Exercise sheet 3
Theoretical Physics 2: SS2016
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Exercise 1 (25 points): Dirac delta distributions

a) (5 points)
Using the properties of the Dirac delta evaluate the integral

$$\int_{1}^{\infty} \sin t \, \delta\left(\frac{t^2}{3} - \frac{\pi^2}{4}\right) \mathrm{d}t$$

b) (10 points)
Consider n point charges located at n different points $\{P_k\}_{k=1}^{n}$ in a plane with P_k having polar coordinates (r_k, θ_k). These points are assumed to be on a curve $r = g(\theta)$. Express the surface charge density using Dirac delta 'functions'. Use this expression for a circle of radius a, where the n charges are separated by its nearest neighbor by an angle α with the first charge on the x axis.

c) (10 points)
Express the three-dimensional charge distributions $\rho(\mathbf{r})$ using Dirac delta functions in cylindrical coordinates for a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

Exercise 2 (20 points): Charged cylinders and tubes

a) (10 points)
An infinitely long solid cylinder with radius a has the homogenous space charge density ρ_0. Solve the potential equation for the in- and outside of the cylinder by using the symmetry of this problem.
b) (5 points)
Use the continuity of potential and field at the cylinder surface to determine the free constants in ϕ. Sketch the resulting potential.
c) (5 points)
An infinitely long hollow cylinder with radius a and infinitely small wall strength has the homogenous surface charge density ρ_0. Solve the potential equation for the in- and outside of the cylinder by using the symmetry of this problem.

Exercise 3 (25 points): Green's Function for two concentric spheres

Obtain the Green's function for two concentric spheres. For an arbitrary potential the function satisfies the equation

$$\nabla_x^2 G(\vec{x}, \vec{x}') = -\delta(\vec{x} - \vec{x}'),$$

with the delta function given by

$$\delta(\vec{x} - \vec{x}') = \frac{1}{r^2}\delta(r - r') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).$$

Consider two concentric spherical surfaces with the radii a and b ($b > a$). Show that the Green's function for this geometry is given by

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)}{2l+1} \frac{1}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right),$$

with $r_<$ denotes the smaller radii among r, r' and $r_>$- the larger one.

Hint: Any function $g(\theta, \phi)$ in spherical coordinates can be decomposed in terms of spherical harmonics as

$$g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \phi) \quad \text{with} \quad A_{lm} = \int \mathrm{d}\Omega Y_{lm}(\theta, \phi) g(\theta, \phi),$$

and the differential equation

$$\frac{d^2}{dr^2} R(r) - \frac{l(l+1)}{r^2} R(r) = 0,$$

has the solution $R(r) = Ar^{l+1} + Br^{-l}$.

Exercise 4 (30 points): Azimuthal symmetric potential

Two concentric spheres have radii a, b, $(b > a)$, and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential $V = const.$ The other hemispheres are at zero potential.

(a) (15 points)

Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta).$$

Find the numerical values of coefficients in series at least up to $l = 4$.

(b) (15 points)

Determine the potential in the region $a \leq r \leq b$ with a help of Green's function in spherical coordinates from the Exercise 3. Compare the result with the result from part a).

Hint: It may be helpful to use the orthogonality of the Legendre polynomials and the relation

$$\int_0^1 P_l(x) dx = \begin{cases} \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!} & l \text{ is odd} \\ 0 & l \text{ is even} \end{cases}$$

with $n!! = n \cdot (n-2) \cdot (n-4) \cdots 3 \cdot 1$ for odd n.