8. Übungsblatt
Theoretische Physik 6: WS 2014/15
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Exercise 1 (50 points): Normalization and completeness of Dirac spinors

By direct inspection prove

(a) (15 points) the normalization condition for spinors w

$$\bar{w}_r(\vec{p})w_{r'}(\vec{p}) = w_r^\dagger(\vec{p})\gamma^0 w_{r'}(\vec{p}) = \delta_{rr'}\lambda_r,$$

(b) (15 points) the completeness of spinors w

$$\sum_{r=1}^{4} \lambda_r(w_r(\vec{p}))_\alpha(\bar{w}_r(\vec{p}))_\beta = \delta_{\alpha\beta}, \quad \alpha, \beta = 1...4.$$

In the following, the normalization of spinors is chosen such that $\bar{u}(p, s_z)u(p, s'_z) = \delta_{s_z s'_z}$ and $\bar{v}(p, s_z)v(p, s'_z) = -\delta_{s_z s'_z}$. Using the explicit form for u, $\bar{u}$, v and $\bar{v}$, show that

(c) (10 points)

$$\begin{aligned} u^\dagger(p, s_z)u(p, s'_z) &= \frac{E_p}{m_0 c^2}\,\delta_{s_z s'_z}, \\ v^\dagger(p, s_z)v(p, s'_z) &= \frac{E_p}{m_0 c^2}\,\delta_{s_z s'_z}, \end{aligned}$$

(d) (10 points)

$$\begin{aligned} \sum_{s_z} u(p, s_z)\bar{u}(p, s_z) &= \frac{\not{p} + m_0 c}{2m_0 c}, \\ \sum_{s_z} v(p, s_z)\bar{v}(p, s_z) &= \frac{\not{p} - m_0 c}{2m_0 c}. \end{aligned}$$

Exercise 2 (30 points): Projection operators

Show that $P_{\pm} = \Lambda_{\pm} = \pm\frac{\not{p} \pm m_0 c}{2m_0 c}$ and $P_{\Sigma+,\Sigma-} = \Sigma_{\pm}(s) = \frac{1 \pm \gamma_5 \not{s}}{2}$ are two complete sets of projection operators, *i.e.* both satisfy the conditions

$$P_i P_j = \delta_{i,j}\, P_i, \qquad \sum_i P_i = 1.$$

Show that $\Lambda_{\pm}$ are the projection operators on positive- and negative-energy solutions, and that $\Sigma_{+,-}$ are the projection operators on positive- and negative-chirality solutions.

Exercise 3 (20 points): Lorentz transformation identity

Verify that for arbitrary proper Lorentz transformation S:

$$S^{-1} = \gamma_0 S^{+} \gamma_0.$$

Exercise 4 (20 bonus points): Gordon identity

Derive the so-called *Gordon identity*

$$\bar{u}(p')\gamma^{\mu}u(p) \;=\; \bar{u}(p')\left[\frac{P^{\mu}}{2M} + \frac{i\sigma^{\mu\nu}q_{\nu}}{2M}\right]u(p),$$

where $P = p' + p$, $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$.