6. Übungsblatt
Theoretische Physik 6: WS 2014/15
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01.12.2014

Exercise 1 (25 points): Electrons in constant magnetic field

To study the behavior of electrons in a constant magnetic field, we have to solve the stationary-state Dirac equation

$$\left(-i\hbar c\vec{\alpha}\cdot\vec{D}+\beta m_0 c^2\right)\psi \quad = \quad E\psi$$

where we have used minimal substitution $\vec{\nabla}\to\vec{D}\equiv\vec{\nabla}-ie\vec{A}/\hbar c$.

(a) (10 points) Verify that

$$\left(\vec{\alpha}\cdot\vec{D}\right)^2 \quad = \quad \vec{D}^2\mathbb{1}+e\vec{\Sigma}\cdot\vec{B}/\hbar c.$$

(b) (15 points) For the particular case $\vec{A}=(0,xB,0)$ and by considering solutions of the form $\psi=e^{i(p_y y+p_z z)/\hbar}u(x)$, show that the energy eigenvalues E of a relativistic electron in constant magnetic induction $\vec{B}$ are given by

$$E^2 \quad = \quad m_0^2c^4+p_z^2c^2+(2n+1)|eB|\hbar c\pm eB\hbar c, \qquad n\in\mathbb{N}.$$

Hint: Remember that the eigenvalues of the harmonic oscillator operator $-\partial_x^2+\omega^2x^2$ are $(2n+1)|\omega|$ with $n\in\mathbb{N}$.

Exercise 2 (60 points): Dirac particle in a scalar potential

Consider a Dirac particle travelling along the z-axis and subject to the scalar square-well potential

$$V(z) = \begin{cases} 0 & \text{when } z < -a/2 \qquad\qquad \text{(region I)} \\ V_0 & \text{when } -a/2 \leq z \leq a/2 \quad \text{(region II)} \\ 0 & \text{when } a/2 < z \qquad\qquad \text{(region III)} \end{cases}$$

where $a > 0$ and $V_0 < 0$. In regions I and III, the eigenvalue Dirac equation takes the form

$$\left(\vec{\alpha} \cdot \hat{\vec{p}}\, c + \beta m_0 c^2\right) \psi \;\; = \;\; E\psi,$$

while in region II, it has the form

$$\left[\vec{\alpha} \cdot \hat{\vec{p}}\, c + \beta (m_0 c^2 + V_0)\right] \psi \;\; = \;\; E\psi.$$

In this second region one can consider that due to the potential, the particle has now an effective mass $m_{eff} = m_0 + V_0/c^2$.

(a) (10 points) Write down the general solution $\psi(z)$ in the three regions with the spin in the z-direction.
Hints: A plane-wave solution with momentum $\vec{p}$, mass m and spin label s can be written as

$$u(\vec{p}, s) \;\; = \;\; A \begin{pmatrix} \chi_s \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\, \chi_s \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)},$$

where A is some complex number and χ_s a two-component spinor. For the spin projection along the z-axis, χ_s is an eigenstate of the Pauli matrix σ_3. Do not forget that plane waves can travel in both directions. Remember that the matrices $\vec{\alpha}$ and β in standard representation are given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

(b) (20 points) Impose the continuity condition at $z = \pm a/2$. Express the coefficient of the incoming plane wave of region I in terms of the plane waves of region III. Define for convenience the dimensionless quantity

$$\gamma \equiv \frac{k_1 c}{E + m_0 c^2}\, \frac{E + m_{eff} c^2}{k_2 c},$$

where k_1 is the momentum in regions I and III, and k_2 is the momentum in region II.

(c) (30 points) Consider the special case $|m_{eff}c^2| < |E| < m_0c^2$ corresponding to bound states. Show that these states satisfy

$$k_2 \cot\left(\frac{k_2 a}{\hbar}\right) = -\left(\frac{m_0 V_0}{\kappa_1} + \kappa_1\right),$$

where $\kappa_1 = -ik_1$.
Hints: Show that in the case considered, there could be neither an incoming wave in region I nor an outgoing wave in region III. Show then that continuity requires

$$\Im\left(\frac{1+\gamma}{1-\gamma}\, e^{-\frac{i}{\hbar} k_2 a}\right) = 0,$$

and that $\gamma = i\Gamma$ is imaginary leading then to

$$\cot\left(\frac{k_2 a}{\hbar}\right) = \frac{1-\Gamma^2}{2\Gamma}.$$

Exercise 3 (15 points): Helicity operator

The helicity operator is defined as $\hat{\Lambda} = \vec{S} \cdot \frac{\vec{p}}{|\vec{p}|}$.

(a) (5 points) Show that for a spin-1/2 the possible values of helicity are $\pm\frac{\hbar}{2}$.

(b) (10 points) Derive the eigenvalues of the helicity operator for a particle with momentum $p = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$.