Bonus exercises
Theoretical Physics 2: SS2016
Lecturer: Prof. M. Vanderhaghen
Assistant: Leonardo de la Cruz

12.07.2016
Until 20.07.2016 at 14:00

Exercise 1 (35 bonus points): Dielectric cylinder in a uniform electric field

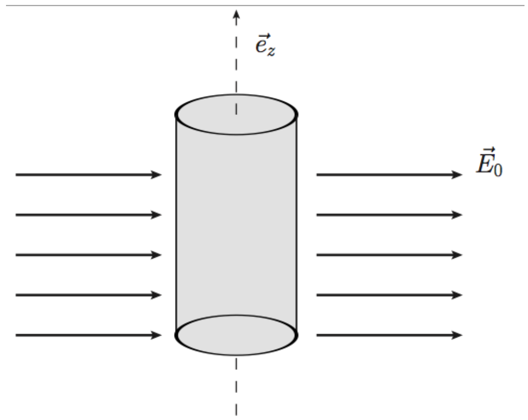
(a) (12 points)
Show by the separation of variables in the cylindrical coordinates that the solution of the Laplace equation for the cylindrical symmetry (there is no dependence on z coordinate) is given by

$$\Phi(r,\phi) = A_0 \ln r + B_0 + \sum_{n=1}^{\infty} \left[A_n r^n + B_n r^{-n}\right] \left[C_n \sin(n\phi) + D_n \cos(n\phi)\right].$$

Consider a (infinitely) long cylinder with radius R and dielectric constant $\varepsilon > 1$ in the homogeneous electric field $\mathbf{E}_0$ ($\mathbf{E}_0 \perp \mathbf{e}_z$).

(b) (6 points)
Write down the boundary conditions for the potential Φ.
(c) (12 points)
Find the electric potential inside and outside the cylinder.
(d) (5 points)

Find the electric field inside and outside the cylinder.



Exercise 2 (30 bonus points): Electric and magnetic fields in laboratory frame

Consider a (infinite) long straight wire with negligibly small cross section area with a line charge density λ in its reference frame S'. The frame S' moves with a constant velocity $\mathbf{v} = v\mathbf{e}_z$ with respect to the laboratory frame S.

(a) (13 points)

Find the electric and magnetic fields in the wire rest frame, S', in cylindrical coordinates and find the fields in the laboratory frame S.

(b) (7 points)

Write down the charge density ρ and the current density $\mathbf{j}$ in S' and find the values in the laboratory frame S.

(c) (10 points)

Using the charge and current density in the laboratory frame S find the electric and magnetic field in the laboratory frame S. Compare the result with the result from **a)**.

Exercise 3 (30 bonus points): Cylindrical cavity

Consider the volumetric resonator of cylindrical form with a radius R_a and height h. The two cylinder heads at $z = 0$ and $z = h$ are made from conducting material.

(a) (10 points)
Separate variables in the wave equation

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

for z-component with substitution $E_z = R(r)Q(\phi)Z(z)e^{-i\omega t}$.

(b) (10 points)
Find the solutions $Q(\phi)$, $Z(z)$ and $R(r)$ in the cylindrical resonator for TM modes (e.g., $B_z = 0$).

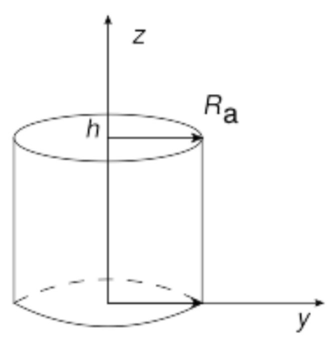
Hint: Recall, that the solution of the equation

$$x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + (x^2 - k^2)y = 0$$

is given by the Bessel functions $J_k(x)$.

(c) (10 points)
Find the resonant frequencies ω_{lmn}. Find three lowest resonant modes for $R_a = h$.

Hint: The q-th Null of the Bessel function of order p x_{pq} is given by: $x_{01} = 2.405,\ x_{02} = 5.520,\ x_{11} = 3.832,\ x_{21} = 5.136, ...$



Exercise 4 (35 bonus points): $\frac{\lambda}{2}$-antenna

(a) (15 points)
The vector potential of the $\frac{\lambda}{2}$-antenna which is oriented symmetrically along the z-axis is given by

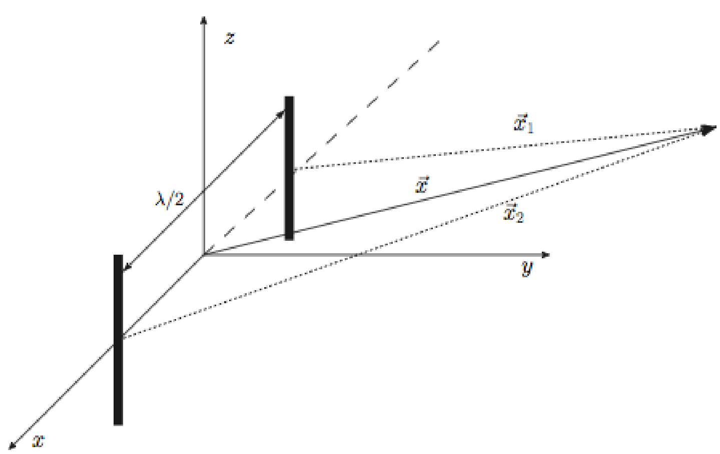
$$\mathbf{A} = \frac{I_0}{2\pi\omega|\mathbf{x}|} \cos\left[\omega\left(t - \frac{|\mathbf{x}|}{c}\right)\right] \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\mathbf{e}_z. \quad (1)$$

The correspondent magnetic field $\mathbf{B}$ can be found by

$$\mathbf{B} \simeq -\frac{1}{c}\hat{x} \times \frac{\partial \mathbf{A}}{\partial t}. \quad (2)$$

Determine the time-averaged power of radiation per solid angle of the antenna $< \frac{dP}{d\Omega} >$.

Consider two parallel $\frac{\lambda}{2}$-antennas which are both located symmetrically along the z-axis at $x = \frac{\lambda}{4}$ and $x = -\frac{\lambda}{4}$ (with $y = 0$).



(b) (20 points)
Compute the time averaged power of radiation per solid angle for the case when both antennas radiate in phase and the oscillation of the antennas is shifted by 180^0. You can use the Eq. (1) as a vector potential for each of two antennas. Write down the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ through $\mathbf{x}$ and $\mathbf{e}_x$ and evaluate

$|\mathbf{x}_1|$ and $|\mathbf{x}_2|$, neglecting the terms of the order higher than λ. Evaluate the total $\mathbf{A}$ considering only terms of order $\frac{1}{|\mathbf{x}|}$.

Hint: For this purpose you can approximate the angle for both antennas, θ_1 and θ_2, by θ.

Summary of formulas:

Addition theorem:

$$\begin{aligned}\sin(x \pm y) &= \sin(x)\cos(y) \pm \sin(y)\cos(x)\\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y)\end{aligned}$$

Spherical coordinates:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$$

Cylindrical coordinates:

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z\\ \nabla^2\Phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\partial\Phi}{\partial^2 z}\\ \nabla\times\mathbf{V} &= \left(\frac{1}{r}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\mathbf{e}_\phi + \frac{1}{r}\left(\frac{\partial(rV_\phi)}{\partial r} - \frac{\partial V_r}{\partial\phi}\right)\mathbf{e}_z\end{aligned}$$

Relation between Heaviside-Lorentz and SI units:

$$\begin{aligned}\mathbf{E}_{HL} &\longleftrightarrow \sqrt{\epsilon_0}\mathbf{E}_{SI}\\ \mathbf{B}_{HL} &\longleftrightarrow \frac{1}{\sqrt{\mu_0}}\mathbf{E}_{SI}\\ q_{HL} &\longleftrightarrow \frac{1}{\sqrt{\epsilon_0}}q_{SI}\\ \mathbf{j}_{HL} &\longleftrightarrow \frac{1}{\sqrt{\epsilon_0}}\mathbf{j}_{SI}\\ \mathbf{m}_{HL} &\longleftrightarrow \sqrt{\mu_0}\mathbf{m}_{SI}\end{aligned}$$