1. Übungsblatt
Theoretische Physik 6: WS 2014/15
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1. Exercise (70 points): Non-relativistic neutron and electron gases

At extremely high density, inverse beta decay, $e^- + p^+ \to n + \nu$, converts virtually all of the protons and electrons inside a star into neutrons. These stars (called neutron stars) are stabilised against gravitational collapse by the degeneracy pressure of their neutrons (this is the pressure of a free neutron gas). Assuming constant density, the radius R of such an object can be calculated as follows:

(a) (5 points) Write the total neutron energy in terms of the radius R, the number of neutrons N and the neutron mass M.

(b) (15 points) Calculate the gravitational energy of a uniformly dense sphere. Express the result in terms of the gravitational constant G, R, N and M. Note that the gravitational energy is negative (attractive).

(c) (15 points) Find the radius for which total energy (a) + (b) is a minimum. R is a function of $\hbar, G, M$, and N. Express R numerically as function of N.

(d) (5 points) Determine the radius of a neutron star (in km) with the mass of the sun (the mass of the sun is $M_\odot = 1.989 \times 10^{30}$ kg, the nucleon mass is $M = 1.674 \times 10^{-27}$ kg). Compare this radius with the Earth radius.

(e) (10 points) Determine the Fermi energy for the neutron star in (d) (in MeV) and compare it with the rest energy of a neutron. Are these neutrons relativistic?

(e) (20 points) Other cold stars (called white dwarfs) are stabilised against gravitational collapse by the degeneracy pressure of their electrons (this is the pressure of a free electron gas). Repeat (a) – (e) for white dwarfs, here q is the number of electrons per nucleon and N is the number of nucleons (protons and neutrons). The electron mass m can be neglected in comparison with the nucleon mass M. Assume $q = 1/2$ for numerical evaluation. Are the electrons relativistic?

2. Exercise (30 points): relativistic electron gas

Extend the theory of a free electron gas to the relativistic electrons. Replace the classical kinetic energy $E = p^2/2m$ with the relativistic formula:

$$E = \sqrt{p^2c^2 + m^2c^4} - mc^2, \tag{1}$$

where momentum and wave vector are related in the following way: $\vec{p} = \hbar\vec{k}$. In the extreme relativistic limit we have: $E \approx pc = \hbar kc$.

(a) (10 points) Calculate E_{tot} of the electron gas in the extreme relativistic limit.

(b) (20 points) Repeat (a) and (b) of exercise 1. Notice that in this case there is no stable minimum, regardless of R. If the total energy is positive, degeneracy forces exceed gravitational forces, and the star will expand. If the total energy is negative, gravitational forces win out, and the star will collapse. Find the critical number of nucleons N_c, such that gravitational collapse occurs for $N > N_c$. This is called the Chandrasekhar limit: $N_c \approx 2 \times 10^{57}$. What is the mass of such a star compared to the mass of the sun? Stars heavier than this will not form white dwarfs, but collapse further, becoming neutron stars.