4. Übungsblatt
Theoretische Physik 6: WS 2014/15
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Exercise 1 (50 points): Scalar theory with $SO(2)$ invariance

Consider the following Lagrangian density of two real scalar fields $\phi_1(x)$, $\phi_2(x)$:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4!}\left(\phi_1^2 + \phi_2^2\right)^2 . \tag{1}$$

(a) (10 points) Identify the corresponding equations of motion.

(b) (10 points) Show that the above Lagrangian is invariant under the transformations

$$\phi_1 \to \phi_1' = \phi_1 \cos\theta - \phi_2 \sin\theta, \tag{2}$$

$$\phi_2 \to \phi_2' = \phi_1 \sin\theta + \phi_2 \cos\theta. \tag{3}$$

(c) (10 points) Calculate the Noether current j_μ and show explicitly that its divergence vanishes for fields ϕ_i that satisfy the equations of motion.

(d) (10 points) Show explicitly that the Noether charge Q is a conserved quantity, provided the surface integral $\int \mathrm{d}S\, \vec{n}\cdot\vec{j}$ vanishes.

(e) (10 points) Construct the Hamiltonian density H, and show that the Poisson bracket with the zero component of the Noether current vanishes (again assuming $\int \mathrm{d}S\, \vec{n}\cdot\vec{j} = 0$):

Exercise 2 (50 points): Pionic atoms

A pionic atom is formed when a negative pion π^-, which is a spin-0 boson, is stopped in matter and is captured by an atom. The incident pion slows down by successive electromagnetic interactions with the electrons and nuclei, and when it reaches the typical velocity of atomic electrons, the pion is captured by ejecting a bound electron from its Bohr orbit. Let us approximate the potential between the nucleus and the pion by a square-well $V = -V_0$ for $r \leq R$ and $V = 0$ for $r > R$, where R is the nucleus radius.

(a) (20 points) Using the minimal substitution $p_\mu \to p_\mu - \frac{e}{c} A_\mu$, with $A_\mu = (V, \vec{0})$, show that the Klein-Gordon equation leads to the following radial equation for the field

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2 \right] u(r) = 0,$$

where $k^2 = \frac{1}{\hbar^2 c^2} \left[(\epsilon - eV)^2 - m_\pi^2 c^4 \right]$ with ϵ the energy of the pion.
Hint: Use the Klein-Gordon field in the following factorized form $\phi(\vec{x}, t) = u(r)\, Y_{lm}(\Omega)\, e^{-\frac{i}{\hbar}\epsilon t}$ with $Y_{lm}(\Omega)$ the standard spherical harmonic functions.

(b) (20 points) Since for a bound state we have $k^2 > 0$ for $r \leq R$ and $k^2 < 0$ for $r > R$, solve the equation for an s-state ($l = 0$) in both regions.
Hint: Use the *Ansatz* $u(r) = v(r)/r$.

(c) (10 points) Match the solutions by imposing equal logarithmic derivatives $\frac{1}{u_i}\frac{du_i}{dr} = \frac{1}{u_o}\frac{du_o}{dr}$ at $r = R$, and show that this amounts to solve the transcendental equation

$$k_i \cot(k_i R) \;=\; -k_o,$$

where $k_i^2 = \frac{1}{\hbar^2 c^2} \left[(\epsilon + eV_0)^2 - m_\pi^2 c^4 \right]$ and $k_o^2 = \frac{1}{\hbar^2 c^2} \left(m_\pi^2 c^4 - \epsilon^2 \right)$.