Exercise sheet 4
Theoretical Physics 2: SS2016
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Exercise 1 (25 points): Legendre polynomials and associate functions

a) (5 points)
Using the generating function of the Legendre polynomials

$$\frac{1}{(1 - 2xt + t^2)^{1/2}} = \sum_{l=0}^{\infty} P_l(x)t^l, \tag{1}$$

derive the recurrence relation

$$(2l + 1)xP_l - (l + 1)P_{l+1} - lP_{l-1} = 0. \tag{2}$$

b) (10 points)
Show that

$$P_m^m(\cos\theta) = (2m - 1)!! \, \sin^m\theta, \qquad m = 0, 1, 2, \ldots, \tag{3}$$

where the functions P_l^m are defined by the relations

$$P_l^m(x) = (1 - x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x), \qquad m = 0, 1, 2, \ldots, l.$$

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c) (10 points)

Given $\mathbf{r}_1$ and $\mathbf{r}_2$ two vectors having polar angles (θ_1, ϕ_1) and (θ_2, ϕ_2), prove the addition theorem of spherical harmonics

$$P_l(\cos \theta) = \frac{4\pi}{2l + 1} \sum_{m=-1}^{+l} Y_{lm}^*(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2), \tag{4}$$

where $\theta = \angle(\mathbf{r}_1, \mathbf{r}_2)$ is the angle between these two vectors and the spherical harmonics are given by

$$Y_{lm}(\theta, \phi) = (-1)^m \left[\frac{(2l + 1)}{4\pi} \frac{(l - m)!}{(l + m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi}, \qquad m \geq 0 \tag{5}$$

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi). \tag{6}$$

Exercise 2 (20 points): Multipole expansion of point charges

Consider the following arrangement of point charges in Cartesian coordinates: two located along the $y-$ axis at $+a$ and $-a$ each of charge $-q$, and one charge along the z axis at $z = a$ of charge q as shown in the Figure below.
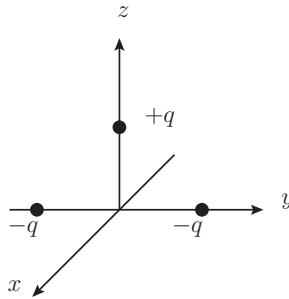
Find the approximate electric field at points far away from the origin. Express your answer in spherical coordinates and include the two lowest orders in the multipole expansion.

Exercise 3 (20 points): Quadrupole moment of an ellipsoid

Determine the quadrupole moment of an uniformly charged ellipsoid with semi-axes lengths a, b, c.

Exercise 4 (35 points): Dielectric sphere

A sphere of radius R and homogeneous dielectric material ϵ is placed in a uniform electric field $\vec{E}_0$.

(a) (20 points)
Find the boundary conditions of the potential Φ and compute the potential inside and outside the sphere.

(b) (15 points)
Find the electric field inside and outside the sphere and the charge density ρ inside the sphere.