3. Übungsblatt
Theoretische Physik 6: WS 2014/15
Dozent: Prof. M. Vanderhaeghen

10.11.2014

Exercise 1. (10 points): Fermion wave function

In terms of the single-fermion states ψ_{is}, $i = 1, 2$ and $s =\uparrow, \downarrow$, write all possible 2- and 3-fermion states.

Exercise 2. (20 points): Number of fermions

Show that for fermions, the particle number operator $N = \sum_i a_i^\dagger a_i$ commutes with the Hamiltonian

$$H = \sum_{i,j} \langle i|H_0|j\rangle\, a_i^\dagger a_j + \frac{1}{2} \sum_{i,j,k,l} \langle i,j|V|k,l\rangle\, a_i^\dagger a_j^\dagger a_l a_k.$$

Hint: Prove first that $[AB, C] = A[B, C] + [A, C]B = A\{B, C\} - \{A, C\}B$.

Exercise 3. (30 points): Fermionic operators

Let a and b be two operators satisfying the anticommutation relations

$$\{a, b\} = 1, \qquad \{a, a\} = \{b, b\} = 0. \tag{1}$$

(a) (5 points) Show that

$$a^2 = b^2 = 0. \tag{2}$$

(b) (10 points) Prove that $ba(1 - ba) = 0$. What does it mean for the eigenvalues of $N = ba$?

(c) (10 points) Prove the relations $[N, a] = -a$ and $[N, b] = b$. Applying these on the eigenstates of N $|0\rangle$ and $|1\rangle$, show that with suitable choice of phases

$$a|0\rangle = 0, \qquad a|1\rangle = |0\rangle, \qquad b|0\rangle = |1\rangle, \qquad b|1\rangle = 0. \tag{3}$$

(d) (5 points) Show that (3) implies that $b = a^\dagger$ is the hermitian conjugate operator of a (*i.e.* $\langle m|b|n\rangle = \langle n|a|m\rangle^*$, $\forall\, m, n$).

Clearly, all the important fermion properties (2) and (3) are just consequences of the anticommutation relations (1).

Exercise 4. (40 points): Ground state energy of high-density electron gas in first order perturbation theory

The Hamiltonian of a homogeneous electron gas is given by $\hat{H} = \hat{H}_0 + \hat{H}_1$ with:

$$\begin{aligned}
\hat{H}_0 &= \frac{\hbar^2}{2m} \sum_{\vec{k},s} \left|\vec{k}\right|^2 a^\dagger_{\vec{k},s} a_{\vec{k},s}\,, \\
\hat{H}_1 &= \frac{e^2}{2V} \sum_{\vec{k},\vec{p}} \sum_{\vec{q}\neq\vec{0}} \sum_{s,s'} \frac{4\pi}{|\vec{q}\,|^2}\, a^\dagger_{\vec{k}+\vec{q},s} a^\dagger_{\vec{p}-\vec{q},s'} a_{\vec{p},s'} a_{\vec{k},s}\,.
\end{aligned}$$

In the high-density limit, $\hat{H}_1$ is a perturbation to $\hat{H}_0$. Using techniques of perturbation theory, it is possible to estimate in this regime the ground state energy of the interacting electron gas.

(a) (5 points) Express the Fermi momentum k_F in terms of the inter-particle spacing r_0.

(b) (15 points) Determine $\frac{E^{(0)}}{N}$ in terms of k_F.

Hints:

- $E^{(0)} = \langle \Psi_0|\hat{H}_0|\Psi_0\rangle$.
- All states with momenta $\left|\vec{k}\right| \leq k_f$ are occupied. Therefore the particle number $n_{\vec{k},s}$ can be expressed trough a Θ – function.
- In the limit that the volume of the system becomes infinite, the sums over states can be replaced by integrals:

$$\sum_{\vec{k},s} f_s(\vec{k}\,) \to \frac{V}{(2\pi)^3} \sum_s \int d^3\vec{k}\, f_s(\vec{k}\,). \tag{4}$$

- The zeroth order describes the free electron gas.

(c) (20 Punkte) Show that

$$E^{(1)} = -\frac{4\pi e^2 V}{(2\pi)^6} \int d^3\vec{k}\, \theta\left[k_F - \left|\vec{k}\right|\right] \int d^3\vec{q}\, \frac{1}{|q|^2}\, \theta\left[k_F - |\vec{k}+\vec{q}|\right].$$

Hints:

- $E^{(1)} = \langle \Psi_0 | \hat{H}_1 | \Psi_0 \rangle$.
- The creation and annihilation operators are acting on the ground state Ψ_0. Which states need to be occupied in order to have a non-vanishing matrix element?
- The creation and annihilation operators need to be paired in a way that the matrix element is non-vanishing. Due to the restriction $\vec{q} \neq 0$ there is only one option to combine the operators.