Exercise sheet 2
Theoretical Physics 2: SS2016
Lecturer: Prof. M. Vanderhaeghen
Assistant: Leonardo de la Cruz

25.04.2016

Exercise 1 (20 points): Electric and magnetic fields equations

Using the Maxwell equations for an arbitrary charge density $\rho(t, \mathbf{r})$ and an arbitrary current density $\mathbf{j}(t, \mathbf{r})$, find an equation for the electric field $\mathbf{E}$ depending on ρ and $\mathbf{j}$ (but not on $\mathbf{B}$) and an equation for the magnetic field $\mathbf{B}$ depending on ρ and $\mathbf{j}$ (but not on $\mathbf{E}$).

Exercise 2 (20 points): Electric dipole radiation

At large distances from its source, electric dipole radiation is described by the fields

$$\mathbf{E} = a_E \sin\theta \frac{e^{i(kr-\omega t)}}{r}\mathbf{e}_\theta, \qquad \mathbf{B} = a_B \sin\theta \frac{e^{i(kr-\omega t)}}{r}\mathbf{e}_\phi. \tag{1}$$

Show that Maxwell's equations in absence of sources

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

are satisfied, if we take

$$\frac{a_E}{a_B} = \frac{\omega}{ck} = 1 \tag{3}$$

Hint. Since r is large, terms of order r^{-2} may be dropped.

Exercise (30 points): Additional symmetries of the Maxwell equations

a) (20 points)
Consider the transformations C, P, and T, given by

T	(time reversal)	$t' = -t$	$\mathbf{x}' = \mathbf{x}$	$\rho'(t, \mathbf{x}) = \rho(t', \mathbf{x}')$
P	(parity)	$t' = t$	$\mathbf{x}' = -\mathbf{x}$	$\rho'(t, \mathbf{x}) = \rho(t', \mathbf{x}')$
C	(charge conjugation)	$t' = t$	$\mathbf{x}' = \mathbf{x}$	$\rho'(t, \mathbf{x}) = -\rho(t', \mathbf{x}')$

where $\rho(x)$ is the charge density, and $\mathbf{x}$, t are the position and time coordinates respectively. How do the current density $\mathbf{j}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ have to transform in order for the Maxwell equations still to hold?

b) (10 points)
Show that the Maxwell equations in vacuum (in absence of any sources) are invariant under the duality transformations $D : \mathbf{B} \to \mathbf{E}, \quad \mathbf{E} \to -\mathbf{B}$. What is D^2?

Exercise 4 (30 points): Gauge invariance

A vector potential $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{r}) = \begin{cases} -yB\mathbf{e}_x & \text{for} \quad x^2 + y^2 < R^2 \\ \frac{BR^2}{2(x^2+y^2)}(-y\mathbf{e}_x + x\mathbf{e}_y) - \frac{1}{2}B(y\mathbf{e}_x + x\mathbf{e}_y) & \text{for} \quad x^2 + y^2 > R^2 \end{cases}$$

a) (20 points)
Find the functions Λ_1, Λ_2 for the gauge transformations

$$\mathbf{A}_{1,2}(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla\Lambda_{1,2}(\mathbf{r})$$

which are tranforming the vector potential in the inner region ($x^2+y^2 < R^2$) into

$$\mathbf{A}_1(\mathbf{r}) = \frac{B}{2}(-y\mathbf{e}_x + x\mathbf{e}_y) \quad \text{and} \quad \mathbf{A}_2(\mathbf{r}) = B(x-y)\mathbf{e}_x.$$

Find the potential $\mathbf{A}_{1,2}$ in the outer region.

b) (10 points)
Which vector potentials satisfy the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$?