III

SPECIAL RELATIVITY

1) PRINCIPLES OF SPECIAL RELATIVITY

2) LORENTZ TRANSFORMATION OF $\bar{E}$ & $\bar{B}$ FIELDS

3) LORENTZ COVARIANCE OF MAXWELL EQUATIONS

1) PRINCIPLES OF SPECIAL RELATIVITY

⇒ RELATIVITY PRINCIPLE (EINSTEIN 1905)

LAWS OF NATURE ARE IDENTICAL IN EVERY INERTIAL FRAME

↳ FRAMES IN WHICH FREE PARTICLES (ON WHICH NO FORCES ACT) MOVE WITH CONSTANT VELOCITY

↳ INERTIAL FRAMES MOVE RELATIVE TO EACH OTHER ALONG A STRAIGHT LINE & WITH UNIFORM VELOCITY

⇒ PRINCIPLE OF FINITE SPEED OF LIGHT

* MAX. SPEED OF ANY INFORMATION IS GIVEN BY THE SPEED OF LIGHT $c = 2.99792458\ 10^8\ m/s$

(CLASSICAL MECHANICS CORRESPONDS WITH $c \to \infty$)

* SPEED OF LIGHT IS SAME IN EVERY INERTIAL FRAME

⇒ SPACE - TIME

* CONSEQUENCE OF ABOVE PRINCIPLES.

↳ TIME IS NOT ABSOLUTE, DEPENDS ON REFERENCE FRAME

↳ VELOCITIES CANNOT JUST BE ADDED

↳ SPACE & TIME HAVE TO BE TREATED ON "EQUAL FOOTING"

⇓

4 DIM "SPACE-TIME" NOTATION

* SPACE-TIME POINT

$$x^\mu = (x^0, x^1, x^2, x^3) \quad \text{GREEK INDICES } \mu, \nu, \ldots = 0,1,2,3 \text{ (SPACE-TIME)}$$

$$= (x^0, \bar{x})$$

WITH $x^0 = ct$

$$\bar{x} = (x^1, x^2, x^3) \quad \text{LATIN INDICES } i, j, \ldots = 1,2,3 \text{ (SPACE)}$$

* SPACE-TIME DISTANCE / 4-DISTANCE

CONSEQUENCE OF CONSTANT SPEED OF LIGHT

4-DISTANCE BETWEEN 2 SPACE TIME POINTS

$$x_a = (x_a^0, x_a^1, x_a^2, x_a^3)$$

&

$$x_b = (x_b^0, x_b^1, x_b^2, x_b^3)$$

$$s_{ab}^2 \equiv (x_a^0 - x_b^0)^2 - (x_a^1 - x_b^1)^2 - (x_a^2 - x_b^2)^2 - (x_a^3 - x_b^3)^2$$

IS INVARIANT (i.e. SAME FOR ANY INERTIAL OBSERVER)

* COVARIANT / CONTRAVARIANT COMPONENTS

↳ CONTRAVARIANT COMPONENTS x^μ

↳ COVARIANT " $x_\mu \equiv g_{\mu\nu} x^\nu$

(SUM OVER $\nu = 0,1,2,3$)

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$x_\mu = (x_0, x_1, x_2, x_3)$$

$$= (ct, -\bar{x})$$

↳ SPACE-TIME DISTANCE

$$s_{ab}^2 = g_{\mu\nu} (x_a - x_b)^\mu (x_a - x_b)^\nu$$

$$= (x_a - x_b)^\mu (x_a - x_b)_\mu$$

$$\equiv (x_a - x_b)^2$$

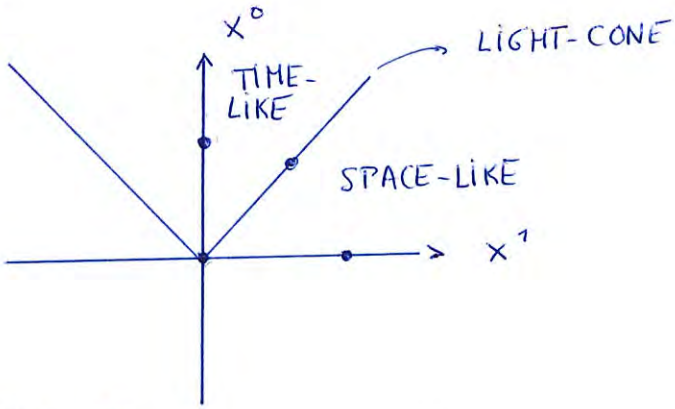
$s_{ab}^2 = 0$: LIGHT-LIKE DISTANCE
(CAN ONLY BE CONNECTED BY PARTICLE TRAVELING AT SPEED OF LIGHT ⇒ DEFINES LIGHT-CONE)

$s_{ab}^2 > 0$: TIME-LIKE DISTANCE
(CAN BE TRAVELED BY PARTICLE MOVING WITH SPEED $v < c$)

$s_{ab}^2 < 0$: SPACE-LIKE DISTANCE
(CANNOT BE CONNECTED BY A PHYSICAL INFORMATION)

e.g. $x_b = 0$

$x_a = x$

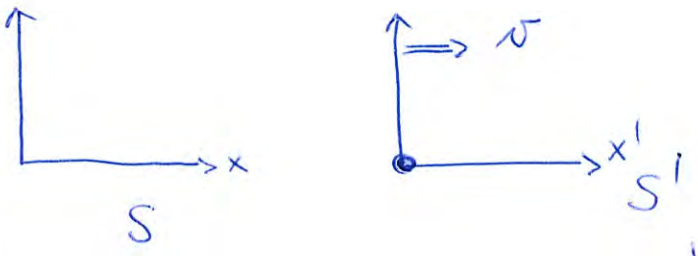


* PROPER TIME τ

TIME OF A CLOCK IN A REFERENCE SYSTEM IN WHICH CLOCK IS AT REST

$$\left.\begin{aligned} dx^0 &= c\,d\tau \\ dx^i &= 0 \end{aligned}\right\} \quad ds^2 = c^2 d\tau^2$$

⇒ TIME DILATATION



v: RELATIVE MOTION OF S' w.r.t S

$y' = y$

$z' = z$

CLOCK AT REST IN ORIGIN OF S'

↳ INVARIANCE OF INTERVAL

$$(x^0)^2 - x^2 = (x'^0)^2 - (x')^2$$

$$\| \quad 0$$

↳ FROM POINT OF VIEW OF S

x : POSITION OF CLOCK

$$x = vt \equiv \beta x^0 \qquad \boxed{\beta \equiv \frac{v}{c}}$$

$\beta \leq 1$!

$$(x^0)^2 - \beta^2 (x^0)^2 = (x'^0)^2$$

$$(x^0)^2 = \frac{1}{1-\beta^2} (x'^0)^2$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1.$$

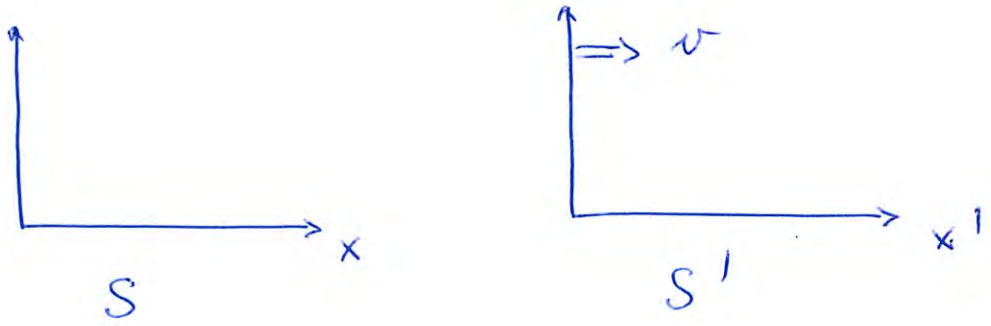
$$t = \gamma\, t' \qquad t \geq t'$$

TIME SEEN BY S IS STRETCHED (TIME DILATATION)

MOVING CLOCK TICKS SLOWER

$$x = \beta\gamma\, x'^{0}$$

⇒ LORENTZ TF FROM INVARIANCE OF SPACE-TIME INTERVAL



↳ GENERAL LINEAR TF

$$x^0 = a\,x'^0 + b\,x'$$

$$x = c\,x' + d\,x'^0$$

$$\left.\begin{aligned} y &= y' \\ z &= z' \end{aligned}\right\}$$ SPECIAL CASE LORENTZ TF ALONG x-AXIS

↳ SPECIAL CASE: CLOCK AT REST IN S'

$$x' = 0$$

$$x^0 = \gamma\,x'^0 \quad \rightarrow \quad a = \gamma$$

$$x = \beta\gamma\,x'^0 \quad \rightarrow \quad d = \beta\gamma$$

↳

$$\begin{cases} x^0 = \gamma\,x'^0 + b\,x' \\ x = c\,x' + \beta\gamma\,x'^0 \end{cases}$$

INVARIANCE OF $(x^0)^2 - x^2 - y^2 - z^2$

$$(x^0)^2 - x^2 - y^2 - z^2 = (x'^0)^2 - (x')^2 - (y')^2 - (z')^2$$

$$(\gamma x'^0 + b x')^2 - (c x' + \beta\gamma x'^0)^2 = (x'^0)^2 - (x')^2$$

$$\begin{cases} \gamma^2(1-\beta^2) = 1 \\ -b^2 + c^2 = 1 \\ b\gamma - c\beta\gamma = 0 \end{cases}$$

$\Downarrow$

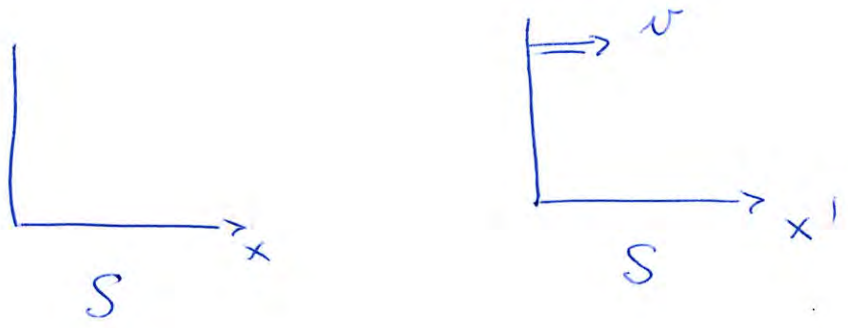
$$\underline{b = \beta c}$$

$$c^2(1-\beta^2) = 1 \quad \Rightarrow \quad c^2 = \frac{1}{1-\beta^2}$$

$$\underline{c = \gamma} \Rightarrow \underline{b = \beta\gamma}$$

$$\boxed{\begin{aligned} x^0 &= \gamma(x'^0 + \beta x') \\ x &= \gamma(x' + \beta x'^0) \end{aligned}} \iff \boxed{\begin{aligned} x'^0 &= \gamma(x^0 - \beta x) \\ x' &= \gamma(x - \beta x^0) \end{aligned}}$$

⇒ "ADDITION OF VELOCITIES"



BULLET FIRED FROM MOVING FRAME IN x' DIRECTION

↳ WHAT IS THE SPEED SEEN BY x ?

$\Delta x, \Delta x'$ DISTANCE TRAVELED BY BULLET IN EACH FRAME

$\Delta t, \Delta t'$: TIME TRAVELED BY BULLET IN EACH FRAME

$$\Delta x = \gamma (\Delta x' + \beta c \Delta t')$$

$$\Delta t = \gamma (\Delta t + \frac{\beta}{c} \Delta x')$$

$$v_S = \frac{\Delta x}{\Delta t} = \frac{\gamma (v_{S'} + \beta c)}{\gamma (1 + \frac{\beta}{c} v_{S'})}$$

↑ SPEED SEEN BY S

$v_{S'} \equiv \frac{\Delta x'}{\Delta t'}$ SPEED SEEN BY S'

$$\beta = \frac{v}{c}$$

$$v_s = \frac{v_{s'} + v}{1 + \frac{v v_{s'}}{c^2}}$$

NOTE ↳ $v_s \leq c$!

↳ FOR $v = c \Rightarrow v_s = c$

$v_{s'} = c \Rightarrow v_s = c$

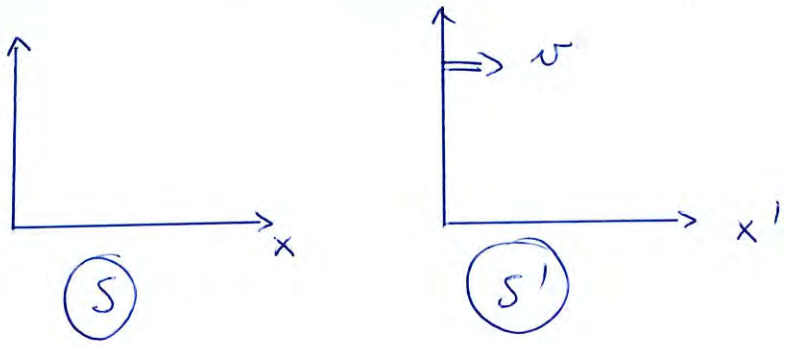
↳ FOR $v \ll c$ OR $v_{s'} \ll c$

(NON-RELATIVISTIC LIMIT)

$$v_s \simeq v_{s'} + v$$

APPROXIMATE ADDITION OF VELOCITIES

⇒ LORENTZ TRANSFORMATION



$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\beta = v/c \leq 1$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 1$$

↳ $\cosh\eta = \gamma$

↳ $\beta = \tanh\eta \Rightarrow \beta\gamma = \sinh\eta$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

(EXERCISES)

⇒ CONSEQUENCES: LENGTH CONTRACTION, TIME DILATATION, ADDITION OF VELOCITIES

⇒ 4 - VECTORS

↳ $A^\mu (A^0, \bar{A})$ WITH $\bar{A}(A^1, A^2, A^3)$

4 - COMPONENT VECTORS WHICH TRANSFORM UNDER A LORENTZ TF (i.e. WHEN GOING TO A DIFFERENT INERTIAL FRAME) AS THE 4 - COORDINATES x^μ

$$A^\mu \rightarrow A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

↳ A^μ : CONTRAVARIANT COMPONENTS

A_μ : COVARIANT COMPONENTS

↳ $A^2 \equiv A_\mu A^\mu$ IS LORENTZ INVARIANT (FRAME INDEPENDENT)

$$A^2 \rightarrow A'^2 = g_{\mu\nu} A'^\mu A'^\nu$$

$$= \underbrace{g_{\mu\nu} \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'}}_{\| \text{ (EXERCISE)}} A^{\mu'} A^{\nu'}$$

$$g_{\mu'\nu'}$$

$$= A_{\mu'} A^{\mu'} = A^2$$

↳ $A \cdot B = A_\mu B^\mu$ IS LORENTZ INVARIANT (LORENTZ SCALAR)

$= A^0 B^0 - \bar{A} \cdot \bar{B}$

↳ GRADIENT FIELD

* SCALAR FIELD $\Phi = \Phi(x^\mu) = \Phi(t, \bar{x})$

* 4 - VECTOR GRADIENT FIELD

$$A_\mu \equiv \frac{\partial}{\partial x^\mu} \Phi(t, \bar{x}) \equiv \partial_\mu \Phi = (\partial_0 \Phi, \partial_i \Phi)$$

$$= \left(\frac{1}{c} \frac{\partial \Phi}{\partial t}, \bar{\nabla} \Phi \right)$$

NOTE: $\rightsquigarrow$ $d\Phi = (\partial_\mu \Phi)\, dx^\mu$ IS INVARIANT

($d\Phi$: DIFFERENTIAL; $\partial_\mu \Phi$: COVARIANT)

$$= \frac{\partial \Phi}{\partial x^\mu} dx^\mu$$

$$\rightsquigarrow \partial^\mu \Phi = (\partial^0 \Phi, \partial^i \Phi) = \left(\frac{1}{c} \frac{\partial \Phi}{\partial t}, -\vec{\nabla} \Phi \right)$$

$\Rightarrow$ **LORENTZ TENSORS**

↳ $T^{\mu\nu}$ 2° RANK

$T^{\lambda\mu\nu}$ 3° RANK

...

TRANSFORM UNDER LORENTZ TF AS

$$T^{\mu\nu\ldots} \rightarrow T'^{\mu\nu\ldots} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \ldots T^{\mu'\nu'\ldots}$$

↳ TENSOR WITH ONE OR MORE COVARIANT INDICES

$$T^\mu{}_\nu = g_{\nu\lambda} T^{\mu\lambda}$$

$$T_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} T^{\kappa\lambda}$$

↳ SYMMETRIES UNDER EXCHANGE OF LORENTZ INDICES

SYMMETRIC UNDER EXCHANGE $S^{\mu\nu} = S^{\nu\mu}$ OR $S^\mu{}_\nu = S_\nu{}^\mu$

ANTI-SYMMETRIC UNDER EXCHANGE $A^{\mu\nu} = -A^{\nu\mu}$

$(A^{00} = A^{ii} = 0)$

↳ TRACE OF TENSOR

$$T = T^\mu{}_\mu = g_{\mu\nu} T^{\mu\nu}$$ LORENTZ SCALAR

↳ METRIC TENSOR $g_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{(SYMMETRIC)}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g^{\mu}{}_{\nu} = g^{\mu\mu'} g_{\mu'\nu}$$

$$= \delta^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

↳ TOTAL ANTI-SYMMETRIC TENSOR $\varepsilon_{\mu\nu\kappa\lambda}$

DEF. $\varepsilon_{0123} \equiv +1$

TOTAL ANTI-SYMM UNDER PERMUTATION OF 2 - INDICES

GENERALIZATION OF $\varepsilon_{ijk} = \begin{cases} +1 & \text{EVEN} \\ -1 & \text{ODD} \\ 0 & \end{cases}$ } PERM. OF (1, 2, 3)

NOTE: $\varepsilon^{0123} = g^{00} g^{11} g^{22} g^{33} \varepsilon_{0123}$

$= (+1)(-1)(-1)(-1) \underbrace{\varepsilon_{0123}}_{+1}$

$= -1$

↳ DUAL TENSOR

$\tilde{A}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} A^{\kappa\lambda}$ $\quad \tilde{A}$ DUAL TO A

$A, \tilde{A}$ ARE ANTI-SYMMETRIC

↳ FIELD TENSOR ANTI-SYMM. DERIVATIVE OF 4-VECTOR

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

* $F_{0i} = \frac{1}{c} \frac{\partial A_i}{\partial t} - \partial_i A_0$

↳ DEFINES COMPONENTS OF 3-VECTOR E^i

$$\bar{E} = -\frac{1}{c} \frac{\partial \bar{A}}{\partial t} - \bar{\nabla} A_0$$

* SPATIAL COMPONENTS

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

DUAL TO $\bar{\nabla} \times \bar{A}$

$$B^i \equiv + \frac{1}{2} \varepsilon^{ijk} F_{jk}$$

$$= + \varepsilon^{ijk} \partial_j A_k$$

$$= (\bar{\nabla} \times \bar{A})^i$$

$\varepsilon^{ijk} = - \varepsilon_{ijk}$

CHOOSE $\varepsilon_{123} = + 1$

* $\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu)$$

$\partial_\mu \partial^\mu \equiv \Box$ (d'ALEMBERT OPERATOR)

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \bar{\nabla}^2$$

↑ LAPLACE OPERATOR

$\Rightarrow$ **IMPROPER LORENTZ TF.** $\quad \Lambda^{\mu}{}_{\nu}\,\Lambda_{\mu}{}^{\nu'} = g_{\nu}{}^{\nu'}$

↳ PROPER LORENTZ TF $\quad \det(\Lambda) = +1$

↳ IMPROPER LORENTZ TF $\quad \det(\Lambda) = -1$

* SPATIAL INVERSION (REFLECTION) $\quad \begin{cases} x'^{i} = -x^{i} \\ t' = t \end{cases}$

* TIME INVERSION $\quad \begin{cases} t' = -t \\ x'^{i} = x^{i} \end{cases}$

↳ AXIAL VECTOR (PSEUDO-VECTOR) $\quad \tilde{A}^{\mu}$

SPATIAL COMPONENTS DO NOT CHANGE SIGN UNDER SPATIAL INVERSION

e.g. $\quad \bar{c} = \bar{a} \times \bar{b}$

↳ PSEUDO SCALAR

VECTOR A^{μ}

AXIAL-VECTOR $\tilde{B}^{\mu}$

$A_{\mu}\,\tilde{B}^{\mu}$ CHANGES SIGN UNDER SPATIAL INVERSION

⇒ RELATIVISTIC KINEMATICS

↳ 4 - MOMENTUM

$$p^\mu \left(\frac{E}{c}, \bar{p} \right) \qquad p^\mu_{R.F.} \left(m_0 c, 0 \right)$$

WITH $E = mc^2$ $\qquad m = \dfrac{m_0}{\sqrt{1-\beta^2}}$

$\bar{p} = m\bar{v}$

$$p_\mu p^\mu = \frac{E^2}{c^2} - \bar{p}^2$$

$$= m_0^2 c^2$$

NOTE $\quad E^2 = m^2 c^4$

$$= \frac{m_0^2 c^4}{1-\beta^2}$$

$$E^2 - \beta^2 E^2 = m_0^2 c^4$$

$$\beta E = \frac{m v}{c} c^2 = cp$$

$$E^2 = c^2 \bar{p}^2 + m_0^2 c^4$$

↳ RELATIVISTIC KINETIC ENERGY T

$$T \equiv E - m_0 c^2 = mc^2 - m_0 c^2$$

$$= m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right)$$

NON-REL. LIMIT $\underset{\beta \ll 1}{\simeq}$ $m_0 c^2 \left(\frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + O(\beta^6) \right) = \frac{1}{2} m_0 v^2 + \dots$

2) LORENTZ TRANSFORMATION OF $\bar{E}$ & $\bar{B}$ FIELDS

* E.M. POTENTIAL

4-VECTOR $A^\mu = (A^0, \bar{A}) = (\Phi, \bar{A})$

GAUGE TF: $A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi$

$$\begin{cases} A^0 \to A'^0 = A^0 - \frac{1}{c}\frac{\partial \chi}{\partial t} \\ \bar{A} \to \bar{A}' = \bar{A} + \bar{\nabla}\chi \end{cases}$$

* E.M. FIELD TENSOR

DEFINE E.M. FIELD TENSOR SUCH THAT IT IS INVARIANT UNDER GAUGE TF.

$$\boxed{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$$

$$= \partial^\mu (A^\nu - \partial^\nu \chi) - \partial^\nu (A^\mu - \partial^\mu \chi)$$

$$= \partial^\mu A^\nu - \cancel{\partial^\mu \partial^\nu \chi} - \partial^\nu A^\mu + \cancel{\partial^\nu \partial^\mu \chi}$$

$$= F^{\mu\nu}$$

$F^{\mu\nu} = -F^{\nu\mu} \Rightarrow$ 6 INDEPENDENT COMPONENTS

* ELEMENTS OF $F^{\mu\nu}$ DEFINE $\bar{E}$ & $\bar{B}$ FIELDS

↳ $F^{0i} = \partial^0 A^i - \partial^i A^0$

$= \left(\frac{1}{c} \frac{\partial \bar{A}}{\partial t} + \bar{\nabla} \Phi \right)^i$

$= - \bar{E}^i$

$F_{0i} = \bar{E}^i$

↳ $F^{12} = \partial^1 A^2 - \partial^2 A^1$

$= - (\bar{\nabla} \times \bar{A})^3$

$= - \bar{B}_z$

↳ $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

↳ $F_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$

↳ BECAUSE $F^{\mu\nu}$ IS INVARIANT UNDER GAUGE TF.

⇓

$\bar{E}$ & $\bar{B}$ FIELDS ARE INVARIANT UNDER GAUGE TF.

* DUAL FIELD TENSOR

$$\tilde{F}_{\alpha\beta} \equiv -\frac{1}{2}\,\varepsilon_{\alpha\beta\mu\nu}\,F^{\mu\nu}$$

WITH $\varepsilon_{0123} = +1$

$\varepsilon_{\alpha\beta\mu\nu}$ TOTALLY ANTI-SYMM. TENSOR

$$\tilde{F}_{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

e.g. $\hookrightarrow \tilde{F}_{01} = -\frac{1}{2}\,\varepsilon_{01\mu\nu}\,F^{\mu\nu}$

$$= -\frac{1}{2}\left(\varepsilon_{0123}\,F^{23} - \varepsilon_{0132}\,F^{32}\right)$$

$$= -F^{23}$$

$$= B_x$$

$\hookrightarrow \tilde{F}_{12} = -\frac{1}{2}\,\varepsilon_{12\mu\nu}\,F^{\mu\nu}$

$$= -\frac{1}{2}\left(\varepsilon_{1203}\,F^{03} - \varepsilon_{1230}\,F^{30}\right)$$

$$= -F^{03}$$

$$= E_z$$

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$\tilde{F}^{\alpha\beta}$ IS OBTAINED FROM $F^{\alpha\beta}$

BY THE REPLACEMENT

DUALITY TRANSFORM. $\begin{cases} \bar{E} \to \bar{B} \\ \bar{B} \to -\bar{E} \end{cases}$

* LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

$F^{\mu\nu}$ TRANSFORMS AS 2° RANK LORENTZ TENSOR.

i.e. UNDER LORENTZ TF.

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F^{\mu'\nu'}$$

$\rightsquigarrow$ FOR BOOST ALONG x-AXIS

$$\hookrightarrow \Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

MATRIX NOTATION

$$F' = \Lambda F \Lambda^{T}$$

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'} F^{\mu'\nu'} (\Lambda^{T})_{\nu'}{}^{\nu}$$

NOTE $(\Lambda^{T})^{\nu}{}_{\nu'} = \Lambda_{\nu'}{}^{\nu}$

$$\Lambda F \Lambda^T = \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} +\beta\gamma E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & -\beta\gamma E_x & -B_z & B_y \\ (\gamma E_y - \beta\gamma B_z) & (-\beta\gamma E_y + \gamma B_z) & 0 & -B_x \\ (\gamma E_z + \beta\gamma B_y) & (-\beta\gamma E_z - \gamma B_y) & B_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E_x & -\gamma(E_y - \beta B_z) & -\gamma(E_z + \beta B_y) \\ E_x & 0 & -\gamma(B_z - \beta E_y) & \gamma(B_y + \beta E_z) \\ \gamma(E_y - \beta B_z) & \gamma(B_z - \gamma E_y) & 0 & -B_x \\ \gamma(E_z + \beta B_y) & -\gamma(B_y + \beta E_z) & B_x & 0 \end{bmatrix}$$

IN VECTOR NOTATION : FOR GENERAL BOOST $\overline{\beta}$

$\overline{a}_{\parallel}$: COMPONENT $\parallel$ TO BOOST DIRECTION (HERE x-AXIS)

$\overline{a}_{\perp}$: COMPONENT $\perp$ TO BOOST DIRECTION

$\overline{E}'_{\parallel} = \overline{E}_{\parallel}$	$\overline{E}'_{\perp} = \gamma(\overline{E}_{\perp} + \overline{\beta} \times \overline{B})$
$\overline{B}'_{\parallel} = \overline{B}_{\parallel}$	$\overline{B}'_{\perp} = \gamma(\overline{B}_{\perp} - \overline{\beta} \times \overline{E})$

CHECK FOR $\bar{\beta} = \beta \bar{e}_x$

e.g. $E'_y = \gamma \left(E_y + \beta \underbrace{(\bar{e}_x \times \bar{B})_y}_{-B_z} \right)$

$\overset{!}{=} \gamma (E_y - \beta B_z)$

$B'_z = \gamma \left(B_z - \beta \underbrace{(\bar{e}_x \times \bar{E})_z}_{E_y} \right)$

$\overset{!}{=} \gamma (B_z - \beta E_y)$

∴ UNDER LORENTZ TF:

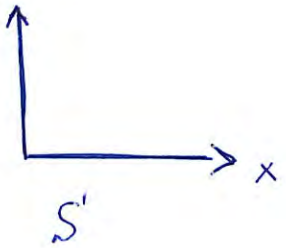
ELECTRIC & MAGNETIC FIELDS TRANSFORM INTO EACH OTHER

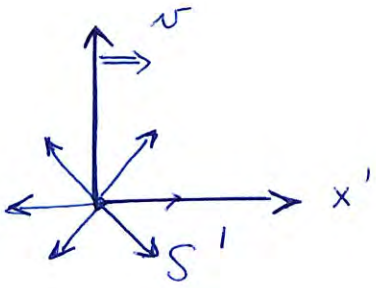
↳ $\bar{E}$ & $\bar{B}$ FIELDS ARE FRAME DEPENDENT !

$\bar{E}$ FIELD FOR ONE OBSERVER

BECOMES $\bar{B}$ FIELD FOR ANOTHER OBSERVER.

* EXAMPLE : $\bar{E}, \bar{B}$ FIELDS OF MOVING POINT CHARGE





IN S' POINT CHARGE q IS AT REST

S' MOVES WITH VELOCITY v w.r.t S

? WHAT ARE $\bar{E}$ & $\bar{B}$ FIELDS AS SEEN BY OBSERVER S ?

$\Rightarrow$ IN S' : $\bar{E}'$ POINT RADIALLY OUTWARD

$$\bar{E}' = \frac{q}{4\pi} \frac{\bar{x}'}{x'^3}$$

$$\bar{B}' = 0$$

($\bar{\nabla} \cdot \bar{E} = \rho$ IN HEAVYSIDE-LORENTZ UNITS)

$\Rightarrow$ IN S : (S MOVES W.R.T S' WITH VELOCITY $-\bar{v}$)

$$\bar{E}_{\parallel} = \bar{E}'_{\parallel} \qquad \bar{E}_{\perp} = \gamma (\bar{E}'_{\perp} - \bar{\beta} \times \bar{B}')$$

$$\bar{B}_{\parallel} = \bar{B}'_{\parallel} \qquad \bar{B}_{\perp} = \gamma (\bar{B}'_{\perp} + \bar{\beta} \times \bar{E}')$$

$$E_x = E'_x \qquad \bar{E}_\perp = \gamma\left(\bar{E}'_\perp\right)$$

$$B_x = 0 \qquad \bar{B}_\perp = \beta\gamma\left(\bar{e}_x \times \bar{E}'\right)_\perp$$

AT $t = t' = 0$ BOTH FRAME COINCIDE

$$\begin{cases} x = \gamma(x' + vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \frac{v}{c^2}x'\right) \end{cases}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

$$b^2 = y^2 + z^2 = y'^2 + z'^2$$

$$E_x = \frac{q}{4\pi} \frac{\gamma(x - vt)}{\left[\gamma^2(x - vt)^2 + b^2\right]^{3/2}}$$

$$E_y = \frac{q}{4\pi} \frac{\gamma y}{\left[\gamma^2(x - vt)^2 + b^2\right]^{3/2}}$$

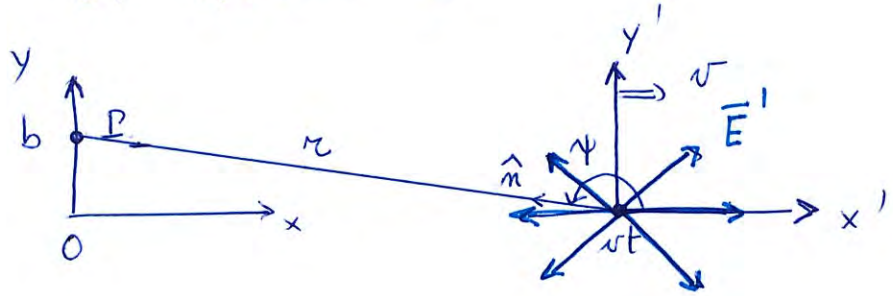
$$E_z = \frac{q}{4\pi} \frac{\gamma z}{\left[\gamma^2(x - vt)^2 + b^2\right]^{3/2}}$$

$$B_x = 0$$

$$B_y = -\frac{q}{4\pi}\,\beta\gamma\,\frac{z}{\left[\gamma^2(x-vt)^2 + b^2\right]^{3/2}}$$

$$B_z = +\frac{q}{4\pi}\,\beta\gamma\,\frac{y}{\left[\gamma^2(x-vt)^2 + b^2\right]^{3/2}}$$

e.g. FIELD AS SEEN IN POINT P(t, $x=0$, $y=b$, $z=0$) OF SYSTEM S



IN P :

$$E_x = -\frac{q}{4\pi}\,\frac{\gamma v t}{\left[\gamma^2 v^2 t^2 + b^2\right]^{3/2}}$$

$$E_y = \frac{q}{4\pi}\,\frac{\gamma b}{\left[\gamma^2 v^2 t^2 + b^2\right]^{3/2}}$$

$$E_z = 0$$

$$B_x = 0$$

$$B_y = 0$$

$$B_z = \frac{q}{4\pi}\,\frac{\beta\gamma b}{\left[\gamma^2 v^2 t^2 + b^2\right]^{3/2}} = \beta E_y$$

REMARKS :

1) NON-REL. LIMIT FOR E_y ($\beta = 0$, $\gamma = 1$)

$$E_y = \frac{q}{4\pi} \frac{b}{(v^2 t^2 + b^2)^{3/2}}$$

2) EXTREME REL. LIMIT $\beta \to 1$, $v \to c$

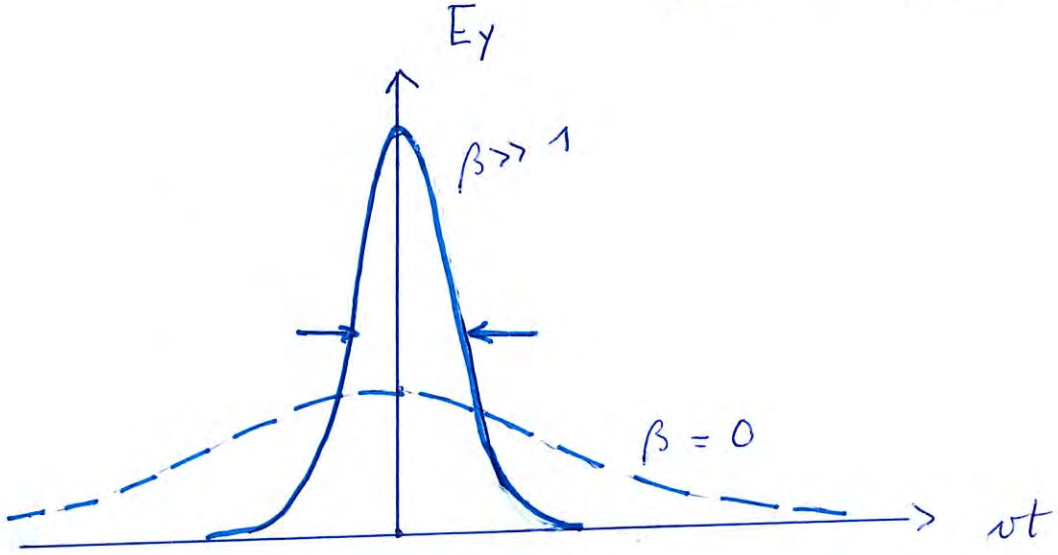
$$E_y = B_z$$

3) PEAK VALUE AT $t = 0$

$$E_y(t=0) = \frac{q}{4\pi} \frac{\gamma}{b^2}$$

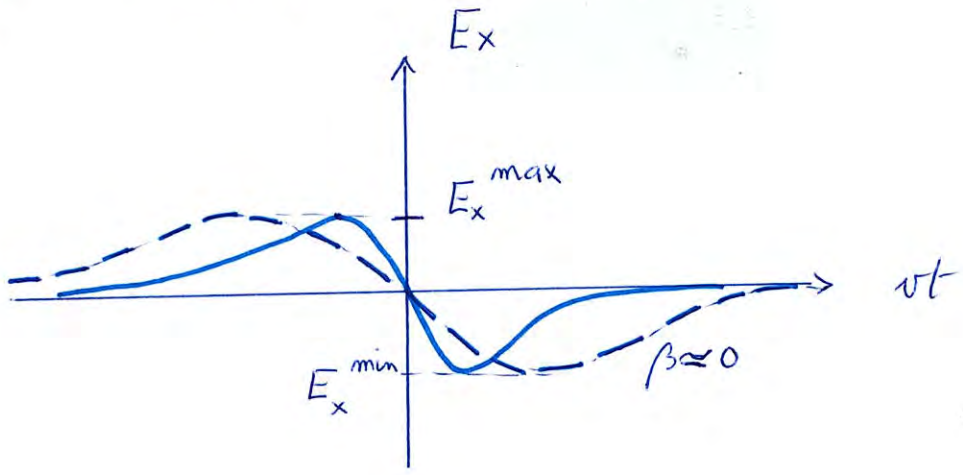
$$= \gamma \underbrace{\left(\frac{q}{4\pi} \frac{1}{b^2} \right)}$$

NON-RELATIVISTIC VALUE OF FIELD



WIDTH OF PEAK $\Delta t \sim \frac{b}{\gamma v}$

4) $E_x(t)$



MAX FIELD $\dfrac{\partial E_x}{\partial(vt)} = 0 \rightsquigarrow vt = \pm \dfrac{b}{\sqrt{2}\gamma}$

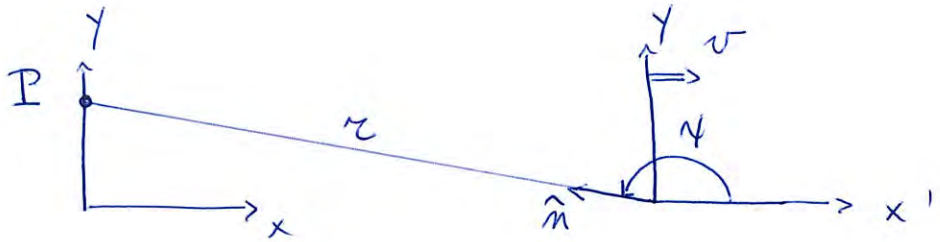
$$E_x^{min} = E_x\left(vt = +\frac{b}{\sqrt{2}\gamma}\right) = -\frac{9}{4\pi}\,\frac{1}{b^2}\,\frac{2}{\sqrt{27}}$$

$$E_x^{max} = -E_x^{min}$$

⇑ MIN/MAX. IS INDEPENDENT OF v

BUT OCCURS AT SMALLER TIMES FOR $v >>$

5) FIELDS RELATIVE TO PRESENT POSITION OF CHARGE $x = vt$ IN LAB. (SYSTEM S)



$$\cos\psi = \hat{n}\cdot\hat{v} \qquad \hat{v} \equiv \frac{\bar{v}}{v}$$

$$r^2 = b^2 + v^2t^2$$

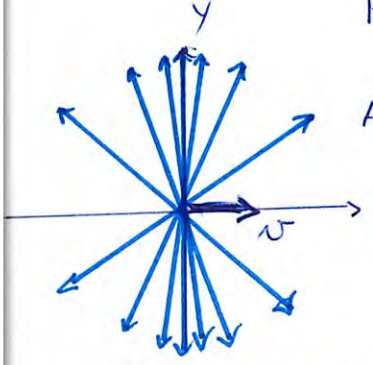
$$\gamma^2 v^2 t^2 + b^2 = \gamma^2 r^2 + (1-\gamma^2)\, b^2$$

$$\downarrow \qquad b^2 = r^2 \sin^2\psi$$

$$= r^2\left[\gamma^2 - \beta^2\gamma^2 \sin^2\psi\right]$$

$$= r^2\gamma^2\left[1 - \beta^2 \sin^2\psi\right]$$

$$\boxed{\bar{E} = \frac{q}{4\pi}\,\frac{\bar{r}}{r^3\gamma^2\left(1-\beta^2\sin^2\psi\right)^{3/2}}}$$



SPATIAL SNAPSHOT

RADIAL BUT ONLY ISOTROPIC FOR $\beta = 0$

ALONG DIRECTION OF MOTION $(\psi = 0, \pi)$

$\hookrightarrow E \sim \frac{1}{\gamma^2}$

$\perp$ TO DIRECTION OF MOTION $(\psi = \pi/2) \rightarrow E \sim \gamma$

3) LORENTZ COVARIANCE OF MAXWELL EQUATIONS

(i.e. EQS. HAVE SAME FORM IN EVERY INERTIAL FRAME)

* 4-VECTOR CURRENT

$$J^\mu \equiv (c\rho, \bar{J}) = (J^0, \bar{J}) \quad \text{WITH} \quad \partial_\mu J^\mu = 0$$

(CURRENT CONSERVATION)

* MAXWELL EQ. WITH CURRENTS

$$\begin{cases} \bar{\nabla}\cdot\bar{E} = \rho \\ \bar{\nabla}\times\bar{B} = \frac{1}{c}\bar{J} + \frac{1}{c}\frac{\partial \bar{E}}{\partial t} \end{cases}$$

$\Downarrow$ IN TERMS OF POTENTIALS, USING LORENTZ GAUGE

$$\begin{cases} \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi = \rho \\ \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2\bar{A} = \frac{1}{c}\bar{J} \end{cases}$$

$\Updownarrow$

$$\begin{cases} \partial_\mu \partial^\mu A^0 = \frac{1}{c} J^0 \\ \partial_\mu \partial^\mu A^i = \frac{1}{c} J^i \end{cases}$$

$\therefore\quad \partial_\mu \partial^\mu A^\nu = \frac{1}{c} J^\nu$

THIS CAN BE WRITTEN USING FIELD TENSOR

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu}$$

INHOMOGENEOUS MAXWELL EQS.

INDEED $\quad \partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu)$$

LORENTZ CONDITION $\frac{1}{c}\frac{\partial \Phi}{\partial t} + \bar{\nabla}.\bar{A} = 0$

IN COVARIANT NOTATION $\partial_\mu A^\mu = 0$

$\therefore\ \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu$

* MAXWELL EQS. WITHOUT CURRENTS :

$$\begin{cases} \bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \\ \bar{\nabla}.\bar{B} = 0 \end{cases}$$

CORRESPONDS WITH 2 PREVIOUS EQS.

WITH $\rho = 0$, $\bar{J} = 0$

AND DUALITY TF $\bar{E} \to \bar{B}$, $\bar{B} \to -\bar{E}$

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu}$$

$\therefore\ \partial_\mu \tilde{F}^{\mu\nu} = 0$ HOMOGENEOUS MAXWELL EQS.

* IN SUMMARY : MAXWELL EQS. IN COVARIANT FORM

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu$$

WITH $\tilde{F}_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$