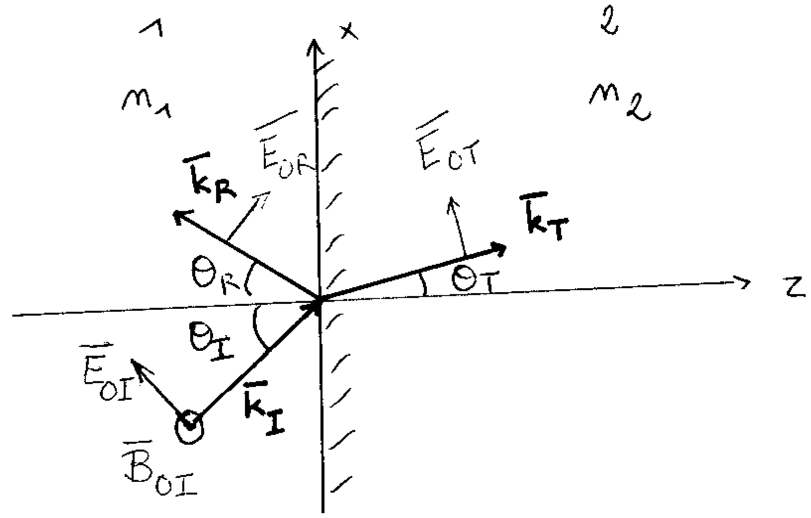
$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \left(\frac{E_{0T}}{E_{0I}} \right)^2 = \frac{n_2}{n_1} \frac{4 n_1^2}{(n_1 + n_2)^2}$$

$$= \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

NOTE $R + T \overset{!}{=} 1$ e.g. $\left.\begin{array}{ll} \text{AIR} & n_1 = 1 \\ \text{GLASS} & n_2 = 1.5 \end{array}\right\}$ $R = 0.04$, $T = 0.96$

⇒ REFLECTION & TRANSMISSION : OBLIQUE INCIDENCE



↳ INCIDENT WAVE I in 1

$$\bar{E}_I = \bar{E}_{0I}\, e^{i(\bar{k}_I \cdot \bar{x} - \omega t)}$$

$$\bar{B}_I = n_1 \left(\hat{k}_I \times \bar{E}_I \right)$$

↳ REFLECTED WAVE R in 1

$$\bar{E}_R = \bar{E}_{0R}\, e^{i(\bar{k}_R \cdot \bar{x} - \omega t)}$$

$$\bar{B}_R = n_1 \left(\hat{k}_R \times \bar{E}_R \right)$$

$$|\bar{k}_I| = |\bar{k}_R| = n_1 \frac{\omega}{c}$$

↳ TRANSMITTED WAVE T in 2

$$\bar{E}_T = \bar{E}_{0T}\, e^{i(\bar{k}_T \cdot \bar{x} - \omega t)}$$

$$\bar{B}_T = n_2 \left(\hat{k}_T \times \bar{E}_T \right)$$

$$|\bar{k}_T| = n_2 \frac{\omega}{c} = \frac{n_2}{n_1} |\bar{k}_I|$$

↳ BOUNDARY CONDITIONS AT $z = 0$

ON LEFT $\bar{E}_I + \bar{E}_R$, $\bar{B}_I + \bar{B}_R$

ON RIGHT $\bar{E}_T$, $\bar{B}_T$

⇝ EXPONENTIAL FACTORS MUST MATCH AT $z = 0$!

$$\bar{k}_I \cdot \bar{x} = \bar{k}_R \cdot \bar{x} = \bar{k}_T \cdot \bar{x} \qquad \text{FOR } z = 0 \quad \forall\, x, y$$

$$\Downarrow$$

$$(k_I)_x = (k_R)_x = (k_T)_x$$
$$(k_I)_y = (k_R)_y = (k_T)_y$$

⇝ e.g. if $(k_I)_y = 0$ $\qquad \bar{k}_I$ LIES IN XZ - PLANE

$$\Downarrow$$

$(k_R)_y = (\bar{k}_T)_y = 0 \rightarrow \bar{k}_R, \bar{k}_T$ ALSO LIE IN XZ-PLANE

1° LAW : INCIDENT, REFLECTED & TRANSMITTED WAVE VECTORS FORM A PLANE (PLANE OF INCIDENCE).

⇝
$$(k_I)_x = |\bar{k}_I| \sin \theta_I$$
$$(k_R)_x = |\bar{k}_R| \sin \theta_R = |\bar{k}_I| \sin \theta_R$$
$$(k_T)_x = |\bar{k}_T| \sin \theta_T = \frac{n_2}{n_1} |\bar{k}_I| \sin \theta_T$$

$$(k_I)_x = (k_R)_x \Longrightarrow \sin \theta_I = \sin \theta_R$$

2° LAW : $\underline{\underline{\theta_I = \theta_R}}$ ANGLE OF INCIDENCE = ANGLE OF REFLECTION

LAW OF REFLECTION

$$\rightsquigarrow \quad (k_I)_x = (k_T)_x$$

$$\Downarrow$$

3° LAW $\quad n_1 \sin \theta_I = n_2 \sin \theta_T$

LAW OF REFRACTION (SNELL'S LAW)

THESE ARE 3 LAWS OF GEOMETRICAL OPTICS

$\rightsquigarrow$ BOUNDARY CONDITIONS FOR FIELDS

1) $\mathcal{D}_{\perp 1} = \mathcal{D}_{\perp 2}$

$$\varepsilon_1 (\bar{E}_{0I} + \bar{E}_{0R})_z = \varepsilon_2 (\bar{E}_{0T})_z$$

2) $\mathcal{B}_{\perp 1} = \mathcal{B}_{\perp 2}$

$$(\bar{\mathcal{B}}_{0I} + \bar{\mathcal{B}}_{0R})_z = (\bar{\mathcal{B}}_{0T})_z$$

3) $E_{\parallel 1} = E_{\parallel 2}$

$$(\bar{E}_{0I} + \bar{E}_{0R})_{x,y} = (\bar{E}_{0T})_{x,y}$$

4) $\frac{1}{\mu_1} \mathcal{B}_{\parallel 1} = \frac{1}{\mu_2} \mathcal{B}_{\parallel 2}$

$$\frac{1}{\mu_1} (\bar{B}_{0I} + \bar{B}_{0R})_{x,y} = \frac{1}{\mu_2} (\bar{B}_{0T})_{x,y}$$

WE CAN CHOOSE POLARIZATION OF INCIDENT WAVE $\overline{E}_{0I}$ IN XZ PLANE

(OTHER CASE CAN BE TREATED ANALOGOUSLY)

$$\overline{E}_{0I} = E_{0I} \left(\cos\theta_I \, \overline{e}_x - \sin\theta_I \, \overline{e}_z \right)$$

* i.e. $(\overline{E}_{0I})_y = 0$

$$\Downarrow$$

$$\overline{B}_{0I} = B_{0I} \, \overline{e}_y = n_1 E_{0I} \, \overline{e}_y \quad (B_{0I} > 0)$$

* $(\overline{E}_{0R})_y = (\overline{E}_{0T})_y$

$$(\overline{B}_{0R})_z = (\overline{B}_{0T})_z$$

AS $\overline{B}_{0R} = n_1 (\hat{k}_R \times \overline{E}_{0R})$

$$(\overline{B}_{0R})_z = n_1 \sin\theta_R \, (\overline{E}_{0R})_y$$

AS $\overline{B}_{0T} = n_2 (\hat{k}_T \times \overline{E}_{0T})$

$$(\overline{B}_{0T})_z = n_2 \sin\theta_T \, (\overline{E}_{0T})_y$$

$$(\overline{B}_{0R})_z = (\overline{B}_{0T})_z \Rightarrow n_1 \sin \underset{\underset{\theta_I}{\parallel}}{\theta_R} = n_2 \sin\theta_T$$

<u>OK</u> 3° LAW

$$\left.\begin{aligned}
(\overline{B}_{0R})_x &= n_1 \cos\theta_R \, (\overline{E}_{0R})_y \\
(\overline{B}_{0T})_x &= -n_2 \cos\theta_T \, \underbrace{(\overline{E}_{0T})_y}_{(\overline{E}_{0R})_y} \\
&= \frac{\mu_2}{\mu_1} (\overline{B}_{0R})_x
\end{aligned}\right\} \begin{aligned}
&(\overline{E}_{0R})_y = 0 \\
&\quad\Downarrow \\
&(\overline{E}_{0T})_y = 0 \\
&(\overline{B}_{0R})_x = (\overline{B}_{0T})_x = 0
\end{aligned}$$

$$\left\| \begin{aligned} \overline{E}_{OR} &= E_{OR} \left(\cos\theta_I \ \overline{e}_x + \sin\theta_I \ \overline{e}_z \right) \\ \overline{E}_{OT} &= E_{OT} \left(\cos\theta_T \ \overline{e}_x - \sin\theta_T \ \overline{e}_z \right) \end{aligned} \right.$$

$$\left\| \begin{aligned} \overline{B}_{OR} &= - n_1 \ E_{OR} \ \overline{e}_y \\ \overline{B}_{OT} &= n_2 \ E_{OT} \ \overline{e}_y \end{aligned} \right.$$

$*$ $$\varepsilon_1 \left(\overline{E}_{OI} + \overline{E}_{OR} \right)_z = \varepsilon_2 \left(\overline{E}_{OT} \right)_z$$

$$\varepsilon_1 \left(- E_{OI} + E_{OR} \right) \sin\theta_I = - \varepsilon_2 \sin\theta_T \ E_{OT}$$

$*$ $$\left(\overline{E}_{OI} + \overline{E}_{OR} \right)_x = \left(\overline{E}_{OT} \right)_x$$

$$\left(E_{OI} + E_{OR} \right) \cos\theta_I = E_{OT} \cos\theta_T$$

$$\underline{\underline{E_{OI} + E_{OR} = \frac{\cos\theta_T}{\cos\theta_I} E_{OT} \equiv \alpha \ E_{OT}}}$$

$*$ $$\left(\overline{B}_{OI} + \overline{B}_{OR} \right)_y = \frac{\mu_1}{\mu_2} \left(\overline{B}_{OT} \right)_y$$

$$\frac{n_1}{\mu_1} \left(E_{OI} - E_{OR} \right) = \frac{n_2}{\mu_2} E_{OT}$$

$$\underline{\underline{E_{OI} - E_{OR} = \frac{\mu_1 n_2}{\mu_2 n_1} E_{OT} \equiv \beta \ E_{OT}}}$$

$$\boxed{\begin{aligned} E_{OR} &= \frac{\alpha-\beta}{\alpha+\beta} E_{OI} \\ E_{OT} &= \frac{2}{(\alpha+\beta)} E_{OI} \end{aligned}}$$

FRESNEL'S EQUATIONS

TRANSMITTED WAVE IN PHASE WITH I

REFLECTED WAVE
- IN PHASE WITH I : $\alpha > \beta$
- OUT OF PHASE WITH I : $\alpha < \beta$ (180°)

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I}}{\cos\theta_I}$$

SPECIAL CASES :

- $\theta_I = 0°$ (NORMAL INCIDENCE)

$\alpha = 1$

$$\begin{cases} E_{OR} = \frac{1-\beta}{1+\beta} E_{OI} \\ E_{OT} = \frac{2}{1+\beta} E_{OI} \end{cases}$$

cf. ABOVE

- $\theta_I = 90°$ (GRAZING INCIDENCE)

$\alpha \to \infty$

$$\begin{cases} E_{OR} = E_{OI} \\ E_{OT} = 0 \end{cases}$$

TOTAL REFLECTION

- BREWSTER ANGLE (Θ_B): NO REFLECTION

$$\alpha = \beta \qquad \begin{cases} E_{OR} = 0 \\ E_{OT} = \dfrac{2}{\beta} E_{OI} \end{cases}$$

$$\alpha^2 = \beta^2$$

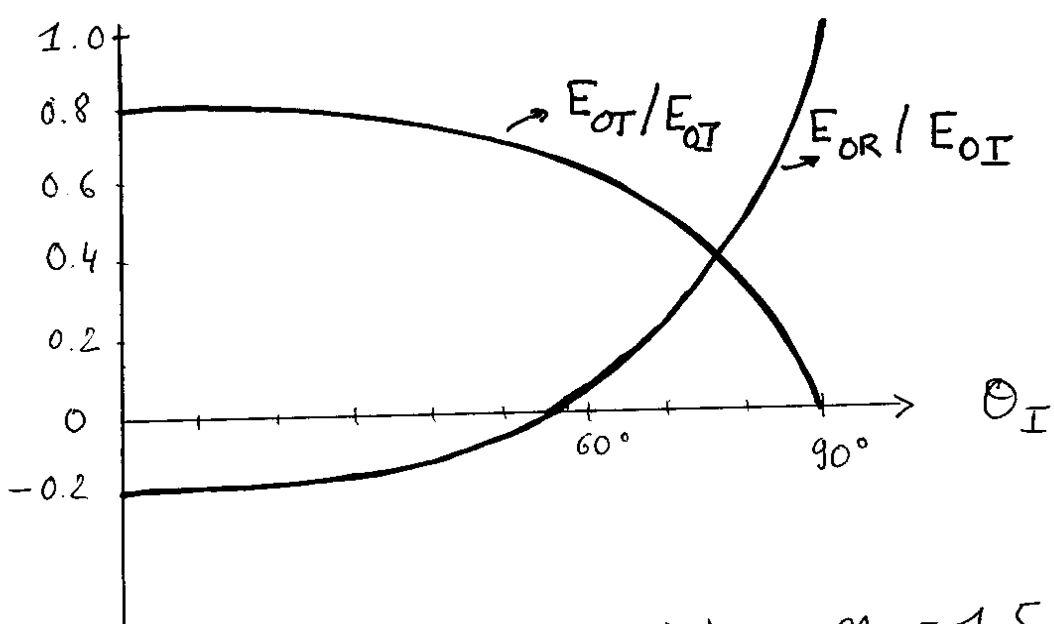
$$1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\Theta_I = \beta^2 \cos^2\Theta_I$$

$$\sin^2\Theta_B = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$$

FOR $\mu_1 \simeq \mu_2 \Longrightarrow \beta \simeq \dfrac{n_2}{n_1}$

$$\sin^2\Theta_B = \frac{1-\beta^2}{\frac{1}{\beta^2} - \beta^2} = \frac{\beta^2}{1+\beta^2}$$

$$\boxed{\tan\Theta_B \cong \frac{n_2}{n_1}}$$



FOR $n_1 = 1$ (AIR), $n_2 = 1.5$ (GLASS)

$\Theta_B \simeq 56°$

↳ REFLECTION & TRANSMISSION COEFFICIENTS

I INTENSITY = POWER / UNIT AREA STRIKING THE INTERFACE

$$I = \langle S_z \rangle$$

$$= \langle \bar{S} . \bar{e}_z \rangle$$

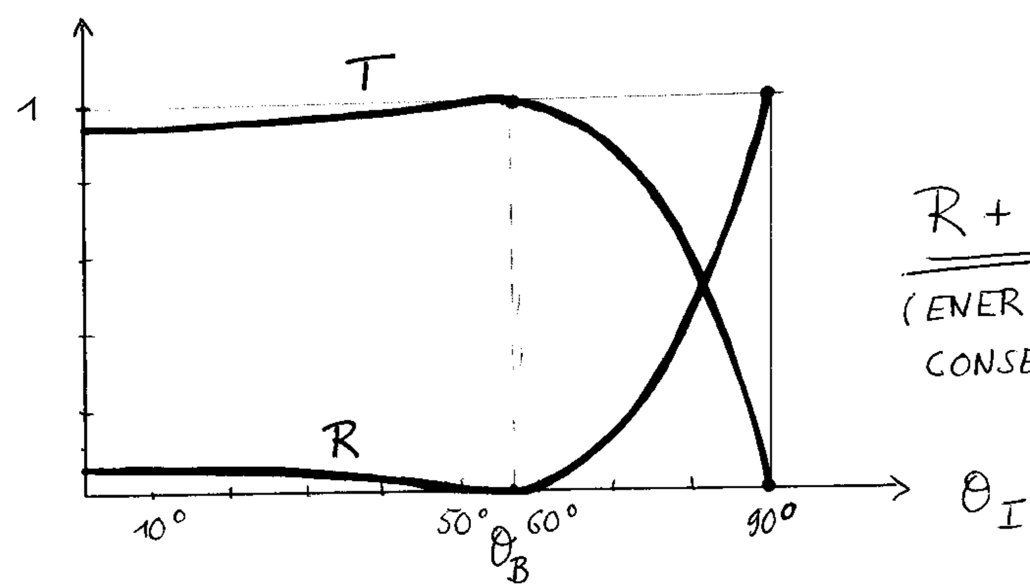
$$I_I = \frac{1}{2} \varepsilon_1 v_1 E_{0I}^2 \cos\theta_I$$

$$I_R = \frac{1}{2} \varepsilon_1 v_1 E_{0R}^2 \cos\theta_I$$

$$I_T = \frac{1}{2} \varepsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

$$R = \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \cdot \frac{E_{0T}^2}{E_{0I}^2} \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

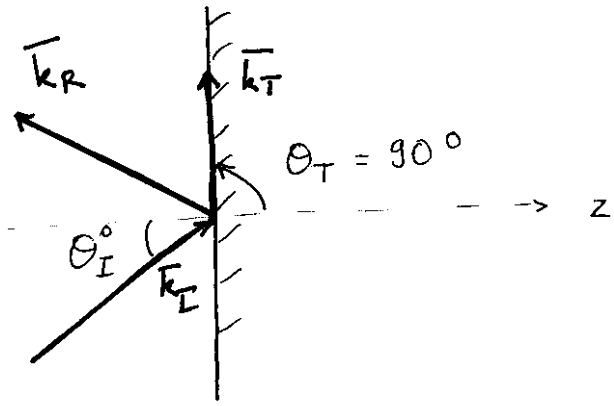


$$\underline{\underline{R + T = 1}}$$

(ENERGY CONSERVATION)

↳ TOTAL INTERNAL REFLECTION

FOR $n_1 > n_2$



⇝ $\theta_T = 90°$ $\Rightarrow$ $n_1 \sin\theta_I^0 = n_2$

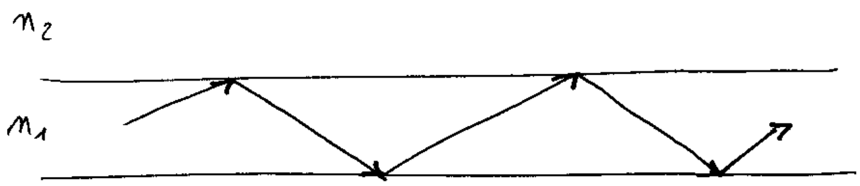
$$\sin\theta_I^0 = \frac{n_2}{n_1}$$

$R = 1, \quad T = 0$

⇝ FOR $\theta_I > \theta_I^0$

T WAVE DECAYS EXPONENTIALLY IN Z - DIR WITHOUT PROPAGATION

⇝ APPLICATION : OPTICAL FIBERS



⇒ DISPERSION

↳ IN DIELECTRIC MEDIA $n \simeq \sqrt{\varepsilon}$

IS NOT CONSTANT BUT DEPENDS ON FREQUENCY ω OF WAVE ⇒ SPEED OF WAVE DEPENDS ON ITS FREQUENCY

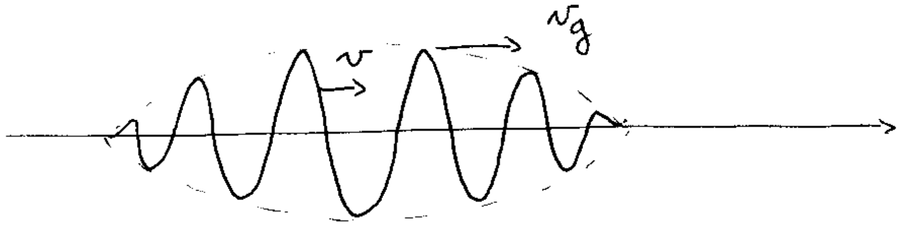
WAVES OF DIFFERENT ω TRAVEL AT DIFFERENT SPEEDS

⇓

WAVE PACKET SPREADS OUT
(DISPERSIVE MEDIUM)

ONE WAVE : SPEED $v = \frac{\omega}{k}$ (PHASE VELOCITY)

WAVE PACKET : GROUP VELOCITY $v_g = \frac{d\omega}{dk}$



↳ CLASSICAL MODEL OF DISPERSION

e^- IN A DIELECTRIC MATERIAL (BOUND TO MOLECULES)

* RESTORING FORCE $F_{binding} = -kx = -m\omega_0^2 x$

MASS e^- ; "SPRING" FREQUENCY

* DAMPING FORCE $F_{damping} = -m\gamma \frac{dx}{dt}$

* DRIVING FORCE : IN PRESENCE OF e.m. WAVE e^- FEELS FORCE

$$F_{driving} = qE = qE_0 e^{-i\omega t}$$

NEWTON'S 2° LAW

$$m \frac{d^2x}{dt^2} = F_{binding} + F_{damping} + F_{driving}$$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma \frac{dx}{dt} + \frac{q}{m} E_0 e^{-i\omega t}$$

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{q}{m} E_0 e^{-i\omega t}$$

IN STEADY STATE $\underline{\underline{x(t) = x_0 e^{-i\omega t}}}$

$$x_0 \left(-\omega^2 - i\omega\gamma + \omega_0^2\right) = \frac{q}{m} E_0$$

$$x_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\omega\gamma} E_0$$

* IN MATERIAL : N MOLECULES / UNIT VOLUME

FOR EACH MOLECULE : f_j e^- WITH ω_j, γ_j

MACROSCOPIP POLARIZATION :

$$\bar{P} = \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \bar{E}$$

$\Downarrow$

NOTE : PHYSICAL POLARIZATION IS REAL PART OF $\bar{P}$

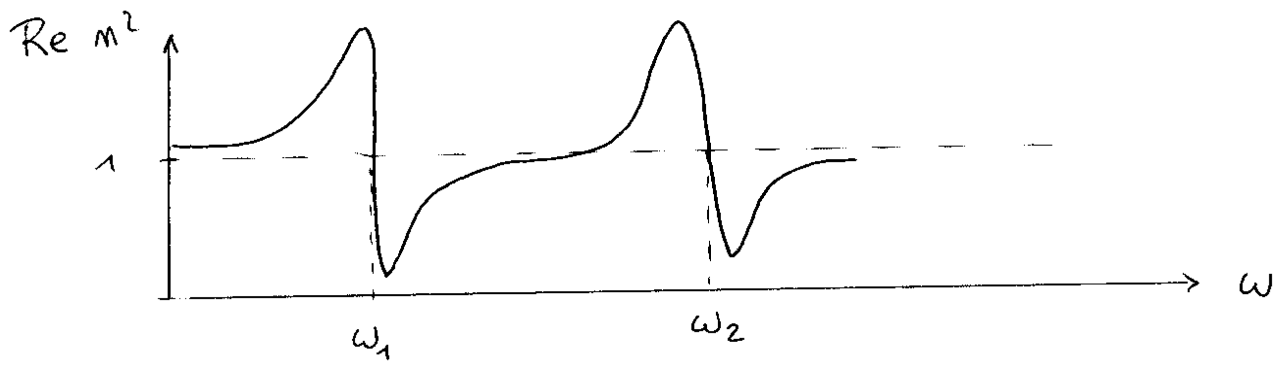
$$\overline{P} = \chi \overline{E}$$

$$\varepsilon = 1 + \chi$$

$$\boxed{\varepsilon = 1 + \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}$$

COMPLEX PERMITTIVITY (SELLMEIER'S EQ.)

FOR $\mu \simeq 1 \Rightarrow n^2 = \varepsilon$.



$\Rightarrow$ FOR RESONANT FREQUENCIES

$$\mathrm{Re}\ n^2(\omega = \omega_j) = 1$$

$\Rightarrow$ FOR $\omega \ll \omega_j$ OR $\omega \gg \omega_j$ (AWAY FROM RESONANCES)

ONE CAN NEGLECT DAMPING $\gamma_j \simeq 0$

$$n^2 \simeq 1 + \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

↳ WAVES PROPAGATING THROUGH DISPERSIVE MEDIUM

e.g. CASE OF ONE RESONANT FREQUENCY

$$\operatorname{Re} n^2(\omega) = 1 + \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2}$$

IN FOLLOWING, WE ONLY CONSIDER REAL PART OF n

WAVE VECTOR $$k(\omega) = \frac{\omega}{c} n(\omega)$$

$$\frac{dk}{d\omega} = \frac{1}{c}\left\{ n + \omega \frac{dn}{d\omega} \right\}$$

$$= \frac{n}{c}\left\{ 1 + \omega \frac{1}{n} \frac{dn}{d\omega} \right\}$$

$$\downarrow \qquad n(\omega) = \sqrt{1 + \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2}}$$

$$= \frac{n}{c}\left\{ 1 + \frac{\omega}{\left(1 + \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2}\right)} \cdot \frac{Nq^2}{m} \frac{\omega}{(\omega_0^2 - \omega^2)^2} \right\}$$

$$= \frac{1}{cn}\left\{ 1 + \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2} + \frac{Nq^2}{m} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2} \right\}$$

$$= \frac{1}{cn}\left\{ 1 + \frac{Nq^2}{m} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2} \right\}$$

$$\boxed{v_g = \frac{c\sqrt{1 + \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2}}}{1 + \frac{Nq^2}{m} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2}}}$$

$v_g < c$

$\forall \omega$

3) ELECTROMAGNETIC WAVES IN CONDUCTORS

⇒ FIELD EQUATIONS

↳ IN CONDUCTOR ⇒ FREE CHARGE DENSITY ρ_f
FREE CURRENT DENSITY $\bar{J}_f$

ARE NOT ZERO

OHM'S LAW : $\bar{J}_f = \sigma \bar{E}$

⇓

MAXWELL'S EQ. FOR LINEAR MEDIA
IN PRESENCE OF ρ_f, $\bar{J}_f = \sigma \bar{E}$

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho_f}{\varepsilon} \qquad \bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c} \frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \frac{\mu}{c} \sigma \bar{E} + \frac{1}{c} \varepsilon \mu \frac{\partial \bar{E}}{\partial t}$$

\+ CONTINUITY EQ. FOR FREE CHARGE

$$\frac{\partial \rho_f}{\partial t} + \bar{\nabla} \cdot \bar{J}_f = 0$$

$$\Downarrow \quad \bar{J}_f = \sigma \bar{E}$$

$$\frac{\partial \rho_f}{\partial t} = -\sigma \bar{\nabla} \cdot \bar{E}$$

$$\Downarrow$$

$$\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon} \rho_f$$

$$\rho_f(t) = \rho_f(0)\, e^{-\frac{\sigma}{\varepsilon} t}$$

FREE CHARGE DENSITY DISSIPATES WITH TIME CONSTANT $\tau = \frac{\varepsilon}{\sigma}$ (CHARGE WILL FLOW TO SURFACE)

$\Rightarrow$ PERFECT CONDUCTOR $\sigma \to \infty$, $\tau \to 0$

$\Rightarrow$ INSULATOR / POOR CONDUCTOR $\sigma \ll$, $\tau \gg$

(TYPICAL $\tau \gg \frac{1}{\omega}$)

FREQUENCY OF WAVE

PERFECT CONDUCTORS : $t \gg \tau \Rightarrow \rho_f = 0$

$$\bar{\nabla}.\bar{E} = 0 \qquad \bar{\nabla}.\bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \frac{\mu}{c}\sigma \bar{E} + \frac{1}{c}\varepsilon\mu\frac{\partial \bar{E}}{\partial t}$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$$

$$\bar{\nabla}(\underbrace{\cancel{\bar{\nabla}.\bar{E}}}_{0}) - \nabla^2 \bar{E} = -\frac{\mu\varepsilon}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} - \frac{\mu}{c^2}\sigma\frac{\partial \bar{E}}{\partial t}$$

$$\boxed{\frac{\mu\varepsilon}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} + \frac{\mu}{c^2}\sigma\frac{\partial \bar{E}}{\partial t} - \nabla^2 \bar{E} = 0}$$

EXTRA TERM DUE TO FREE CURRENT DENSITY IN CONDUCTOR

ANALOGOUSLY FOR $\bar{B}$

$$\boxed{\frac{\mu\varepsilon}{c^2}\frac{\partial^2\bar{B}}{\partial t^2} + \frac{\mu}{c^2}\sigma\frac{\partial\bar{B}}{\partial t} - \nabla^2\bar{B} = 0}$$

↳ PLANE WAVE SOLUTIONS IN CONDUCTOR

FOR WAVES PROPAGATING IN Z - DIRECTION

$$\bar{E} = \bar{E}_0 \, e^{i(\tilde{k}z - \omega t)}$$

$$\bar{B} = \bar{B}_0 \, e^{i(\tilde{k}z - \omega t)}$$

⇒ FOR $\sigma = 0$: $\tilde{k} = n\frac{\omega}{c}$

WITH $n = \sqrt{\varepsilon\mu}$ INDEX OF REFRACTION

⇒ FOR $\sigma \neq 0$: TRY SAME SOLUTION

$$-\frac{\mu\varepsilon}{c^2}\omega^2 - \frac{\mu}{c^2}\sigma i\omega + \tilde{k}^2 = 0$$

↳ $\tilde{k}$ IS COMPLEX

DEFINE $\tilde{k} = k + i\underset{\substack{\uparrow \\ \text{KAPPA}}}{K}$ k, K REAL

$$\begin{cases} k^2 - K^2 = \dfrac{\mu\varepsilon}{c^2}\omega^2 & \text{REAL PART} \\[2ex] 2kK = \dfrac{\mu}{c^2}\sigma\omega & \text{IM. PART} \end{cases}$$

$$(k^2)^2 - \frac{\mu\varepsilon}{c^2}\,\omega^2 k^2 - \left(\frac{\mu}{c^2}\,\frac{\sigma}{2}\,\omega\right)^2 = 0$$

$$k^2 = \frac{\mu\varepsilon}{2c^2}\,\omega^2 + \frac{1}{2}\sqrt{\left(\frac{\mu\varepsilon}{c^2}\,\omega^2\right)^2 + 4\left(\frac{\mu\varepsilon}{c^2}\,\frac{\sigma}{2\varepsilon}\,\omega\right)^2}$$

(ONLY + SOLUTION ALLOWED $k^2 > 0$)

$$k^2 = \frac{\mu\varepsilon}{2c^2}\,\omega^2\left\{1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right\}$$

$$\boxed{k = \sqrt{\frac{\mu\varepsilon}{2}}\,\frac{\omega}{c}\left[1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}}$$

$$\kappa = \frac{1}{2k}\cdot\frac{\mu}{c^2}\,\sigma\,\omega$$

$$= \frac{1}{k}\cdot\left(\frac{\mu\varepsilon}{2}\right)\frac{\omega^2}{c^2}\left(\frac{\sigma}{\varepsilon\omega}\right)$$

KAPPA $$\boxed{\kappa = \sqrt{\frac{\mu\varepsilon}{2}}\,\frac{\omega}{c}\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}}$$

$$\left\| \begin{aligned} \bar{E} &= \bar{E}_0 \, e^{-\kappa z} \, e^{i(kz - \omega t)} \\ \bar{B} &= \bar{B}_0 \, e^{-\kappa z} \, e^{i(kz - \omega t)} \end{aligned} \right.$$

$\uparrow$

WAVE DAMPED IN Z - DIRECTION

$\Rightarrow$ **SKIN DEPTH**

$*$ 2 EXTREME CASES

1) VERY POOR CONDUCTOR : $\sigma \ll \varepsilon\omega$

$$k \simeq \sqrt{\frac{\mu\varepsilon}{2}} \, \frac{\omega}{c} \left[1 + \left(1 + \frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2 \right) \right]^{1/2}$$

$$\simeq \sqrt{\varepsilon\mu} \, \frac{\omega}{c} \left[1 + \cancel{\frac{1}{2}\left(\frac{\sigma}{2\varepsilon\omega}\right)^2} \right]$$

NEGLECT
SMALL QUADRATIC TERM

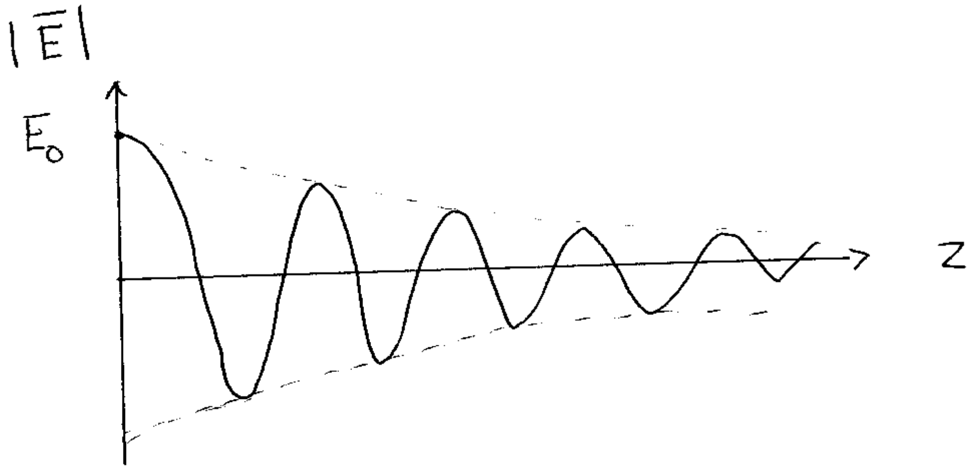
$$\kappa \simeq \sqrt{\frac{\mu\varepsilon}{2}} \, \frac{\omega}{c} \left[\cancel{-1} + \cancel{1} + \frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2 \right]^{1/2}$$

$$= \sqrt{\varepsilon\mu} \, \frac{\omega}{2c} \, \frac{\sigma}{\varepsilon\omega}$$

$$= \sqrt{\frac{\mu}{\varepsilon}} \, \frac{\sigma}{2c}$$

$$\tilde{k} = n \frac{\omega}{c} + i \sqrt{\frac{\mu}{\varepsilon}} \, \frac{\sigma}{2c}$$

$$n = \sqrt{\mu\varepsilon}$$



$$K = \sqrt{\frac{\mu}{\varepsilon}} \frac{\sigma}{2c}$$

DISTANCE OVER WHICH WAVE AMPLITUDE DECREASES BY $\frac{1}{e}$

$$\Delta z = \frac{1}{K} = \frac{2c}{\sigma} \sqrt{\frac{\varepsilon}{\mu}}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{\varepsilon\mu}} \frac{c}{\omega} \qquad \text{(WAVE LENGTH)}$$

$\Delta z \gg \lambda$ WHEN $\sigma \ll \varepsilon\omega$

'OSCILLATORY' DAMPED

2) VERY GOOD CONDUCTOR $\quad \sigma \gg \varepsilon\omega$

$$k \simeq \sqrt{\frac{\mu\varepsilon}{2}}\ \frac{\omega}{c}\ \sqrt{\frac{\sigma}{\varepsilon\omega}}$$

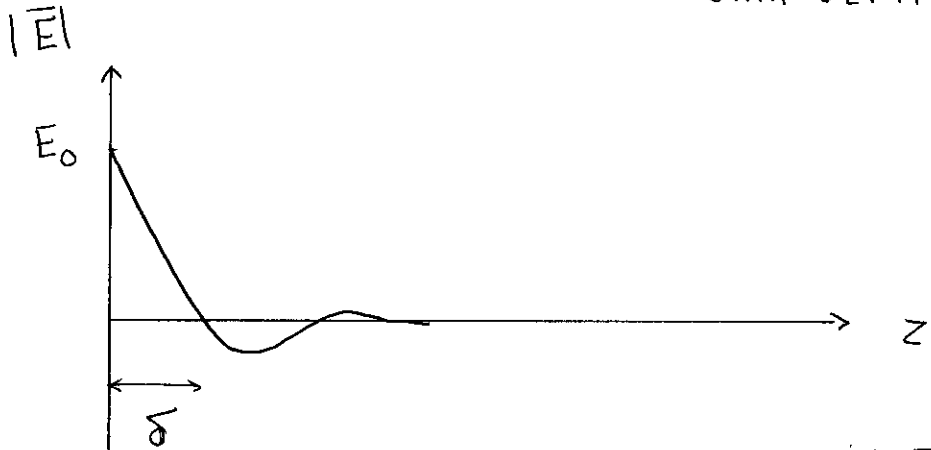
$$K \simeq \sqrt{\frac{\mu\varepsilon}{2}}\ \frac{\omega}{c}\ \sqrt{\frac{\sigma}{\varepsilon\omega}}$$

$$\tilde{k} = (1+i)\ \frac{1}{c}\ \sqrt{\frac{\sigma\mu}{2}\omega}$$

STRONGLY DAMPED

$$\Delta z = \frac{1}{K} = \boxed{c\sqrt{\frac{2}{\sigma\mu\omega}} \equiv \delta}$$

SKIN DEPTH



↳ VERY THIN SURFACE LAYER ⇒ SKIN EFFECT

$$\lambda = \frac{2\pi}{k} = 2\pi\,\delta$$

$$\delta = \frac{\lambda}{2\pi} \sim \frac{1}{\sqrt{\omega}}$$

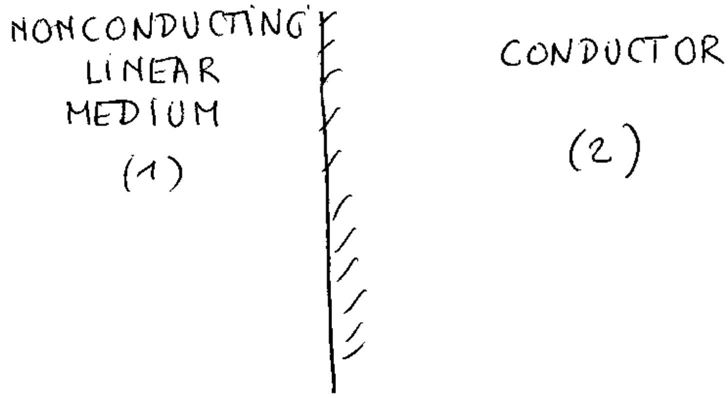
HIGH-FREQUENCY e.m. WAVES CANNOT PENETRATE CONDUCTOR

LONG WAVES PENETRATE DEEPER THAN SHORT WAVES / HIGH-FREQUENCY WAVES

↳ ATTENUATION: JOULE HEATING

⇒ REFLECTION AT CONDUCTING SURFACE

↳ BOUNDARY CONDITIONS



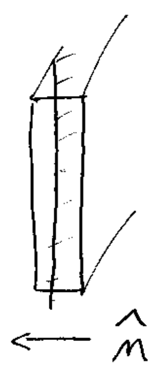
⇓

AT INTERFACE

σ_f : FREE SURFACE CHARGE DENSITY

$\overline{K}_f$: " " CURRENT "

⇝ $\overline{\nabla} \cdot \overline{D} = \rho_f$



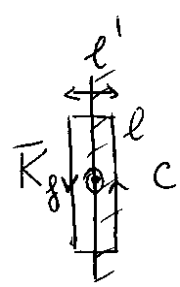
$E_\perp \equiv \overline{E} \cdot \hat{n}$

$$\varepsilon_1 E_{\perp 1} - \varepsilon_2 E_{\perp 2} = \sigma_f$$

⇝ $B_{\perp 1} = B_{\perp 2}$

⇝ $E_{\parallel 1} = E_{\parallel 2}$

⇝ $\frac{1}{\mu} \overline{\nabla} \times \overline{B} = \frac{1}{c} \overline{J}_f + \frac{1}{c} \varepsilon \frac{\partial \overline{E}}{\partial t}$



↓ // DIR

$$\frac{1}{\mu_1} B_{\parallel 1} - \frac{1}{\mu_2} B_{\parallel 2} = \frac{1}{c}\left(\overline{K}_f \times \hat{n}\right)$$

NOTE $\lim_{l'\to 0} l' J_{f\parallel} = K_{f\parallel}$

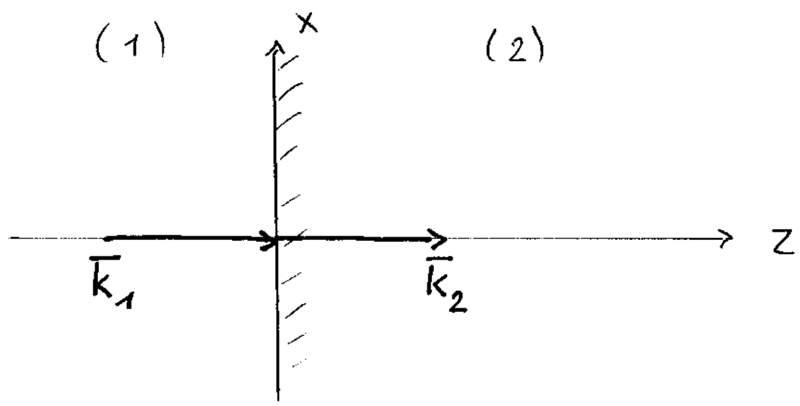
FOR OHMIC CONDUCTOR $\bar{J}_f = \sigma \bar{E}$

$$\lim_{l'\to 0} \sigma l' E_{\parallel} = K_{f\parallel}$$

FINITE $K_{f\parallel}$ WOULD IMPLY INFINITE $E_{\parallel}$ AT BOUNDARY

FOR OHMIC CONDUCTOR $\bar{K}_f = 0$

↳ REFLECTION OF INCOMING WAVE



INCOMING
$$\bar{E}_I = E_{0I}\, e^{i(k_1 z - \omega t)}\, \bar{e}_x$$
$$\bar{B}_I = \frac{c}{v_1} E_{0I}\, e^{i(k_1 z - \omega t)}\, \bar{e}_y$$

REFLECTED
$$\bar{E}_R = E_{0R}\, e^{i(-k_1 z - \omega t)}\, \bar{e}_x$$
$$\bar{B}_R = -\frac{c}{v_1} E_{0R}\, e^{i(-k_1 z - \omega t)}\, \bar{e}_y$$

TRANSMITTED
$$\bar{E}_T = E_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \bar{e}_x$$
$$\bar{B}_T = c\,\frac{\tilde{k}_2}{\omega}\, E_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \bar{e}_y$$

AT INTERFACE $z = 0$

$$\rightsquigarrow \quad E_{\perp 1} = E_{\perp 2} = 0 \quad \Rightarrow \quad \sigma_f = 0$$

$$\rightsquigarrow \quad E_{0I} + E_{0R} = E_{0T} \qquad (E_{\parallel 1} = E_{\parallel 2})$$

$$\rightsquigarrow \quad \frac{1}{\mu_1} B_{\parallel 1} - \frac{1}{\mu_2} B_{\parallel 2} = 0$$

$$\frac{1}{\mu_1} \cdot \frac{c}{v_1}\left(E_{0I} - E_{0R}\right) = \frac{1}{\mu_2}\, c\, \frac{\tilde{k}_2}{\omega}\, E_{0T}$$

$$E_{0I} - E_{0R} = \underbrace{\frac{\mu_1 v_1}{\mu_2\, \omega}\tilde{k}_2}_{\equiv\, \tilde{\beta}}\; E_{0T}$$

$$E_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right) E_{0I}$$
$$E_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right) E_{0I}$$

FOR PERFECT CONDUCTOR $k_2 \to \infty \Rightarrow \tilde{\beta} \to \infty$

$$\hookrightarrow \quad E_{0R} = -E_{0I}, \qquad E_{0T} = 0$$

TOTAL REFLECTION WITH 180° PHASE SHIFT (e.g. SILVER COATING ON GLASS $\Rightarrow$ MIRROR)

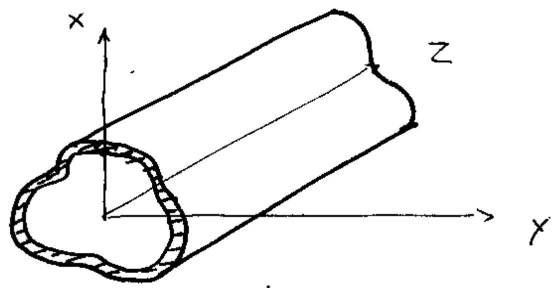
4) WAVE GUIDES & RESONANT CAVITIES

⇒ CYLINDRICAL WAVE GUIDE

↳ UP TO NOW : PLANE WAVES OF INFINITE EXTENT

IN THIS SECTION : STANDING E.M. WAVES IN RESONANT CAVITY
(e.g. ACOUSTIC EXAMPLE : ORGAN PIPE)

IN E.M. CASE : RESONANT CAVITY / WAVE GUIDE
→ INTERIOR IS EMPTY
→ WALLS ARE PERFECT CONDUCTORS



↳ BOUNDARY CONDITIONS

WALLS ARE PERFECT CONDUCTORS
⇒ $\bar{E} = 0$, $\bar{B} = 0$ INSIDE MATERIAL

⇒ AT INNER WALL (ON BOUNDARY)

$$\bar{E}_{\parallel} = 0 \quad , \quad \bar{B}_{\perp} = 0$$

↳ IN INTERIOR OF WAVEGUIDE $\rho_f = 0$, $\bar{J}_f = 0$
$\varepsilon = 1$, $\mu = 1$

$$\begin{cases} -\dfrac{1}{c^2}\dfrac{\partial^2 \bar{E}}{\partial t^2} + \nabla^2 \bar{E} = 0 \\ -\dfrac{1}{c^2}\dfrac{\partial^2 \bar{B}}{\partial t^2} + \nabla^2 \bar{B} = 0 \end{cases}$$

WAVE EQS. IN VACUUM

↳ LOOK FOR SOLUTIONS (MONOCHROMATIC WAVES $\sim \omega$)

THAT PROPAGATE DOWN THE WAVEGUIDE

$$\begin{cases} \bar{E} = \bar{E}_0(x,y)\, e^{i(kz-\omega t)} \\ \bar{B} = \bar{B}_0(x,y)\, e^{i(kz-\omega t)} \end{cases}$$

k IS REAL

⇝ FOR PLANE WAVES IN VACUUM $\bar{E}_0$ = CONSTANT $\bar{k}.\bar{E}_0 = 0$

$$k = |\bar{k}| = \frac{\omega}{c}$$

⇝ NOW WE NEED TO SOLVE FOR $\bar{E}_0(x,y)$

AND DETERMINE EIGENVALUES k

↳ z-COMPONENT E_z, B_z

$$\begin{cases} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right] E_z = 0 \\ \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right] B_z = 0 \end{cases}$$

↳ x, y COMPONENTS

- $\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$ $\Rightarrow$ $$\begin{cases} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c}\frac{\partial B_x}{\partial t} = \frac{i\omega}{c} B_x \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c}\frac{\partial B_y}{\partial t} = \frac{i\omega}{c} B_y \end{cases}$$

$$\frac{\partial E_y}{\partial z} = ik E_y \, , \quad \frac{\partial E_x}{\partial z} = ik E_x$$

$$\begin{cases} \dfrac{\partial E_z}{\partial y} - ik E_y = i\dfrac{\omega}{c} B_x & (1) \\ ik E_x - \dfrac{\partial E_z}{\partial x} = \dfrac{i\omega}{c} B_y & (2) \end{cases}$$

- $\bar{\nabla} \times \bar{B} = + \dfrac{1}{c} \dfrac{\partial \bar{E}}{\partial t}$

$$\Rightarrow \begin{cases} \dfrac{\partial B_z}{\partial y} - ik B_y = - i \dfrac{\omega}{c} E_x & (3) \\ ik B_x - \dfrac{\partial B_z}{\partial x} = - i \dfrac{\omega}{c} E_y & (4) \end{cases}$$

- $$\begin{cases} \dfrac{\omega}{c} B_x + k E_y = - i \dfrac{\partial E_z}{\partial y} & (1) \\ k B_x + \dfrac{\omega}{c} E_y = - i \dfrac{\partial B_z}{\partial x} & (4) \end{cases}$$

SOLUTION :

$$\boxed{\begin{aligned} B_x &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} \left[k \frac{\partial B_z}{\partial x} - \frac{\omega}{c} \frac{\partial E_z}{\partial y} \right] \\ E_y &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} \left[k \frac{\partial E_z}{\partial y} - \frac{\omega}{c} \frac{\partial B_z}{\partial x} \right] \end{aligned}}$$

- ANALOGOUSLY EQS. (2) & (3) YIELD :
[$\omega \to -\omega$ IN (1) & (4)]

$$\boxed{\begin{aligned} B_y &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} \left[k \frac{\partial B_z}{\partial y} + \frac{\omega}{c} \frac{\partial E_z}{\partial x} \right] \\ E_x &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} \left[k \frac{\partial E_z}{\partial x} + \frac{\omega}{c} \frac{\partial B_z}{\partial y} \right] \end{aligned}}$$

$\therefore$ ONCE WE SOLVE FOR E_z, B_z

↳ E_x, E_y, B_x, B_y ARE KNOWN

↳ EQUATIONS FOR E_z, B_z ARE UNCOUPLED

SPECIAL CASES

1) $\underline{E_z = 0}$ TE WAVES : TRANSVERSE ELECTRIC WAVES

↓

$\bar{E} \perp \bar{e}_z$ WAVE VECTOR

2) $\underline{B_z = 0}$ TM WAVES : TRANSVERSE MAGNETIC WAVES

$\bar{B} \perp \bar{e}_z$

3) $\underline{E_z = 0 \ \& \ B_z = 0}$ TEM WAVES

$\bar{E} \perp \bar{e}_z \ , \ \bar{B} \perp \bar{e}_z$

$$\bar{\nabla}.\bar{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

$$(\bar{\nabla} \times \bar{E})_z = 0 \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$$\Rightarrow \bar{\nabla}.\bar{E}_0 = 0 \ , \ \bar{\nabla} \times \bar{E}_0 = 0$$

$$\bar{E}_0 = -\bar{\nabla}\phi \ , \ \nabla^2 \phi = 0$$

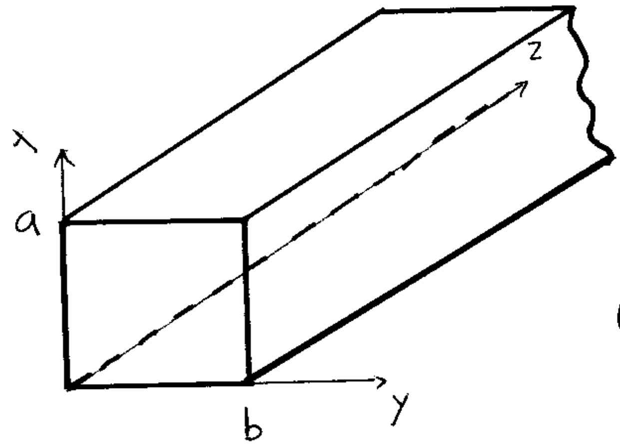
$\Rightarrow$ BOUNDARY CONDITION : SURFACE IS EQUIPOTENTIAL SURFACE (ϕ = CONSTANT)

LAPLACE EQ. $\Rightarrow$ SOLUTION ϕ = CONSTANT EVERYWHERE

$\therefore$ FOR HOLLOW WAVEGUIDE $\Rightarrow \bar{E} = 0$ NO TEM WAVE

TEM MODE

⇒ RECTANGULAR WAVE GUIDE : TM WAVE



(ASSUME $a \geq b$)

↳ TM WAVE $\quad B_z = 0$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right] E_z = 0$$

+ BOUNDARY CONDITION $\quad \bar{E}_{\parallel} = 0$ ON WALLS

$x = 0, \; x = a, \; y = 0, \; y = b$

SEPARATION OF VARIABLES

$$E_z(x, y, z, t) = E_{oz}(x, y)\, e^{i(kz - \omega t)}$$

$$E_{oz}(x, y) = X(x)\, Y(y)$$

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = \left(k^2 - \frac{\omega^2}{c^2}\right) XY$$

2 EQNS.
$$\begin{cases} \dfrac{1}{X} \dfrac{d^2 X}{dx^2} = -k_x^2 \\[2ex] \dfrac{1}{Y} \dfrac{d^2 Y}{dy^2} = -k_y^2 \end{cases}$$

$$\& \quad k^2 + k_x^2 + k_y^2 - \frac{\omega^2}{c^2} = 0$$

↳ SOLUTIONS FOR X & Y

$$\rightsquigarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$$

$$\Downarrow$$

$$X(x) = A \sin(k_x x) + B\cos(k_x x)$$

BOUNDARY CONDITION

$$E_z = 0 \quad \text{FOR} \quad x = 0 \Rightarrow X(0) = 0 \Rightarrow B = 0$$

$$x = a \Rightarrow X(a) = 0$$

$$\Downarrow$$

$$\sin(k_x a) = 0$$

$$\underline{\underline{k_x = m\frac{\pi}{a}, \quad m = 1, 2, 3, \dots}}$$

$$\rightsquigarrow \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$$

$$Y(y) = C\sin(k_y y) + D\cos(k_y y)$$

BOUNDARY CONDITION

$$E_z = 0 \quad \text{FOR} \quad y = 0 \Rightarrow Y(0) = 0 \Rightarrow D = 0$$

$$y = b \Rightarrow Y(b) = 0$$

$$\Downarrow$$

$$\sin(k_y b) = 0$$

$$\underline{\underline{k_y = n\frac{\pi}{b}, \quad n = 1, 2, 3}}$$

↳ TOTAL SOLUTION

⇝ $B_z = 0$

⇝ $E_z = E_0 \sin\left(m \frac{\pi}{a} x\right) \sin\left(n \frac{\pi}{b} y\right) e^{i(kz - \omega t)}$

CONSTANT

SOLUTION IS CALLED TM_{mn} MODE

⇝ WAVENUMBER k IS SOLUTION OF

$$\boxed{k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}}$$

LOWEST FREQUENCY FOR WHICH PROPAGATION CAN OCCUR (i.e. k IS REAL)

$$\omega \geq \omega_{mn} \equiv c\pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \equiv c\,\gamma_{mn}$$

$$\boxed{k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}}$$

$m, n = 1, 2, \ldots$ FOR TM MODE

ω_{mn} IS CALLED CUTOFF FREQUENCY FOR MODE m, n

e.g. FOR TM_{11} MODE

$$\omega_{11} = c\pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

⇝ SOLUTIONS FOR OTHER COMPONENTS

e.g. $E_x = \frac{ik}{\gamma_{mn}^2} \frac{\partial E_z}{\partial x} = \frac{ik\left(\frac{m\pi}{a}\right)}{\gamma_{mn}^2} E_0 \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \cdot e^{i(kz - \omega t)}$

$\Rightarrow$ RECTANGULAR WAVE GUIDE : TE WAVE

$\hookrightarrow$ TE WAVE $\quad E_z = 0$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right] B_z = 0$$

+ BOUNDARY CONDITION $\bar{B}_\perp = 0$ ON WALLS

$$B_x = 0 \qquad x = 0,\ x = a$$

$$B_y = 0 \qquad y = 0,\ y = a.$$

SINCE $B_x \sim \frac{\partial B_z}{\partial x}$, $B_y \sim \frac{\partial B_z}{\partial y}$ $\Rightarrow$ ON WALLS $\frac{\partial X}{\partial x} = 0,\ x = 0, a$; $\frac{\partial Y}{\partial y} = 0,\ x = 0, a$

$$B_{oz}(x, y) = X(x)\, Y(y)$$

TOTAL SOLUTION :

$$\underline{\underline{B_z = B_0 \cos\left(m \frac{\pi}{a} x\right) \cos\left(n \frac{\pi}{b} y\right) e^{i(kz - \omega t)}}}$$

$m = 0, 1, 2, \ldots$

$n = 0, 1, 2, \ldots$

$m = 0$ OR $n = 0$ MODE IS POSSIBLE FOR TE WAVE

HOWEVER $m = n = 0$ MODE IS NOT POSSIBLE $\cancel{TE}_{00}$

SINCE

$$k = \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2}$$

$$\omega_{mn} = c\pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$B_z = \text{CONSTANT} = 0$

$\uparrow$

(ω_{00} WOULD BE ZERO)

$\hookrightarrow$ TE$_{10}$ MODE $\quad m = 1 \ , \ n = 0$

$$B_z = B_0 \cos\left(\frac{\pi}{a} x\right) e^{i(kz - \omega t)}$$

PROPAGATION POSSIBLE FOR

$$\omega \geq \omega_{10} = c\frac{\pi}{a}$$

TYPICAL CHOICE $\quad a = 2b$

NEXT MODES $\quad \omega_{11} = c\frac{\pi}{a}\sqrt{5}$

$$\omega_{20} = c\frac{\pi}{a} 2$$

$$\omega_{01} = c\frac{\pi}{b} = c\frac{\pi}{a} 2$$

$$\vdots$$

FOR $\quad \omega_{10} < \omega < \omega_{01} = \omega_{20}$

$$c\frac{\pi}{a} < \omega < c\frac{\pi}{a} 2$$

ONLY 1 MODE CAN PROPAGATE

TYPICAL : $\quad a \sim 1 \text{ cm}$

$$\omega_{10} \sim 10^{11} \ s^{-1}$$

$$\nu_{10} \sim 10 \text{ GHz}$$

$$B_x = \frac{ik}{\gamma_{10}^2} \frac{\partial B_z}{\partial x}$$

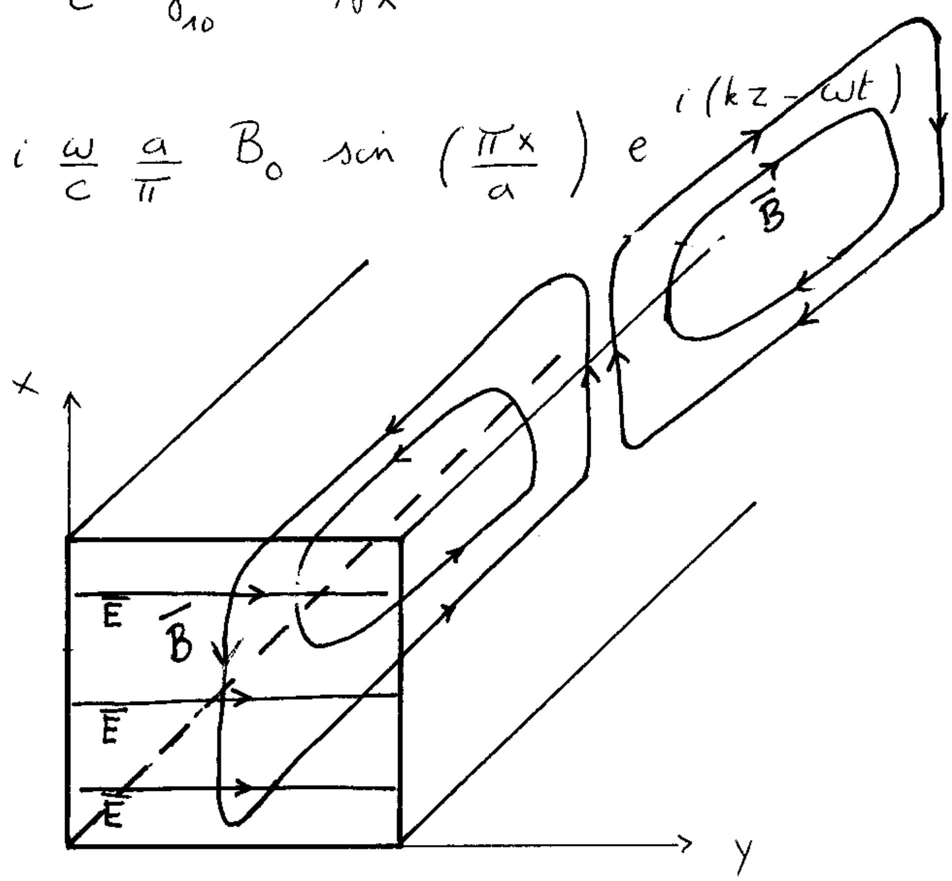
$$= -\frac{ika}{\pi} B_0 \sin\left(\frac{\pi}{a} x\right) e^{i(kz - \omega t)}$$

$$B_y = 0$$

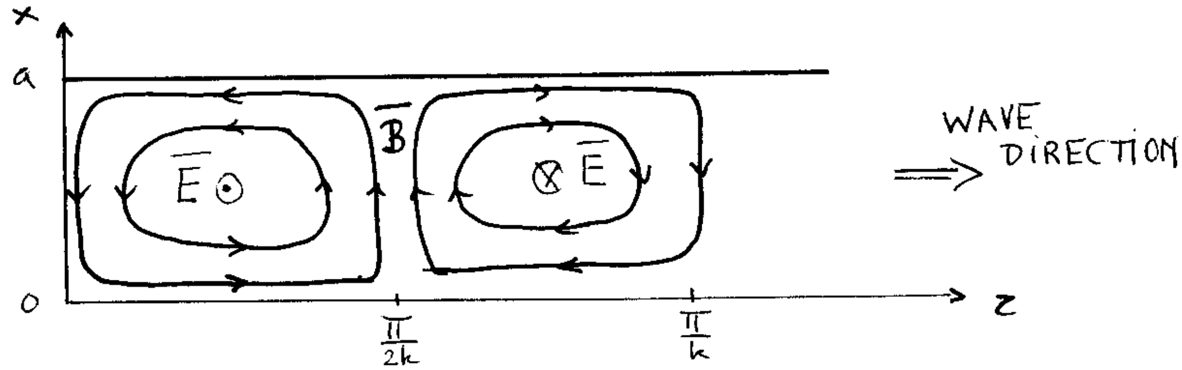
$$E_x = 0$$

$$E_y = -\frac{i\omega}{c} \frac{1}{\gamma_{10}^2} \frac{\partial B_z}{\partial x}$$

$$= + i \frac{\omega}{c} \frac{a}{\pi} B_0 \sin\left(\frac{\pi x}{a}\right) e^{i(kz - \omega t)}$$



CROSS SECTIONAL VIEW : PLANE $y = \frac{b}{2}$



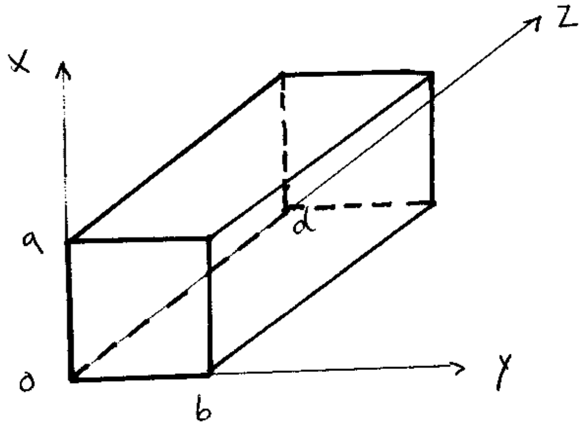
↳ GROUP VELOCITY IN WAVE GUIDE

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2} \qquad \omega > \omega_{mn}$$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\frac{\omega}{\sqrt{\omega^2 - \omega_{mn}^2}}$$

$$\boxed{v_g = c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} < c}$$

⇒ RECTANGULAR RESONANT CAVITY : TM MODE



DIFFERENCE WITH WAVEGUIDE :
CONDUCTING SURFACE NOW CLOSED AT ENDS $z=0$, $z=d$

↳ TM MODE $\quad B_z = 0$

FOR E_z : STANDING WAVES IN z

$$E_z = E_0 \sin\left(m\frac{\pi}{a}x\right)\sin\left(n\frac{\pi}{b}y\right) \cdot \left\{ A\sin(kz) + B\cos(kz)\right\} e^{-i\omega t}$$

BOUNDARY CONDITION AT $z=0$, $z=d$

$\bar{E}_{\parallel} = 0$

$E_x = 0$ & $E_y = 0$ FOR $z=0$, $z=d$.

FOR TM MODE $(B_z = 0)$

$$E_x = \frac{ik}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial E_z}{\partial x} = \frac{1}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial^2 E_z}{\partial z \partial x}$$

$$E_y = \frac{ik}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial E_z}{\partial y} = \frac{1}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial^2 E_z}{\partial z \partial y}$$

$$E_x = E_y = 0 \qquad \begin{array}{l} \text{FOR } z = 0 \\ \text{FOR } z = d \end{array}$$

$$E_x = \frac{1}{\left(\frac{\omega}{c}\right)^2 - k^2} E_0 \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \cdot \left\{ A\, k \cos(kz) - B k \sin(kz) \right\}.$$

$$E_x(z=0) = 0 \implies A = 0$$

$$E_x(z=d) = 0 \implies \sin(kd) = 0$$

$$k = \frac{\ell\pi}{d} \qquad \ell = 0, 1, 2, \ldots$$

0 IS POSSIBLE $\cos(kz)$

$\therefore$ TM MODE

$$\boxed{E_z = E_0 \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \cos\left(\frac{\ell\pi}{d} z\right) \cdot \underbrace{\cos(\omega t)}_{\text{REAL PART OF } e^{-i\omega t}}}$$

$TM_{\ell m n}$ MODE

+ EIGENFREQUENCY REL.

$$\left(\frac{\omega}{c}\right) = \left(\frac{m\pi}{a}\right) + \left(\frac{\ }{b}\right) + \left(\frac{\ }{c}\right)$$

$$E_x = \frac{1}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial^2 E_z}{\partial z \, \partial x}$$

$$\sim \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \quad \left(\frac{\ell\pi}{d} z\right) \cos\omega t$$

$$E_y \sim \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \quad \left(\frac{\ell\pi}{d} z\right) \cos\omega t$$

$$B_x \sim \frac{\partial E_z}{\partial y \, \partial t}$$

$$\sim \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \cos\left(\frac{\ell\pi}{d} z\right) \sin\omega t$$

$$B_y \sim \frac{\partial E_z}{\partial x \, \partial t}$$

$$\sim \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \cos\left(\frac{\ell\pi}{d} z\right) \sin\omega t$$

↳ LOWEST TM MODE : TM_{011}

$$E_z \sim \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\cos\omega t$$

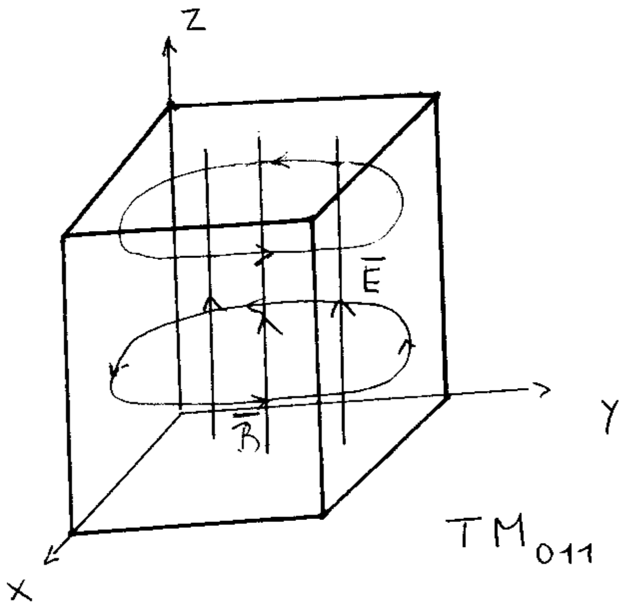
$$E_x = 0$$

$$E_y = 0$$

$$B_x \sim \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\sin\omega t$$

$$B_y \sim \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\sin\omega t$$

$$B_z = 0$$



TM_{011} MODE

RESONANT FREQUENCY

$$\underline{\underline{\omega_{011} = \pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}}$$

⇒ ENERGY DISSIPATION IN A CAVITY

ABOVE SOLUTION : CAVITY WITH PERFECTLY CONDUCTING WALLS

IN REALITY : FINITE σ

↓

DAMPING IN WALLS

↓

AMPLITUDES DECREASE IN TIME BECAUSE WALLS ABSORB ENERGY

↳ IN INTERIOR OF CAVITY

ENERGY DENSITY $T_{00} = \frac{1}{2} (\bar{E}^2 + \bar{B}^2)$

ENERGY IN CAVITY $= T_{00} \cdot V$

↳ VOLUME OF CAVITY

↳ FINITE σ ⇒ FIELDS PENETRATE IN LAYER δ

↑ SKIN DEPTH

ENERGY IN LAYER $\delta \sim T_{00} \cdot (A\delta)$

↳ INTERIOR SURFACE OF CAVITY

THIS IS ~ ENERGY ABSORBED IN METAL PER CYCLE

QUALITY OF CAVITY : Q - FACTOR

$$Q = \frac{\text{ENERGY IN CAVITY}}{\text{ENERGY LOST PER CYCLE}} \sim \frac{V}{A\delta}$$

$$Q = \frac{U}{\underbrace{-T\frac{dU}{d}}_{\text{ENERGY LOST PER CYCLE}}} = -\frac{\omega}{2\pi}\frac{U}{\frac{dU}{d}}$$

$\downarrow$

$$\frac{dU}{d} = -\frac{\omega}{2\pi}\frac{1}{Q}U$$

$$\boxed{U(t) = U_0\, e^{-\omega t/2\pi Q}}$$

ENERGY DECREASES EXP.

DAMPING TIME $\sim \frac{2\pi Q}{\omega} = Q \cdot T$

$\uparrow$ PERIOD OF CAVITY

GOOD CAVITIES $Q \sim 10^4$

SUPERCONDUCTING CAVITIES $Q \sim 10^8$ $\Rightarrow$ GO ON FOR 10^8 CYCLES BEFORE ENERGY IS REDUCED BY e^{-1}

(↳ USED IN ACCELERATORS)