# 13. Übungsblatt
# Theoretische Physik 6: WS2014/15
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## Exercise 1. (40 points): Hyperfine splitting in hydrogen - 21 cm line

The interaction between an electron and a magnetic field is given by

$$\hat{H}_{\text{int}} = -\vec{\mu} \cdot \hat{\vec{B}},$$

where $\vec{\mu}$ is the magnetic moment

$$\vec{\mu} = \frac{e}{2m} \frac{\hbar}{c} \vec{\sigma}$$

of the electron.

**(a) (10 points)** Express $\hat{H}_{\text{int}}$ through its normal mode expansion.
*Hint*: Use

$$\hat{\vec{A}} = \sum_{\vec{k},\sigma} N_k \vec{\epsilon}_{\vec{k},\sigma} \left\{ \hat{a}_{\vec{k},\sigma} e^{i\vec{k}\vec{x}} + \hat{a}^{\dagger}_{\vec{k},\sigma} e^{-i\vec{k}\vec{x}} \right\},$$

where $\vec{\epsilon}_{\vec{k},\sigma}$ is the photon polarisation vector.

**(b) (30 points)** The $1S$-state with $f = 1$ has a slightly higher energy than the $1S$-state with $f = 0$. In the transition between the initial state

$$|\psi_i\rangle = |1S\rangle |\uparrow\rangle_e |\uparrow\rangle_p,$$

and the final state

$$|\psi_f\rangle = |1S\rangle \frac{1}{\sqrt{2}} \{| \uparrow\rangle_e | \downarrow\rangle_p - | \downarrow\rangle_e | \uparrow\rangle_p\} ,$$

a photon with $\lambda \approx 21$cm is emitted. Calculate the lifetime of this transition and give a numerical value.
*Hint*: Calculate the matrix element

$$\langle f|\hat{H}_{\text{int}}|i\rangle$$

in dipole approximation and apply Fermi's golden rule. The expected answer is:

$$\frac{1}{\tau_{i\to f}} = \frac{e^2}{4\pi}\frac{1}{3}\frac{\hbar}{m^2c^2}k_0^2,$$

with $k_0 = 2\pi/\lambda$.

# Exercise 2 (45 points): Mott scattering

Consider unpolarized electron scattering in Quantum Electrodynamics in an external Coulomb field $A^\mu = (-\frac{Ze}{4\pi r}, 0, 0, 0)$. Treat the electron as a massive Dirac particle with mass m.

**(a) (15 points)** Write down the expression for the differential cross-section in the laboratory frame in terms of the matrix element. For the case of a static potential, the energy of initial and final electrons are the same, whereas the electron three-momentum can change its direction. For a potential scattering, we only need to consider the phase space and normalization factors of the electron.

**(b) (5 points)** Integrate over the final electron energy.

**(c) (20 points)** The matrix element can be obtained from the interaction term $\mathcal{L}_1 = -eA^\mu j_\mu$. In Fourier space the Coulomb potential is given by

$$V(q) = -\int \frac{Ze}{4\pi r} e^{i\vec{q}\vec{r}} d^3r = -\frac{Ze}{|\vec{q}|^2}$$

Calculate the squared matrix element for the unpolarized scattering.

**(d) (5 points)** Find the result for the differential cross section in the laboratory frame in terms of the initial electron kinematic variables (energy $w$, and relative velocity $\beta$) and the scattering angle. Evaluate the nonrelativistic limit of cross section formula (this limit is called Rutherford scattering).

## Exercise 3 (15 points):

Consider the following Lagrangian for a real scalar field $\phi$ in 1+1 (one spatial and one time) dimensions

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{\lambda}{4}(\phi^2 - v^2)^2,$$

**(a) (5 points)** Construct the corresponding Hamiltonian and find the condition on classical field configurations $\phi_0(x)$ which minimizes the energy.
*Hint*: Remember that the conjugate momentum $\pi$ of a field $\phi$ is given by $\pi = \partial \mathcal{L}/\partial \dot{\phi}$ and the Hamiltonian by $H = \int dx \left(\pi \dot{\phi} - \mathcal{L}\right)$.

**(b) (5 points)** Find the equations of motions (Euler Lagrange equations) for the field $\phi$.

**(c) (5 points)** The static solution which interpolates between 2 vacuum states

$$\phi_0(x) = v \times \tanh\left(\sqrt{\frac{\lambda}{2}} v x\right),$$

is called the kink solution. Prove that the kink is indeed a solution of the equations of motion.