II

STATIC ELECTROMAGNETIC FIELDS

1) ELECTROSTATICS

2) MULTIPOLE EXPANSION

3) ELECTRIC FIELDS IN MATTER : DIELECTRICS

4) MAGNETOSTATICS

5) MAGNETIC FIELDS IN MATTER

1) ELECTROSTATICS

⇒ BASIC EQS. OF ELECTROSTATICS

$\bar{B} = 0 \quad , \quad \dot{\bar{E}} = 0$

$$\bar{\nabla} \cdot \bar{E} = \rho$$

$$\bar{\nabla} \times \bar{E} = 0 \quad \Rightarrow \quad \bar{E} = -\bar{\nabla}\phi \qquad \phi : \text{SCALAR POTENTIAL}$$

$$\bar{\nabla} \cdot \bar{E} = \rho \quad \Rightarrow \quad \boxed{\nabla^2 \phi = -\rho} \qquad \text{POISSON EQ.}$$

⇒ GREEN'S FUNCTIONS

↳ FOR LOCALIZED POINT-SOURCE

$$\rho(\bar{x}) = \delta^3(\bar{x} - \bar{x}')$$

POISSON EQ: $\nabla_x^2 G(\bar{x}, \bar{x}') = -\delta^3(\bar{x} - \bar{x}')$

SOLUTION OF POISSON EQ. WITH LOCALIZED POINT SOURCE IS CALLED GREEN'S FUNCTION $G(\bar{x}, \bar{x}')$

$G(\bar{x}, \bar{x}') = G(\bar{x}', \bar{x})$ SYMMETRIC

↳ A SOLUTION OF POISSON EQ. FOR A CHARGE DISTR. $\rho(\bar{x})$:

$$\boxed{\phi(\bar{x}) = \int d^3\bar{x}' \; G(\bar{x}, \bar{x}') \, \rho(\bar{x}')}$$

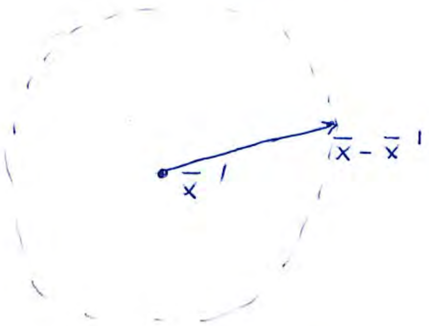
NOTE: THIS SOLUTION IS NOT UNIQUE

⇝ WE CAN ALWAYS ADD SOLUTION OF LAPLACE EQ. $\nabla^2 \phi(\bar{x}) = 0$

↳ SOLUTIONS FOR GREEN'S FUNCTION

$$* \quad \nabla_x^2 \frac{1}{|\bar{x}-\bar{x}'|} = -4\pi\,\delta^3(\bar{x}-\bar{x}')$$

PROOF $\displaystyle\int d^3\bar{x}\ \nabla_x^2 \frac{1}{|\bar{x}-\bar{x}'|} = -4\pi$



CHOOSE $\bar{x}'$ AS ORIGIN

$$\nabla_x^2 \frac{1}{|\bar{x}-\bar{x}'|} = \bar{\nabla}\cdot\left(\bar{\nabla}\frac{1}{|\bar{x}-\bar{x}'|}\right)$$

$$= -\bar{\nabla}\cdot\left(\frac{\bar{x}-\bar{x}'}{|\bar{x}-\bar{x}'|^3}\right)$$

$$\int_V d^3\bar{x}\ \nabla_x^2 \frac{1}{|\bar{x}-\bar{x}'|} \underset{\text{GAUSS THEOREM}}{=} -\int_{\Gamma(V)} d\bar{S}\cdot\frac{\bar{x}-\bar{x}'}{|\bar{x}-\bar{x}'|^3}$$

$$d\bar{S} = 4\pi R^2\,\hat{R}$$

$$\bar{R} \equiv \bar{x}-\bar{x}'$$

$$\overset{\nabla}{\underset{0}{=}} -4\pi$$

$$* \quad \boxed{G(\bar{x},\bar{x}') = \frac{1}{4\pi}\frac{1}{|\bar{x}-\bar{x}'|} + F(\bar{x},\bar{x}')}$$

WITH $\nabla_x^2 F(\bar{x},\bar{x}') = 0$

PARTICULAR SOLUTION

SOLUTION OF HOMOGENEOUS EQ. (LAPLACE EQ.)

⇒ BOUNDARY VALUE PROBLEMS

↳ TO ENSURE UNIQUE & PHYSICALLY WELL-BEHAVED SOLUTION IN BOUNDED REGION:

⇓

SPECIFY BOUNDARY CONDITIONS

(1) DIRICHLET BOUNDARY PROBLEM

FUNCTION ϕ IS GIVEN ON A CLOSED SURFACE S

$$\phi(\bar{x})\Big|_{\bar{x}\in S} \equiv \phi_0(\bar{x})$$

e.g. ON CONDUCTOR ⇝ SURFACE : EQUIPOTENTIAL

(2) NEUMANN BOUNDARY PROBLEM

NORMAL DERIVATIVE IS GIVEN ON A CLOSED SURFACE S

$$\frac{\partial}{\partial \bar{n}}\phi(\bar{x})\Big|_{\bar{x}\in S} \equiv \sigma_0(\bar{x})$$

e.g. INSULATOR ⇝ GIVEN NORMAL COMPONENT OF $\bar{E}$ - FIELD ON SURFACE

$$E_n = -\bar{n}\cdot\bar{\nabla}\phi$$

⇒ FORMAL SOLUTION OF ELECTROSTATIC BOUNDARY VALUE PROBLEMS

↳ CONSIDER

$$\bar{\nabla}_{x'} \cdot \left(\phi(\bar{x}') \bar{\nabla}_{x'} G(\bar{x}, \bar{x}') - G(\bar{x}, \bar{x}') \bar{\nabla}_{x'} \phi \right)$$

$$= \phi(\bar{x}') \nabla^2_{x'} G(\bar{x}, \bar{x}') - G(\bar{x}, \bar{x}') \nabla^2_{x'} \phi$$

$$= -\phi(\bar{x}') \delta^3(\bar{x} - \bar{x}') + G(\bar{x}, \bar{x}') \rho(\bar{x}')$$

↳ $$\int d^3\bar{x}' \; \bar{\nabla}_{x'} \cdot \left(\phi(x') \bar{\nabla}_{x'} G(\bar{x}, \bar{x}') - G(\bar{x}, \bar{x}') \bar{\nabla}_{x'} \phi(x') \right)$$

$$= -\phi(\bar{x}) + \int d^3\bar{x}' \; G(\bar{x}, \bar{x}') \rho(\bar{x}')$$

⇓ GAUSS THEOREM

$$\phi(\bar{x}) = \int d^3\bar{x}' \; G(\bar{x}, \bar{x}') \rho(\bar{x}') + \oint_S \underbrace{da'}_{\substack{\uparrow \\ \text{SURFACE} \\ \text{ELEMENT}}} \left[G(\bar{x}, \bar{x}') \frac{\partial \phi}{\partial n'} - \phi(\bar{x}') \frac{\partial G(\bar{x}, \bar{x}')}{\partial n'} \right]$$

UNIQUE SOLUTION IS FIXED BY BOUNDARY CONDITIONS
↓
DETERMINES SURFACE INTEGRAL

DIRICHLET BOUNDARY CONDITIONS

$$\underline{G_D(\bar{x},\bar{x}') = 0 \quad , \quad \forall\, \bar{x}' \in S}$$

$$\phi(\bar{x}) = \int d^3\bar{x}'\; G_D(\bar{x},\bar{x}')\,\rho(\bar{x}') - \oint_S da'\; \phi(\bar{x}')\, \frac{\partial G_D(\bar{x},x')}{\partial n'}$$

POTENTIAL IS GIVEN ON S

NEUMANN BOUNDARY CONDITIONS

$$\underline{\frac{\partial G_N}{\partial n'}(\bar{x},\bar{x}') = -\frac{1}{S}, \quad \forall\, \bar{x}' \in S}$$

INDEED $\quad \nabla^2_{x'}\, G(\bar{x},\bar{x}') = -\delta^3(\bar{x}-\bar{x}')$

$$\int d^3\bar{x}'\; \bar{\nabla}_{x'}\cdot\left(\bar{\nabla}_{x'} G(\bar{x},\bar{x}')\right) = -1$$

$$\oint_S da'\; \underbrace{\frac{\partial G}{\partial n}(\bar{x},\bar{x}')}_{-\frac{1}{S}} = -1$$

$$-1 \overset{!}{=} -1$$

BOUNDARY CONDITION CONSISTENT

$$\phi(\bar{x}) = \langle \phi \rangle_S + \int d^3\bar{x}'\; G_N(\bar{x},\bar{x}')\,\rho(\bar{x}') + \oint_S da'\; G_N(\bar{x},\bar{x}')\,\frac{\partial \phi}{\partial n'}$$

AVERAGE OVER S

→ EXAMPLE OF DIRICHLET PROBLEM

POTENTIAL OUTSIDE A CONDUCTING SPHERE

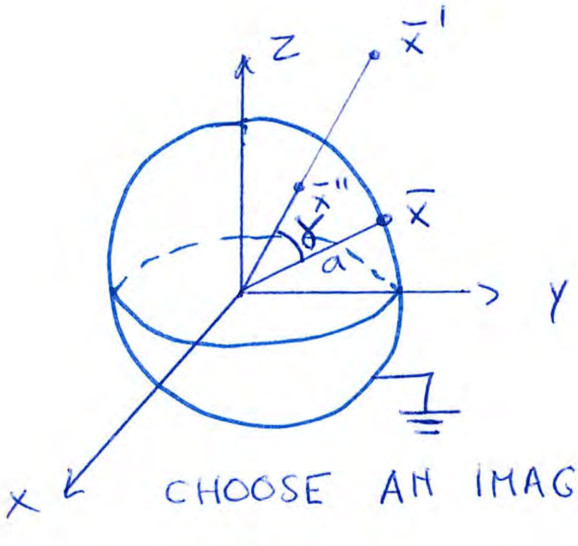
SURFACE OF SPHERE : EQUIPOTENTIAL SURFACE.

* GREEN'S FUNCTION IN PRESENCE OF CONDUCTING SPHERE

GREEN'S FUNCTION : $G(\bar{x}, \bar{x}')$

↳ POTENTIAL IN POINT $\bar{x}$ DUE TO A POINT CHARGE IN $\bar{x}'$ IN PRESENCE OF A CONDUCTING SPHERE WHICH IS GROUNDED $\Rightarrow$ $G_D(\bar{x}, \bar{x}') = 0 \;\; \forall \bar{x} \in S$

POTENTIAL ON SURFACE → ZERO



RADIUS a

CHOOSE AN IMAGE CHARGE q'' IN $\bar{x}''$

$$G_D(\bar{x}, \bar{x}') = \frac{1}{4\pi|\bar{x} - \bar{x}'|} + \frac{q''}{4\pi|\bar{x} - \bar{x}''|}$$

DETERMINE q'' & $\bar{x}''$ SUCH THAT

$$G_D(\bar{x}, \bar{x}') = 0 \qquad \forall \bar{x} \in S$$

$\forall \bar{x} \in S \quad : \quad |\bar{x}| = a$

$$|\bar{x} - \bar{x}'| = \left[|\bar{x}|^2 + |\bar{x}'|^2 - 2|\bar{x}||\bar{x}'| \cos\gamma \right]^{1/2}$$

$$= a \left[1 + \frac{|\bar{x}'|^2}{a^2} - 2\frac{|\bar{x}'|}{a} \cos\gamma \right]^{1/2}$$

$$|\bar{x} - \bar{x}''| = |\bar{x}''| \left[1 + \frac{a^2}{|\bar{x}''|^2} - 2\frac{a}{|\bar{x}''|} \cos\gamma \right]^{1/2}$$

$G_D(\bar{x}, \bar{x}') = 0 \quad \forall \bar{x} \in S$

$\hookrightarrow$ CHOICE $\dfrac{q''}{|\bar{x}''|} = -\dfrac{1}{a}$ AND $\dfrac{a}{|\bar{x}''|} = \dfrac{|\bar{x}'|}{a}$

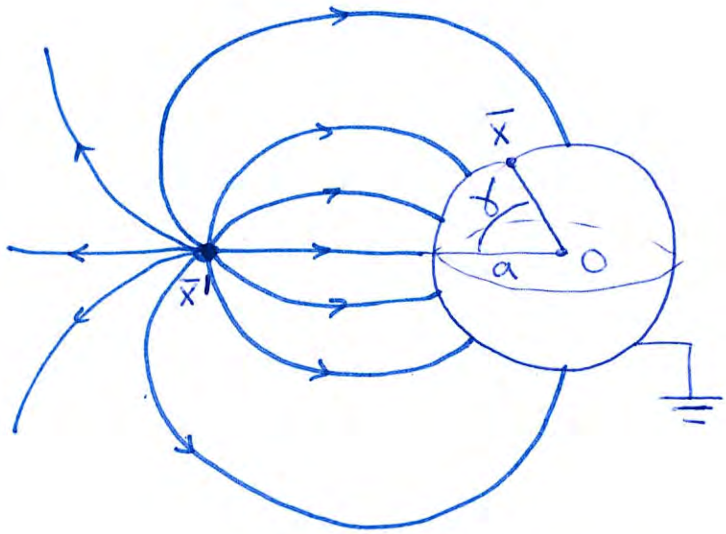
$\boxed{q'' = -\dfrac{a}{|\bar{x}'|}}$, IMAGE CHARGE BIGGER WHEN POINT $\bar{x}'$ CLOSER TO SPHERE

$\boxed{|\bar{x}''| = \dfrac{a^2}{|\bar{x}'|}}$, IMAGE CHARGE CLOSER TO SURFACE WHEN $\bar{x}'$ CLOSER TO SPHERE

$$\therefore \quad G_D(\bar{x}, \bar{x}') = \frac{1}{4\pi|\bar{x} - \bar{x}'|} - \frac{a}{|\bar{x}'|} \frac{1}{4\pi\left|\bar{x} - \frac{a^2}{|\bar{x}'|^2}\bar{x}'\right|}$$

$$\boxed{G_D(\bar{x}, \bar{x}') = \frac{1}{4\pi\left[\bar{x}^2 + \bar{x}'^2 - 2|\bar{x}||\bar{x}'|\cos\gamma\right]^{1/2}} - \frac{1}{4\pi\left[\frac{\bar{x}^2\bar{x}'^2}{a^2} + a^2 - 2|\bar{x}||\bar{x}'|\cos\gamma\right]^{1/2}}}$$

↳ $G_D(\bar{x}, \bar{x}')$ IS POTENTIAL OF A UNIT POINT CHARGE OUTSIDE A GROUNDED CONDUCTING SPHERE



- INSIDE CONDUCTING SPHERE $\bar{E} = 0$

$\Rightarrow$ $\bar{E}$ - FIELD LINES $\perp$ TO CONDUCTOR

↓

BECAUCE SURFACE IS EQUIPOTENTIAL SURFACE

- DUE TO POINT CHARGE ~ INDUCED CHARGE ON SURFACE OF SPHERE

NORMAL COMPONENT OF $\bar{E}$ - FIELD AT SURFACE

$E_n = - \bar{n} . \bar{\nabla}_x G_D$ (RADIALLY OUTWARD FROM SPHERE)

$$\bar{n} . \bar{\nabla}_x G_D(\bar{x}, \bar{x}') \Big|_{\substack{|\bar{x}|=a, \\ \bar{x} \in S}}$$

$$= - \frac{(a - |\bar{x}'| \cos \gamma)}{4\pi \left[a^2 + \bar{x}'^2 - 2a|\bar{x}'| \cos \gamma\right]^{3/2}}$$

$$+ \frac{(|\bar{x}'|^2/a - |\bar{x}'| \cos \gamma)}{4\pi \left[\bar{x}'^2 + a^2 - 2a|\bar{x}'| \cos \gamma\right]^{3/2}}$$

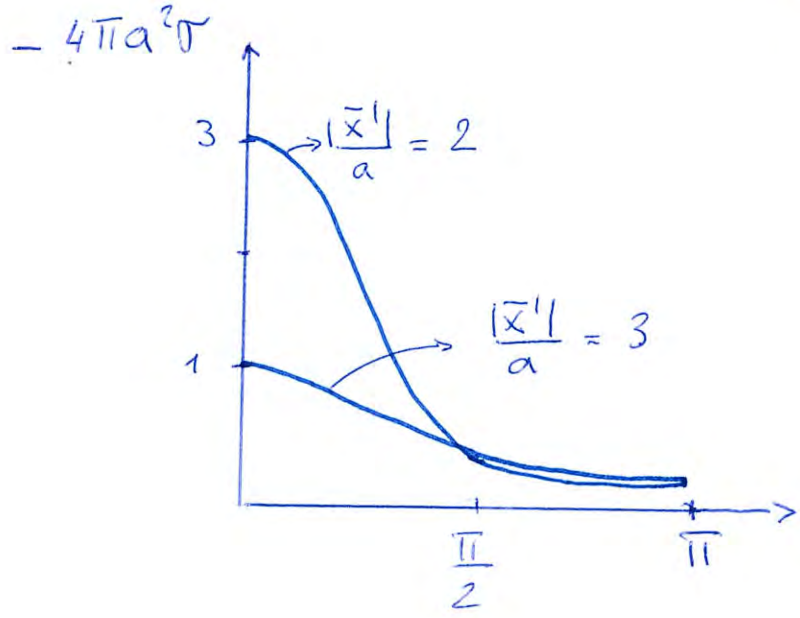
$$\bar{n}\cdot\bar{\nabla}_x\, G_D(\bar{x},\bar{x}')\Big|_{\bar{x}\in S} \equiv \frac{\partial G_D}{\partial n}\Big|_{\bar{x}\in S}$$

$$= \frac{(|\bar{x}'|^2 - a^2)}{4\pi a\left[\bar{x}'^2 + a^2 - 2a|\bar{x}'|\cos\gamma\right]^{3/2}}$$

$$\boxed{-\frac{\partial G_D}{\partial n}\Big|_{\bar{x}\in S} = -\frac{1}{4\pi a^2}\left(\frac{a}{|\bar{x}'|}\right)\cdot\frac{\left(1-\frac{a^2}{|\bar{x}'|^2}\right)}{\left[1+\frac{a^2}{|\bar{x}'|^2} - 2\frac{a}{|\bar{x}'|}\cos\gamma\right]^{3/2}}}$$

PHYSICALLY : INDUCED SURFACE CHARGE DENSITY σ ON GROUNDED CONDUCTING SPHERE

GAUSS LAW : $-\frac{\partial G_D}{\partial n}\Big|_{\bar{x}\in S} = \sigma$



γ : ANGLE BETWEEN $\bar{x}\in S$ AND POINT CHARGE $\bar{x}'$

SECOND TERM IN $G_D(\bar{x},\bar{x}')$ CAN BE INTERPRETED AS POTENTIAL OF INDUCED SURFACE CHARGE DENSITY WHICH GUARANTEES THAT POTENTIAL ON SURFACE IS 0.

- TOTAL CHARGE ON SURFACE

$$Q_{SURFACE} = 2\pi a^2 \int_{-1}^{1} d\cos\gamma \quad \sigma$$

$$= -\frac{2\pi}{4\pi} \left(\frac{a}{|\bar{x}'|}\right) \int_{-1}^{1} d\cos\gamma \frac{\left(1 - \frac{a^2}{|\bar{x}'|^2}\right)}{\left[1 + \frac{a^2}{|\bar{x}'|^2} - 2\frac{a}{|\bar{x}'|}\cos\gamma\right]^{3/2}}$$

$$= -\frac{1}{2}\left(1 - \frac{a^2}{|\bar{x}'|^2}\right) \cdot \frac{1}{\left[1 + \frac{a^2}{|\bar{x}'|^2} - 2\frac{a}{|\bar{x}'|}\cos\gamma\right]^{1/2}} \Bigg|_{-1}^{1}$$

$$= -\frac{1}{2} \cdot 2\frac{a}{|\bar{x}'|}$$

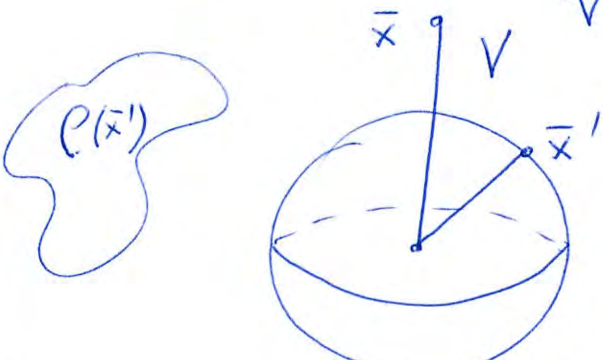
$$\boxed{Q_{SURFACE} = -\frac{a}{|\bar{x}'|}}$$

TOTAL CHARGE ON SURFACE = IMAGE CHARGE !

* GENERAL POTENTIAL OUTSIDE SPHERE WHEN POTENTIAL ON SURFACE IS GIVEN

$$\phi(\bar{x}) = \int_V d^3\bar{x}' \, G_D(\bar{x}, \bar{x}') \, \rho(\bar{x}')$$

$$- \oint_S da' \, \phi(\bar{x}') \, \frac{\partial G_D}{\partial n'}(\bar{x}, \bar{x}')$$



1° TERM : DUE TO EXTERNAL CHARGE DISTR ρ

2° TERM : SURFACE TERM DUE TO INDUCED CHARGE ON SPHERE

IN ABSENCE OF EXTERNAL CHARGES $\rho = 0$

$$\phi(\bar{x}) = + \oint_{S} da' \, \phi(\bar{x}') \underbrace{\left(\frac{-\partial G_D}{\partial n'}(\bar{x}, \bar{x}') \right)}_{\text{INDUCED SURFACE CHARGE DENSITY}}$$

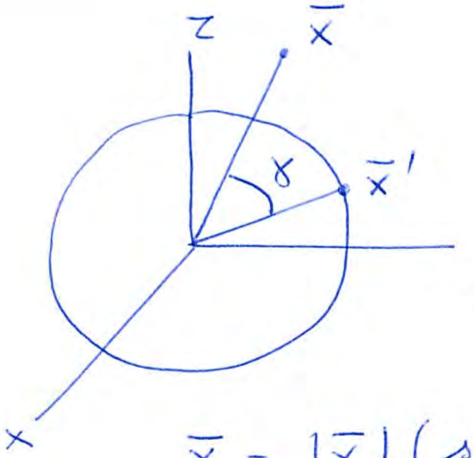
$|\bar{x}'| = a$

$$- \left.\frac{\partial G_D(\bar{x}, \bar{x}')}{\partial n'}\right|_{\bar{x}' \in S} = + \frac{1}{4\pi a^2} \cdot \frac{a(|\bar{x}|^2 - a^2)}{\left[|\bar{x}|^2 + a^2 - 2a|\bar{x}| \cos\gamma \right]^{3/2}}$$

<u>NOTE</u> : OPPOSITE SIGN $-\frac{\partial G_D}{\partial n'}$ AS BEFORE

FOR POINT $\bar{x}$ IN VOLUME V (OUTSIDE SPHERE)

$\frac{\partial}{\partial n'}$ IS DERIVATIVE OUTWARD FROM $V \Rightarrow$ POINTS <u>INTO</u> SPHERE



SPHERICAL COORDINATES

$$\bar{x} = |\bar{x}| (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$\bar{x}' = a (\sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta')$$

$$\cos\gamma = \hat{x} \cdot \hat{x}' = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$$

$$\boxed{\phi(\bar{x}) = \frac{1}{4\pi} \int d\Omega' \, \phi(a, \theta', \phi') \frac{a(|\bar{x}|^2 - a^2)}{(|\bar{x}|^2 + a^2 - 2a|\bar{x}|\cos\gamma)^{3/2}}}$$

$$\int d\Omega' \ldots = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'$$

WHEN POTENTIAL ON SURFACE OF SPHERE

$\phi(a, \theta', \phi')$ IS SPECIFIED

THIS EQUATION ALLOWS TO DETERMINE POTENTIAL

IN POINT $\bar{x}$ OUTSIDE SPHERE

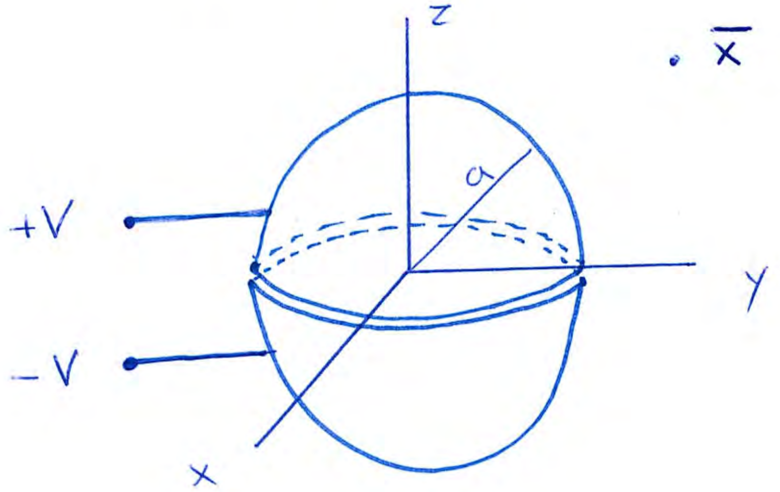
(NOTE: FOR POINT $\bar{x}$ INSIDE SPHERE

REPLACE $|\bar{x}|^2 - a^2 \rightarrow a^2 - |\bar{x}|^2$

SIGN CHANGES)

↳ EXAMPLE:

SPHERE MADE OF 2 HEMISPHERICAL CONDUCTING SHELLS (SEPARATED BY THIN INSULATING RING). HELD AT DIFFERENT POTENTIALS V, $-V$



WHAT IS POTENTIAL $\phi(\bar{x})$ OUTSIDE SPHERE?

$$\phi(|\bar{x}|, \theta, \phi) = \frac{V}{4\pi} \int_0^{2\pi} d\phi' \left\{ \underbrace{\int_0^1 d\cos\theta'}_{\text{UPPER SHELL}} - \underbrace{\int_{-1}^0 d\cos\theta'}_{\text{LOWER SHELL}} \right\}$$

$$\cdot \frac{a(|\bar{x}|^2 - a^2)}{(|\bar{x}|^2 + a^2 - 2a|\bar{x}|\cos\gamma)^{3/2}}$$

IN 2° INTEGRAL: $\theta' \to \pi - \theta'$

$\phi' \to \phi' + \pi$

$$\int_{-1}^{0} d\cos\theta' \ldots \longrightarrow \int_0^1 d\cos\theta'$$

$$\cos\gamma \longrightarrow -\cos\gamma$$

$$\Phi(|\bar{x}|,\theta,\phi) = \frac{V}{4\pi}\cdot\left(\frac{|\bar{x}|^2}{a^2}-1\right)$$

$$\cdot\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \left\{ \left[\frac{|\bar{x}|^2}{a^2}+1-2\frac{|\bar{x}|}{a}\cos\gamma\right]^{-3/2} - \left[\frac{|\bar{x}|^2}{a^2}+1+2\frac{|\bar{x}|}{a}\cos\gamma\right]^{-3/2}\right\}$$

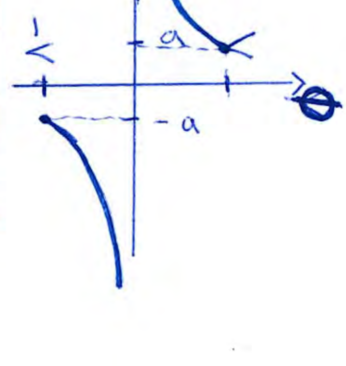
SPECIAL CASE : FOR POINT $\bar{x}$ ALONG Z-AXIS.

$$\Phi(z) = \frac{V}{4\pi}\left(\frac{z^2}{a^2}-1\right)\cdot 2\pi$$

$$\cdot\int_0^1 d\cos\theta' \left\{\left[\frac{z^2}{a^2}+1-2\frac{z}{a}\cos\theta'\right]^{-3/2} - \left[\frac{z^2}{a^2}+1+2\frac{z}{a}\cos\theta'\right]^{-3/2}\right\}$$

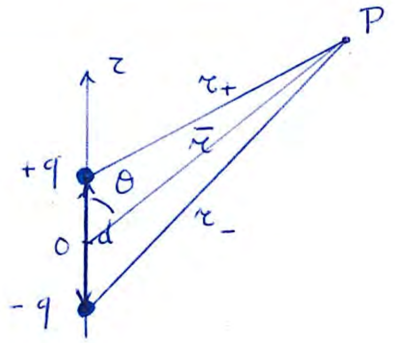
$$= \frac{V}{2}\left(\frac{z^2}{a^2}-1\right)\frac{a}{z}\left\{ \frac{1}{\left[\frac{z^2}{a^2}+1-2\frac{z}{a}\cos\theta'\right]^{1/2}}\Bigg|_0^1 + \frac{1}{\left[\frac{z^2}{a^2}+1+2\frac{z}{a}\cos\theta'\right]^{1/2}}\Bigg|_0^1 \right\}$$

$$\boxed{\Phi(z) = V\left[1 - \frac{(z/a - a/z)}{\sqrt{\frac{z^2}{a^2}+1}}\right]}$$



2) MULTIPOLE EXPANSION

⇒ ELECTRIC DIPOLE



OPPOSITE CHARGES SEPARATED BY DISTANCE d ⤳ FIELD AT LARGE DISTANCE ?

$$\phi(\bar{r}) = \frac{1}{4\pi}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$r_\pm = \left|\bar{r} \mp \frac{\bar{d}}{2}\right| = \left(r^2 + \left(\frac{d}{2}\right)^2 \mp r\, d \cos\theta\right)^{1/2}$$

$$= r\left(1 + \frac{d^2}{4r^2} \mp \frac{d}{r}\cos\theta\right)^{1/2}$$

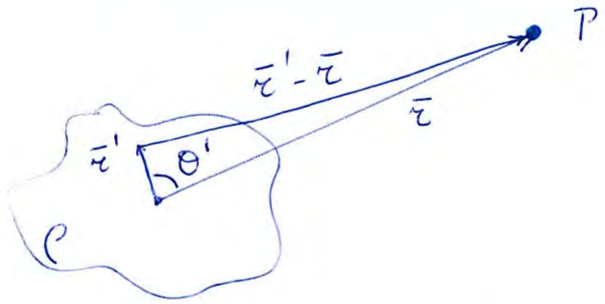
$$\phi(\bar{r}) = \frac{q}{4\pi}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

FOR $r_\pm \gg d$ $\Rightarrow$ $\displaystyle \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$

$$\frac{1}{r_+} - \frac{1}{r_-} \simeq \frac{d}{r^2}\cos\theta$$

$$\boxed{\phi(\bar{r}) \underset{r \gg d}{\simeq} \frac{1}{4\pi}\,\frac{q\,d\cos\theta}{r^2}}$$

⇒ APPROXIMATE POTENTIAL AT LARGE DISTANCES



↳ CHARGE DISTRIBUTION $\rho(\bar{r}')$

POTENTIAL AT LARGE DISTANCES

$$\phi(\bar{r}) = \frac{1}{4\pi} \int d^3\bar{r}' \, \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}$$

↳ $$|\bar{r}-\bar{r}'| = \left(r^2 + r'^2 - 2\, r r' \cos\theta'\right)^{1/2}$$

$$= r\left(1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta'\right)^{1/2}$$

$$= r\sqrt{1+\varepsilon}$$

FOR $r' \ll r \Rightarrow \varepsilon \ll 1.$

$$\frac{1}{|\bar{r}-\bar{r}'|} \underset{r \gg r'}{\simeq} \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + O(\varepsilon^3)\right)$$

$$\simeq \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' - \frac{r'^2}{2r^2} + \frac{3}{8}\cdot 4\frac{r'^2}{r^2}\cos^2\theta' + O\left(\frac{r'^3}{r^3}\right)\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' - \frac{r'^2}{2r^2}\left(1 - 3\cos^2\theta'\right) + O\left(\frac{r'^3}{r^3}\right)\right)$$

↳ IN GENERAL:

$$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta')$$

$$\boxed{\phi(\bar{r}) = \frac{1}{4\pi r}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell}}\int d^3\bar{r}'\; r'^{\ell}\, P_\ell(\cos\theta')\,\rho(\bar{r}')}$$

MULTIPOLE EXPANSION OF POTENTIAL.

↳ $\ell = 0$: MONOPOLE TERM

$$\phi_{MON}(\bar{r}) = \frac{1}{4\pi r}\, Q \quad \text{WITH} \quad \underset{\substack{\Uparrow \\ \text{TOTAL CHARGE}}}{Q} = \int d^3\bar{r}'\,\rho(\bar{r}')$$

↳ $\ell = 1$: DIPOLE TERM

$$\phi_{DIP}(\bar{r}) = \frac{1}{4\pi r^2}\int d^3\bar{r}'\; r'\cos\theta'\,\rho(\bar{r}')$$

$$\downarrow \quad \hat{r}\cdot\hat{r}' = \cos\theta'$$

$$= \frac{1}{4\pi r^2}\,\hat{r}\cdot\int d^3\bar{r}'\;\bar{r}'\,\rho(\bar{r}')$$

$$\boxed{\phi_{DIP}(\bar{r}) = \frac{1}{4\pi}\frac{\hat{r}\cdot\bar{p}}{r^2}}$$

WITH $\boxed{\bar{p} = \int d^3\bar{r}'\;\bar{r}'\,\rho(\bar{r}')}$ DIPOLE MOMENT OF CHARGE DISTRIBUTION

⇝ NOTE : FOR POINT PARTICLES

$$\bar{p} = \sum_{i=1}^{n} q_i \, \bar{r}_i'$$

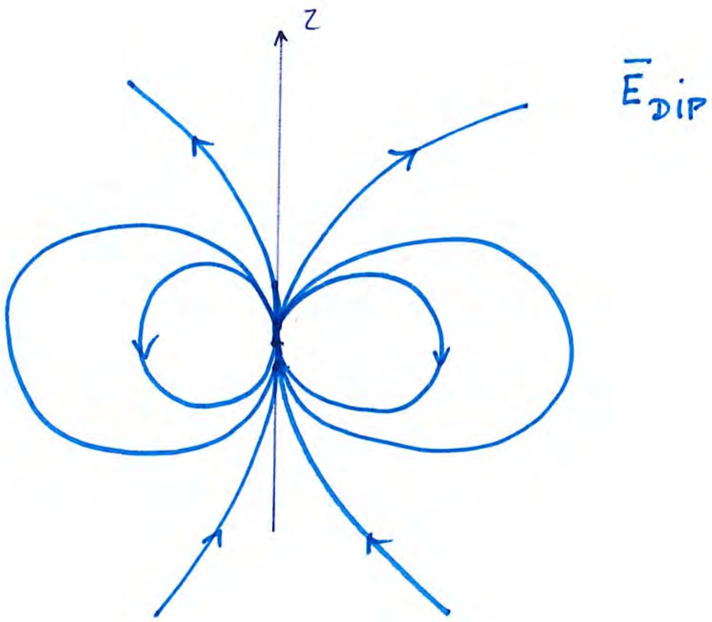
⇝ ELECTRIC DIPOLE FIELD

$$\bar{E}_{DIP} = - \bar{\nabla} \phi_{DIP}$$

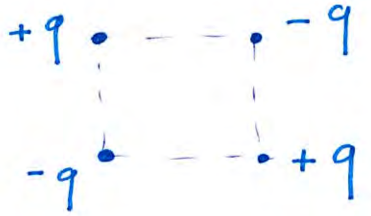
$$= - \frac{1}{4\pi} \bar{\nabla} \left(\frac{\bar{r}}{r^3} \cdot \bar{p} \right)$$

$$= - \frac{1}{4\pi} \frac{1}{r^3} \left(\bar{p} - 3 (\hat{r} \cdot \bar{p}) \hat{r} \right)$$

$$\bar{E}_{DIP} = \frac{1}{4\pi} \frac{1}{r^3} \left(3 (\hat{r} \cdot \bar{p}) \hat{r} - \bar{p} \right)$$



↳ $\ell = 2$ QUADRUPOLE TERM



$$\phi_{QUAD}(\bar{r}) = \frac{1}{4\pi r^3} \int d^3\bar{r}' \, r'^2 \, \frac{1}{2}\left(3\cos^2\theta' - 1\right) \rho(\bar{r}')$$

$$\downarrow \qquad \hat{r}\cdot\hat{r}' = \cos\theta'$$

$$= \frac{1}{4\pi r^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \hat{r}_i \hat{r}_j \int d^3\bar{r}' \, \frac{1}{2}\left(3 r'_i r'_j - r'^2 \delta_{ij}\right) \rho(\bar{r}')$$

$$\boxed{\phi_{QUAD}(\bar{r}) = \frac{1}{4\pi} \frac{1}{2r^3} \hat{r}_i \hat{r}_j Q_{ij}}$$

(SUM OVER REPEATED INDICES)

WITH $$\boxed{Q_{ij} = \int d^3\bar{r}' \left[3 r'_i r'_j - r'^2 \delta_{ij}\right] \rho(\bar{r}')}$$

QUADRUPOLE MOMENT OF CHARGE DISTRIBUTION.

DESCRIBES DEVIATION FROM SPHERICAL SHAPE OF CHARGE DISTRIBUTION.

$$\hookrightarrow \phi(\bar{r}) = \frac{1}{4\pi} \left\{ \frac{Q}{r} + \frac{\bar{p}\cdot\hat{r}}{r^2} + \frac{1}{2r^3} \hat{r}_i \hat{r}_j Q_{ij} + \cdots \right\}$$

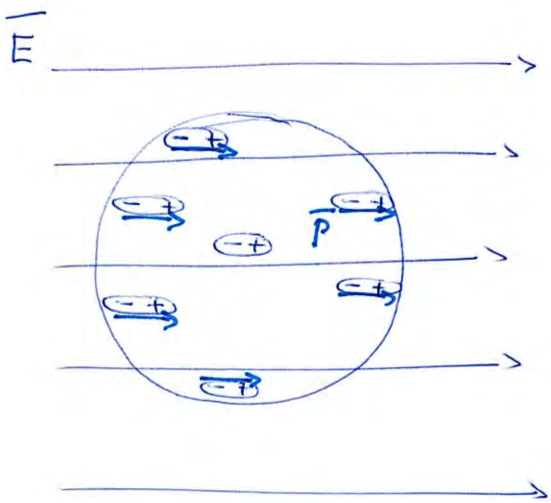
3) ELECTRIC FIELDS IN MATTER : DIELECTRICS

⇒ INDUCED DIPOLES

* MATTER : ELECTRIC PROPERTIES FALL INTO 2 LARGE CLASSES
1) CONDUCTORS
2) INSULATORS (DIELECTRICS)

↳ CONDUCTOR : PLACED IN EXTERNAL $\bar{E}$ - FIELD FREE e^- WILL REARRANGE SUCH AS TO MAKE $\bar{E}$ - FIELD INSIDE CONDUCTOR ZERO (EQUILIBRIUM CONFIGURATION)

↳ DIELECTRIC : CHARGES ARE BOUND WILL BE DISPLACED BY EXTERNAL FIELD
⇒ INDUCED DIPOLE MOMENT



$\bar{p}$: INDUCED ELECTRIC DIPOLE MOMENT OF INDIVIDUAL ATOM OR MOLECULE

FOR NEUTRAL ATOM $\quad \bar{p} = \alpha \bar{E}$

α ↑ ATOMIC POLARIZABILITY

⇒ FIELD OF A POLARIZED OBJECT

* MACROSCOPIC POLARIZATION

$\overline{P}$: DIPOLE MOMENT PER UNIT OF VOLUME

$$\overline{P} = \sum_i N_i \, \overline{p}_i$$

$\overline{p}_i$: INDUCED DIPOLE MOMENT OF TYPE-i

N_i: NUMBER OF MOLECULES OF TYPE-i PER UNIT OF VOLUME

$\sum_i$: SUM OVER ALL MOLECULES OF TYPE-i IN MEDIUM

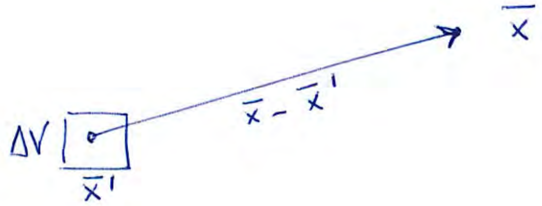
* FIELD

$\rho(\overline{x}')$: CHARGE DENSITY AT $\overline{x}'$

$\overline{P}(\overline{x}')$: DIPOLE MOMENT PER UNIT OF VOLUME AT $\overline{x}'$

IN ABSENCE OF HIGHER MULTIPOLE MOMENTS

WHAT IS $\left\{ \begin{array}{l} \text{FIELD} \\ \text{POTENTIAL} \end{array} \right.$ AT $\overline{x}$?



$$\Delta \Phi(\bar{x}) = \frac{\rho(\bar{x}')\,\Delta V}{4\pi\,|\bar{x}-\bar{x}'|} + \frac{(\bar{x}-\bar{x}')\cdot\bar{P}(\bar{x}')\,\Delta V}{4\pi\,|\bar{x}-\bar{x}'|^3}$$

MONOPOLE FIELD

DIPOLE FIELD

$$\Phi(\bar{x}) = \frac{1}{4\pi}\int d^3\bar{x}' \left[\frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|} + \frac{(\bar{x}-\bar{x}')\cdot\bar{P}(\bar{x}')}{|\bar{x}-\bar{x}'|^3} \right]$$

$$\downarrow \qquad \bar{\nabla}'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = + \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}$$

$$= \frac{1}{4\pi}\int d^3\bar{x}' \left[\frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|} + \bar{P}(\bar{x}')\cdot\bar{\nabla}'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) \right]$$

INTEGRATION BY PARTS

$$= \frac{1}{4\pi}\int d^3\bar{x}' \left[\frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|} - \frac{1}{|\bar{x}-\bar{x}'|}\,\bar{\nabla}'\cdot\bar{P}(\bar{x}') \right]$$

$$\Phi(\bar{x}) = \frac{1}{4\pi}\int d^3\bar{x}'\,\frac{1}{|\bar{x}-\bar{x}'|}\underbrace{\left[\rho(\bar{x}') - \bar{\nabla}'\cdot\bar{P}(\bar{x}')\right]}_{\text{EFFECTIVE CHARGE DENSITY}}$$

INDUCED POLARIZATION CHANGES LOCAL CHARGE DENSITY

⇒ ELECTRIC DISPLACEMENT $\bar{D}$

↳ $$\bar{\nabla}.\bar{E} = \rho(\bar{x}) - \bar{\nabla}.\bar{P}$$

INTRODUCE ELECTRIC DISPLACEMENT $\bar{D}$ (ELEKTRISCHE INDUKTION)

$$\boxed{\bar{D} = \bar{E} + \bar{P}}$$

$$\bar{\nabla}.\bar{D} = \rho$$

MAXWELL EQ. IN MEDIUM (GAUSS LAW)

↳ RELATION BETWEEN $\bar{E}$ & $\bar{P}$

⇝ FOR $\bar{E}$ FIELDS THAT ARE NOT TOO STRONG

LINEAR RELATION $$P_i = \chi_{ij} E_j$$

↳ SUSCEPTIBILITY TENSOR

⇝ FOR ISOTROPIC MEDIA ⇒ χ_{ij} IS DIAGONAL

$$\chi_{ij} = \chi_e \delta_{ij}$$

$$\bar{P} = \chi_e \bar{E}$$

↳ ELECTRIC SUSCEPTIBILITY

$$\bar{D} = \bar{E} + \bar{P} = (1 + \chi_e)\bar{E}$$

$$\equiv \varepsilon \bar{E}$$

↳ DIELECTRIC CONSTANT OR RELATIVE ELECTRIC PERMITTIVITY

VACUUM $\varepsilon = 1$

$$\varepsilon \geq 1$$

⇒ BOUNDARY - VALUE PROBLEMS WITH DIELECTRICS

↳ ELECTROSTATICS IN MEDIA

$$\bar{\nabla} \cdot \bar{D} = \rho$$

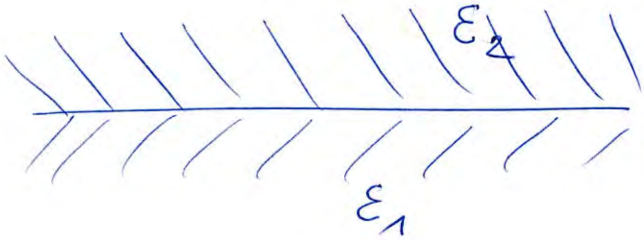
$$\bar{\nabla} \times \bar{E} = 0$$

FOR LINEAR MEDIA $\bar{D} = \varepsilon \bar{E}$ $\sim$ $\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon}$

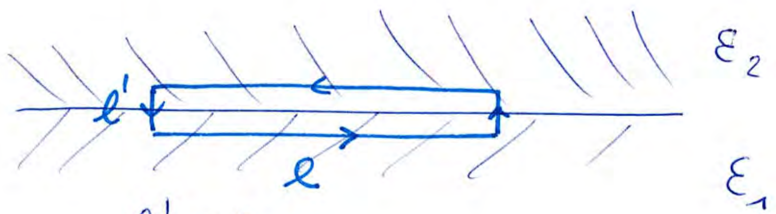
$$\bar{E} = -\bar{\nabla}\phi$$

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon}$$

↳ INTERFACE OF 2 DIELECTRICS



* $\bar{\nabla} \times \bar{E} = 0$



$l' \to 0$

$$\oint \bar{E} \cdot d\bar{l} = 0 \Rightarrow l E_{\parallel 1} - l E_{\parallel 2}$$

PARALLEL COMPONENT OF $\bar{E}$ IS CONTINUOUS

$$E_{\parallel 1} = E_{\parallel 2}$$

$*$ $\bar{\nabla} \cdot \bar{D} = \rho$

$l' \to 0$

$\rightsquigarrow \quad \oint d\bar{S} \cdot \bar{D} = \sigma_F \cdot S$

↳ FREE SURFACE CHARGE DENSITY

$$S \, \bar{n}_{21} \cdot \left(\bar{D}_2 - \bar{D}_1 \right) = \sigma_F \, S$$

⇑
UNIT VECTOR
FROM 1 TO 2

$$\underline{\underline{D_{\perp 2} - D_{\perp 1} = \sigma_F}}$$

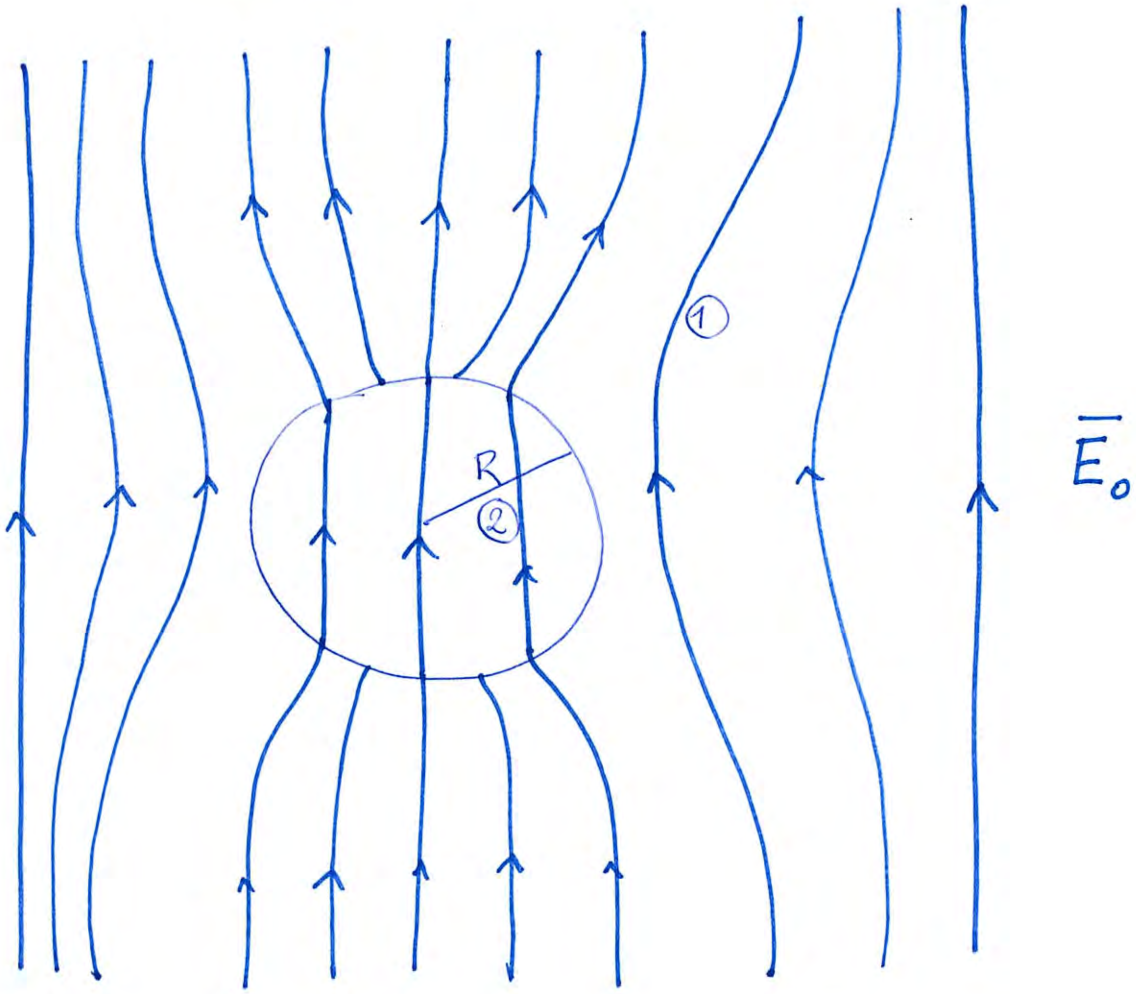
IF $\sigma_F = 0 \Rightarrow D_{\perp 2} = D_{\perp 1}$

NORMAL COMPONENT OF D
CONTINUOUS

$\rightsquigarrow$ BOUNDARY CONDITION FOR $E_\perp$

$$\varepsilon_2 \, E_{\perp 2} - \varepsilon_1 \, E_{\perp 1} = \sigma_F$$

⇒ EXAMPLE : DIELECTRIC SPHERE PLACED IN UNIFORM $\bar{E}$-FIELD



$\varepsilon_1 = 1$ VACUUM

$\varepsilon_2 > 1$

NO FREE SURFACE CHARGE DENSITY $\sigma_F = 0$

QUESTION: WHAT IS $\bar{E}$ FIELD INSIDE SPHERE AS FUNCTION OF CONSTANT EXTERNAL FIELD $\bar{E}_0$?

↳ REGION 1 : VACUUM $\nabla^2 \phi_1 = 0$

↳ REGION 2 (INSIDE SPHERE) : NO CHARGE DENSITY

$$\nabla^2 \phi_2 = 0$$

↳ ASYMPTOTIC VALUE $\bar{E} \to \bar{E}_0 = E_0 \, \bar{e}_z$

$$\phi_1(r, \theta) \xrightarrow[r \gg R]{} -E_0 z = -E_0 r \cos\theta$$

↳ BOUNDARY CONDITIONS.

- ϕ : CONTINUOUS AT $r = R$

$$\phi_1(r = R, \theta) = \phi_2(r = R, \theta)$$

- $E_{\perp 1} = \varepsilon E_{\perp 2}$ AT $r = R$ $(\sigma_F = 0)$

$$\frac{\partial \phi_1}{\partial r}(r = R, \theta) = \varepsilon \frac{\partial \phi_2}{\partial r}(r = R, \theta)$$

↳ POTENTIAL HAS AZIMUTHAL SYMMETRY :

DOES NOT DEPEND ON Φ ANGLE

- BOTH IN REGION 1 & 2 $\phi(r,\theta)$ IS A SOLUTION OF $\nabla^2\phi = 0$

IN SPHERICAL COORDINATES.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\Phi^2} = 0.$$

IF $\phi(r,\theta)$ DOES NOT DEPEND ON Φ

$\Rightarrow$ LAST TERM IS ZERO

$$\underline{\underline{\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0}}$$

SOLUTION BY SEPARATION OF VARIABLES

$$\phi(r,\theta) = R(r)\,\Theta(\theta)$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{ONLY DEPENDS ON } r} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\text{ONLY DEPENDS ON } \theta} = 0$$

EQUALITY MEANS THAT EACH TERM HAS TO BE EQUAL TO CONSTANT (WHICH WE CHOOSE AS $\ell(\ell+1)$; WITH ℓ A NUMBER)

$$\begin{cases} \dfrac{1}{R} \dfrac{d}{dr}\left(r^2 \dfrac{dR}{dr}\right) = \ell(\ell+1) \\[2ex] \dfrac{1}{\Theta} \dfrac{1}{\sin\Theta} \dfrac{d}{d\Theta}\left(\sin\Theta \dfrac{d\Theta}{d\Theta}\right) = -\ell(\ell+1) \end{cases}$$

NOTE: THROUGH SEPERATION OF VARIABLES: INITIAL PARTIAL DIFFERENTIAL EQUATION IS TURNED INTO 2 ORDINARY DIFF. EQUATIONS

↳ r EQUATION

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) R$$

HAS GENERAL SOLUTION

$$R(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

WHERE A_ℓ & B_ℓ NEED TO BE DETERMINED FROM BOUNDARY CONDITIONS.

↳ Θ EQUATION

$$\frac{d}{d\theta}\left(\sin\theta \, \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\sin\theta \, \Theta$$

INTRODUCE $x = \cos\theta$

$$\frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

$$+\cancel{\sin\theta}\,\frac{d}{dx}\Big(\underbrace{\sin^2\theta}_{(1-x^2)}\,\frac{d\Theta}{dx}\Big) = -\ell(\ell+1)\,\cancel{\sin\theta}\,\Theta$$

$$\Updownarrow$$

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \ell(\ell+1)\,\Theta = 0$$

↳ SOLUTIONS ARE GIVEN BY LEGENDRE POLYNOMIALS FOR $\ell = 0, 1, 2, \dots$ $(\ell \in \mathbb{N})$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\vdots$

$$\Theta(\theta) = P_\ell(\cos\theta) \qquad \forall \ell = 0, 1, 2, \dots$$

- GENERAL SOLUTION OF $\nabla^2 \phi = 0$

$$\boxed{\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)}$$

↳ IN REGION 2 : $(r \leq R)$

POTENTIAL CANNOT BE SINGULAR AT $r = 0$

$\Downarrow$

$B_\ell = 0$ FOR $r \leq R$

$$\boxed{\phi_2(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)}$$

$(r \leq R)$

↳ IN REGION 1 : $(r \geq R)$

POTENTIAL HAS TO APPROACH $-E_0 r \cos\theta$ FOR $r \gg R$

$$\boxed{\phi_1(r,\theta) = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)}$$

NOTE TERM WITH A_0 CONSTANT $= 0$

A_1 ↝ CHOSEN AS $-E_0$
TO MATCH BOUNDARY CONDITION

A_ℓ $\ell \geq 2$: r^ℓ POTENTIAL BLOWS UP $\rightarrow A_\ell = 0, \ell \geq 2$

- BOUNDARY CONDITIONS ON SURFACE OF SPHERE

↳ $\Phi_1(r=R,\theta) = \Phi_2(r=R,\theta)$

$$-E_0 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

MUST BE VALID FOR EACH ℓ

$$\ell \neq 1 : \quad \frac{B_\ell}{R^{\ell+1}} = A_\ell R^\ell$$

$$\ell = 1 : \quad A_1 R = \frac{B_1}{R^2} - E_0 R$$

↳ $\dfrac{\partial \Phi_1}{\partial r}(r=R,\theta) = \varepsilon \dfrac{\partial \Phi_2}{\partial r}(r=R,\theta)$

$$-E_0\cos\theta - \sum_{\ell=0}^{\infty} \frac{(\ell+1) B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) = \varepsilon \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta)$$

$$\ell \neq 1 : \quad \frac{(\ell+1) B_\ell}{R^{\ell+2}} = \ell A_\ell R^{\ell-1}$$

$$\ell = 1 : \quad -E_0 - \frac{2}{R^3} B_1 = \varepsilon A_1$$

• SOLUTION :

↳ FOR $\ell \neq 1$: $\underline{\underline{A_\ell = 0, \quad B_\ell = 0.}}$

↳ FOR $\ell = 1$: $\underline{\underline{A_1 = -\dfrac{3}{2+\varepsilon} E_0.}}$

$$B_1 = R^3 (A_1 + E_0)$$

$$\underline{\underline{B_1 = \frac{\varepsilon - 1}{\varepsilon + 2} R^3 E_0}}$$

↳ IN INTERIOR :

$$\boxed{\phi_2(r, \theta) = -\frac{3}{2+\varepsilon} E_0 \, r \cos\theta}$$

$$= -\frac{3}{2+\varepsilon} E_0 \, z$$

$\Downarrow$ $\overline{E}_2$ IN z- DIRECTION

$$\boxed{\overline{E}_2 = \frac{3}{2+\varepsilon} \overline{E}_0}$$

E - FIELD IS UNIFORM INSIDE SPHERE

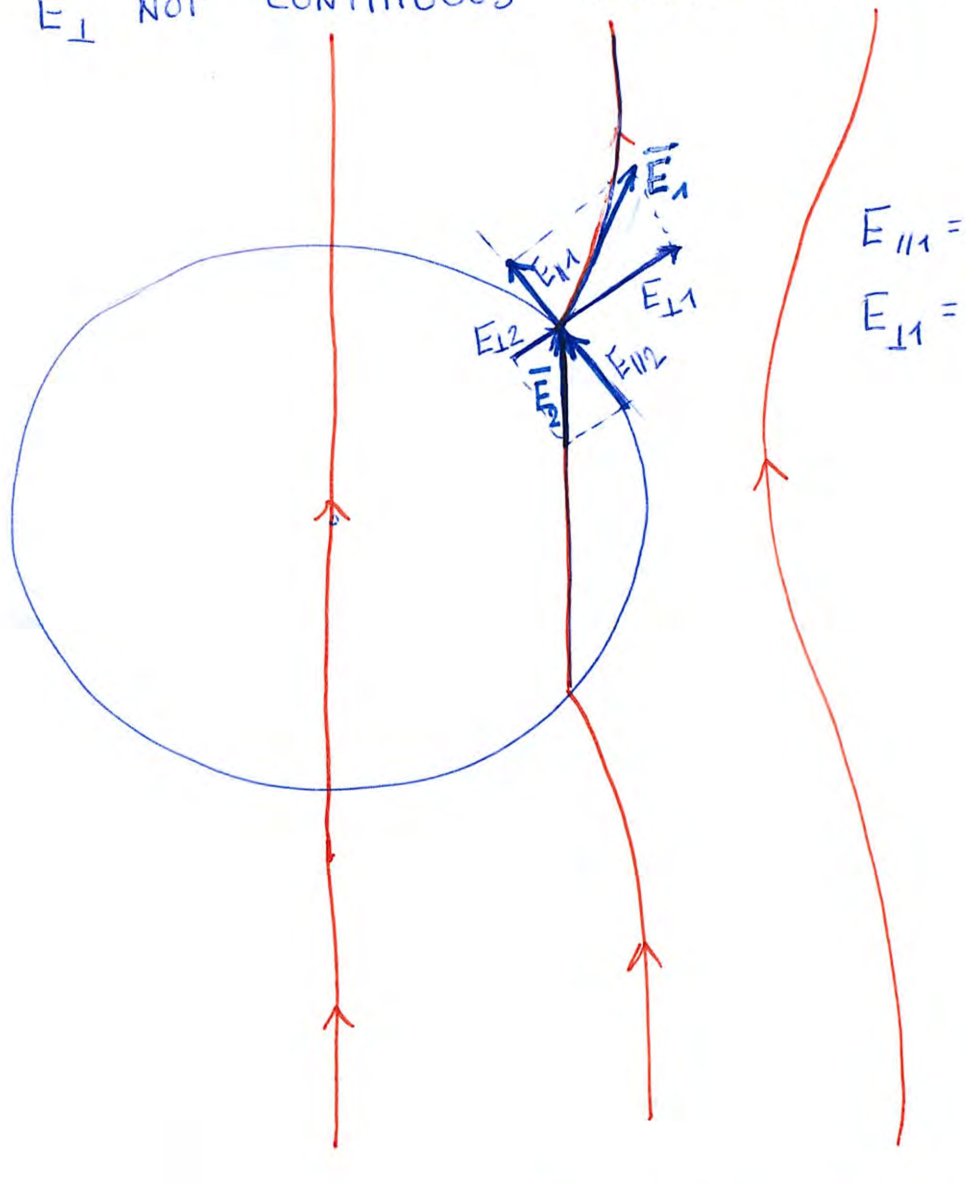
AS $\varepsilon > 1$ $E_2 < E_0$

PARTIAL CANCELLATION OF $\overline{E}$ - FIELD IN DIELECTRIC
(TOTAL " " " " " CONDUCTOR)

↳ IN EXTERIOR

$$\boxed{\phi_1(r,\theta) = -E_0 r \cos\theta \left[1 - \frac{\varepsilon - 1}{\varepsilon + 2}\left(\frac{R}{r}\right)^3\right]}$$

↳ NOTE : $E_\perp$ NOT CONTINUOUS ACROSS SURFACE



$E_{\parallel 1} = E_{\parallel 2}$

$E_{\perp 1} = \varepsilon E_{\perp 2}$

$\bar{E}$ FIELD LINES

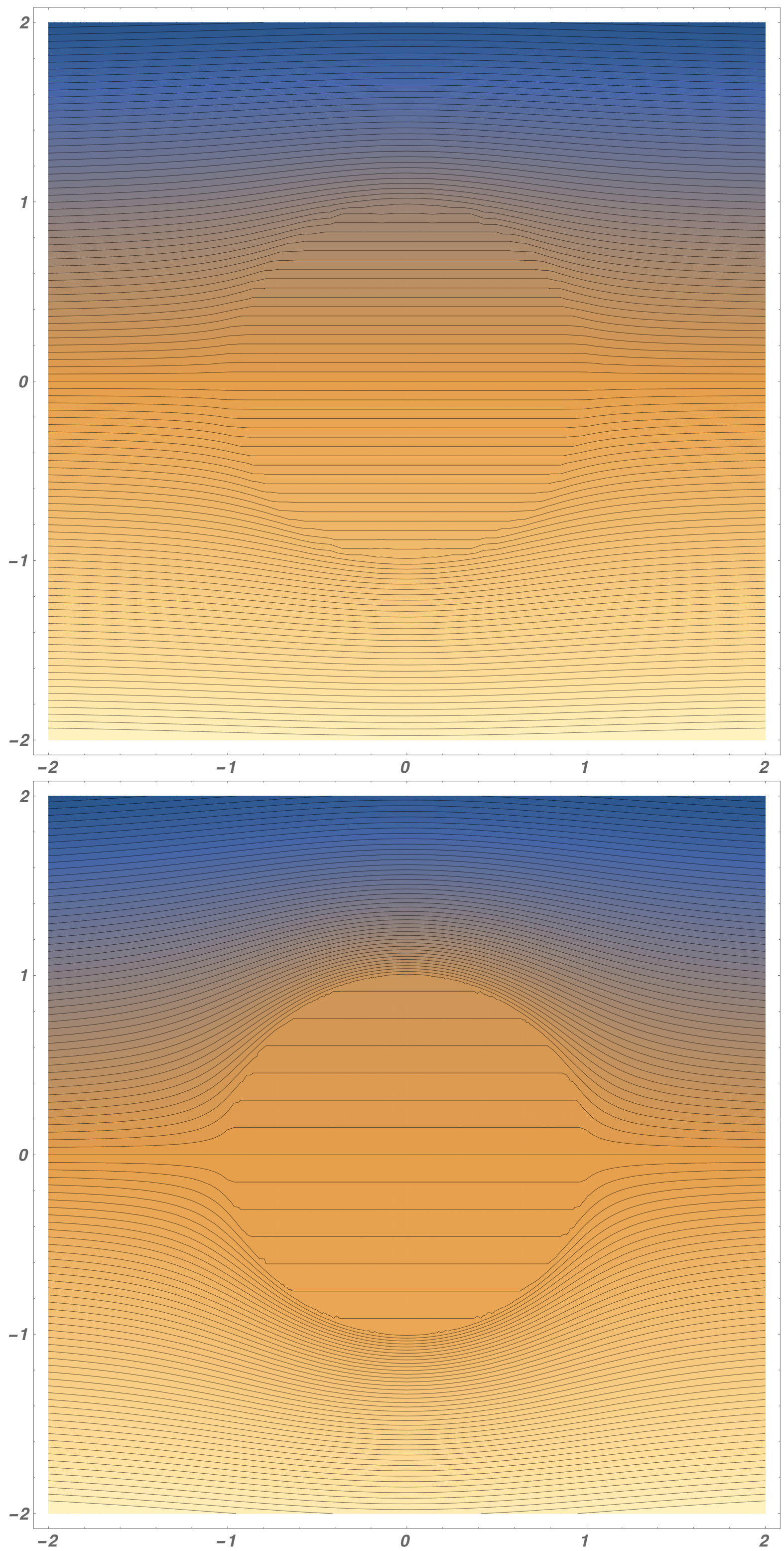


Figure 1: Equipotential lines of an uncharged dielectric sphere (of radius R = 1) for $\varepsilon = 2$ (upper panel) and $\varepsilon = 10$ (lower panel) placed in a uniform electric field, along the vertical axis.

4) MAGNETOSTATICS

⟹ BASIC EQS. FOR MAGNETOSTATICS

* FOR STEADY CURRENTS (CURRENTS THAT DO NOT CHANGE IN TIME)

$$\Downarrow$$

$$\begin{cases} \bar{\nabla} \cdot \bar{B} = 0 \\ \bar{\nabla} \times \bar{B} = \frac{1}{c} \bar{J} \end{cases}$$

CONSTANT MAGNETIC FIELDS

2^0 EQ. $\Rightarrow$ $\underline{\underline{\bar{\nabla} \cdot \bar{J} = 0}}$ (CURRENT CONSERVATION $\dot{\rho} = 0$)

* VECTOR POTENTIAL

1^0 EQ $\Rightarrow$ $\bar{B} = \bar{\nabla} \times \bar{A}$

2^0 EQ $\Rightarrow$ $\bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A} = \frac{1}{c} \bar{J}$

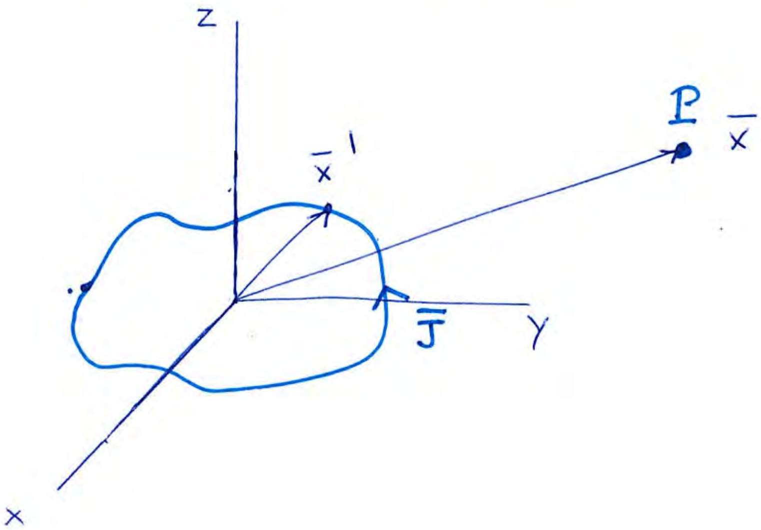
$$\Downarrow$$

CHOICE OF GAUGE $\bar{\nabla} \cdot \bar{A} = 0$ (COULOMB GAUGE)

$$\boxed{\nabla^2 \bar{A} = -\frac{1}{c} \bar{J}}$$

⇒ MAGNETIC DIPOLE MOMENT

↳ CURRENT LOOP $\bar{J}$ LOCALIZED IN SPACE



QUESTION : WHAT IS $\bar{A}(\bar{x})$ IN POINT P : $|\bar{x}| \gg |\bar{x}'|$

POISSON EQ $\quad \nabla^2 \bar{A} = -\frac{1}{c} \bar{J}$

↳ PARTICULAR SOLUTION AS IN ELECTROSTATICS $\rho \Rightarrow \frac{1}{c} J^i$

$$\bar{A}(\bar{x}) = \frac{1}{4\pi c} \int d^3\bar{x}' \, \frac{\bar{J}(\bar{x}')}{|\bar{x} - \bar{x}'|}$$

FOR $|\bar{x}| \gg |\bar{x}'|$, DENOTE $r = |\bar{x}|$

$$\frac{1}{|\bar{x} - \bar{x}'|} \simeq \frac{1}{r} + \frac{\bar{x} \cdot \bar{x}'}{r^3} + \ldots$$

$$\hookrightarrow \bar{A}(\bar{x}) = \frac{1}{4\pi c}\left\{ \frac{1}{r}\int d^3\bar{x}'\ \bar{J}(\bar{x}') + \frac{1}{r^3}\int d^3\bar{x}'\ (\bar{x}.\bar{x}')\ \bar{J}(\bar{x}') + \dots \right\}$$

$\rightsquigarrow$ 1° TERM VANISHES :

$$\int d^3\bar{x}'\ J^i(\bar{x}')$$

$$= \int d^3\bar{x}'\ \bar{J}.\bar{\nabla}'\ x'^i$$

$$\bar{\nabla}'.\bar{J} = 0$$

$$= \int d^3\bar{x}'\ \bar{\nabla}'.(x'^i\ \bar{J})$$

$$= \oint d\bar{S}'.\ x'^i\ \bar{J}$$

ON SURFACE : $\bar{J} = 0$

(CURRENT ONLY NON-ZERO IN BOUNDED REGION)

$\rightsquigarrow$ 2° TERM : DIPOLE TERM

$$\bar{A}_{\text{DIPOLE}}(\bar{x}) = \frac{1}{4\pi c}\frac{1}{r^3}\ x^i \int d^3\bar{x}'\ x'^i\ \bar{J}(\bar{x}')$$

HELP 1

$$\int d^3\bar{x}'\; x'^i\; J^j$$

$$= \int d^3\bar{x}'\; x'^i (\bar{J}\cdot\bar{\nabla}')\, x'^j$$

$$= -\int d^3\bar{x}' \left[\bar{\nabla}'\cdot(x'^i \bar{J})\right] x'^j$$

$$\downarrow \quad \bar{\nabla}\cdot\bar{J} = 0$$

$$= -\int d^3\bar{x}' \left[\underbrace{(\bar{\nabla}' x'^i)\cdot\bar{J}}_{J^i}\right] x'^j$$

$$= -\int d^3\bar{x}'\; x'^j\; J^i$$

HELP 2

$$\bar{x}\times(\bar{x}'\times\bar{J})$$

$$= (\bar{x}\cdot\bar{J})\,\bar{x}' - (\bar{x}\cdot\bar{x}')\,\bar{J}$$

$$\therefore \quad \int d^3\bar{x}'\, x'^i J^j = \frac{1}{2}\int d^3\bar{x}'\left(x'^i J^j - x'^j J^i\right)$$

$$\Downarrow$$

$$\int d^3\bar{x}'\,(\bar{x}\cdot\bar{x}')\,\bar{J} = \frac{1}{2}\int d^3\bar{x}'\left[(\bar{x}\cdot\bar{x}')\bar{J} - (\bar{x}\cdot\bar{J})\,\bar{x}'\right]$$

$$= -\frac{1}{2}\,\bar{x}\times\int d^3\bar{x}'\,(\bar{x}'\times\bar{J})$$

DIPOLE TERM

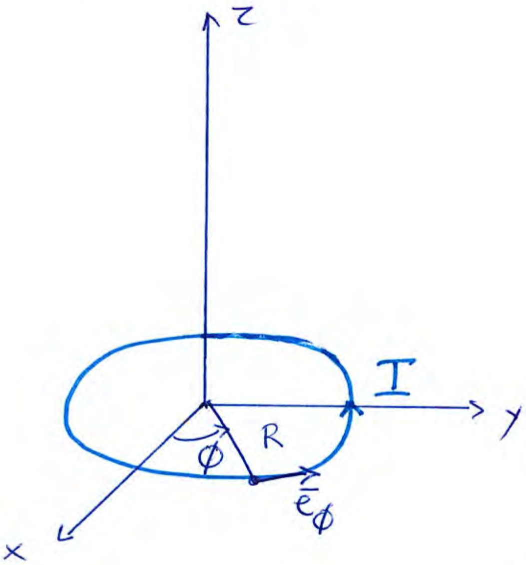
$$\bar{A}_{DIPOLE}(\bar{x}) = \frac{1}{4\pi}\frac{1}{r^3}\left(-\frac{1}{2c}\right)\bar{x} \times \int d^3\bar{x}'(\bar{x}' \times \bar{J})$$

↳ MAGNETIC (DIPOLE) MOMENT OF CURRENT DENSITY

$$\boxed{\bar{m} \equiv \frac{1}{2c}\int d^3\bar{x}' \left[\bar{x}' \times \bar{J}(x')\right]}$$

$$\boxed{\bar{A}_{DIPOLE} = \frac{1}{4\pi}\frac{1}{r^3}(\bar{m} \times \bar{x})}$$

↳ EXAMPLE : RING OF CURRENT (RADIUS R)



$$\bar{J} = \underbrace{\frac{I}{\Delta S}}_{\text{CURRENT } I \text{ PER UNIT SURFACE}} \cdot \bar{e}_\phi$$

$$\bar{e}_\phi = -\sin\phi\,\bar{e}_x + \cos\phi\,\bar{e}_y$$

$$\bar{x}' \times \bar{e}_\phi = R\left\{\cos^2\phi + \sin^2\phi\right\}\bar{e}_z$$

$$= R\,\bar{e}_z$$

$$\bar{m} = \frac{1}{2c} \underbrace{\int_0^{2\pi} d\phi \; R\,(\Delta S)}_{\int d^3\bar{x}'} \cdot \frac{I}{(\Delta S)} R \; \bar{e}_z$$

$$= \frac{1}{2c} \; 2\pi R^2 \; I \; \bar{e}_z$$

$$\boxed{\bar{m} = \frac{1}{c} \underbrace{\pi R^2}_{} \; I \; \bar{e}_z}$$

AREA OF CURRENT LOOP

MAGNETIC MOMENT POINTS PERPENDICULAR TO CIRCULAR CURRENT LOOP

⇒ MAGNETIC DIPOLE FIELD

$$\bar{A}_{\text{DIPOLE}}(\bar{x}) = \frac{1}{4\pi} \frac{1}{r^3} \left(\bar{m} \times \bar{x} \right)$$

$$\bar{B}_{\text{DIPOLE}} = \bar{\nabla} \times \bar{A}_{\text{DIPOLE}}$$

$$B^i = \varepsilon_{ijk} \nabla^j A^k \qquad (\text{NOTE } \varepsilon_{123} = +1)$$

$$= \underbrace{\varepsilon_{ijk}\,\varepsilon_{klm}}\, \nabla^j \cdot \frac{1}{4\pi}\frac{1}{r^3}\, m^l\, x^m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \nabla^j \left[\frac{1}{4\pi r^3}\, m^l\, x^m\right]$$

$$= \frac{1}{4\pi}\left\{ m^i\, \nabla^j\left(\frac{x^j}{r^3}\right) - \bar{m}\cdot\bar{\nabla}\left(\frac{x^i}{r^3}\right)\right\}$$

$$\downarrow \qquad \nabla^j\left(\frac{x^i}{r^3}\right) = -\frac{3x^i x^j}{r^5} + \frac{\delta^{ij}}{r^3}$$

$$= \frac{1}{4\pi}\left\{ m^i \underbrace{\cancel{\left[-\frac{3}{r^3} + \frac{3}{r^3}\right]}}_{0} - m^j\left[-\frac{3x^i x^j}{r^5} + \frac{\delta^{ij}}{r^3}\right]\right\}$$

$$= \frac{1}{4\pi}\left[\frac{3\,(\bar{m}\cdot\bar{x})\,\bar{x}}{r^5} - \frac{\bar{m}}{r^3}\right]^i$$

$$\bar{B}_{\text{DIPOLE}} = \frac{1}{4\pi}\cdot\frac{1}{r^3}\left(3\,(\hat{r}\cdot\bar{m})\,\hat{r} - \bar{m}\right)$$

$\hat{r}$: UNIT VECTOR IN DIRECTION $\bar{x}$

5) MAGNETIC FIELDS IN MATTER

* MACROSCOPIC MAGNETIZATION

e^- IN MATTER HAVE → ORBITAL MOTION
→ INTRINSIC SPIN ⇒ MAGNETIC MOMENT

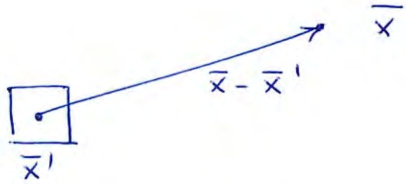
$\overline{M}$: MAGNETIC DIPOLE MOMENT PER UNIT OF VOLUME

$$\overline{M} = \sum_i N_i \langle \overline{m}_i \rangle$$

↳ MICROSCOPIC MAGN MOMENT DUE TO ATOM / MOLECULE OF TYPE i

↓ # ATOMS / MOLECULES OF TYPE i

* VECTOR POTENTIAL



$\overline{J}(\overline{x}')$, $\overline{M}(\overline{x}')$ ⇒ WHAT IS $\overline{A}(\overline{x})$

$$\Delta \overline{A}(\overline{x}) = \frac{1}{4\pi c}\left\{ \frac{\overline{J}(\overline{x}')\,\Delta V}{|\overline{x}-\overline{x}'|} + \frac{c\,\overline{M}(\overline{x}') \times (\overline{x}-\overline{x}')\,\Delta V}{|\overline{x}-\overline{x}'|^3} \right\}$$

$$\bar{A}(\bar{x}) = \frac{1}{4\pi c}\int d^3\bar{x}' \left\{ \frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|} + \frac{c\bar{M}(\bar{x}') \times (\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3} \right\}$$

$$\downarrow \quad \bar{\nabla}'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = \frac{\bar{x}-\bar{x}'}{|\bar{x}-\bar{x}'|^3}$$

$$= \frac{1}{4\pi c}\int d^3\bar{x}' \left\{ \frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|} + c\bar{M}(\bar{x}') \times \bar{\nabla}'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) \right\}$$

INTEGRATION BY PARTS.

(SURFACE TERM VANISHES AS $\bar{M}$ IS LOCALIZED)

$$= \frac{1}{4\pi c}\int d^3\bar{x}' \frac{1}{|\bar{x}-\bar{x}'|} \underbrace{\left\{ \bar{J}(\bar{x}') + c\bar{\nabla}' \times \bar{M}(\bar{x}') \right\}}_{\substack{\text{EFFECTIVE CURRENT DENSITY} \\ \bar{J}_{eff}}}$$

* MAGNETIC FIELD $\bar{H}$

$\rightarrow \quad \bar{\nabla} . \bar{B} = 0$

$\rightarrow \quad \bar{\nabla} \times \bar{B} = \frac{1}{c} \bar{J}_{eff}$ (IN MATTER)

$$= \frac{1}{c}\bar{J} + \bar{\nabla} \times \bar{M}$$

$$\Downarrow$$

$$\boxed{\bar{H} \equiv \bar{B} - \bar{M}}$$

$$\bar{\nabla} \times \bar{H} = \frac{1}{c} \bar{J}$$

* RELATION BETWEEN $\bar{B}$ & $\bar{M}$

↳ ISOTROPIC DIAMAGNETIC / PARAMAGNETIC MEDIA

$$\bar{B} = \mu \bar{H}$$ LINEAR

↳ MAGNETIC PERMEABILITY

$\mu > 1$: PARAMAGNETIC

$\mu < 1$: DIAMAGNETIC

EQUIVALENTLY THIS MEANS

$$\bar{M} = \chi_m \bar{H}$$

$$\bar{B} = (1 + \chi_m) \bar{H}$$

$\chi_m > 0$: PARAMAGNETIC

$\chi_m < 0$: DIAMAGNETIC : INDUCED CURRENTS OPPOSE EXTERNAL FIELD

↳ FERROMAGNETIC MATERIALS

NON-LINEAR RELATION $\bar{B} = f(\bar{H})$

(HYSTERESIS LOOP)