Equilibrium Labor Turnover, Firm Growth and Unemployment

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Abstract

This paper considers a dynamic, non-steady state environment in which wage dispersion exists. Workers do not observe firm productivity and firms do not commit to future wages, but there is on-the-job search for higher paying jobs. The model allows for firm turnover (new start-up firms are created, some existing firms die) and firm specific productivity shocks. In a separating equilibrium, more productive firms signal their type by paying strictly higher wages in every state of the market. Workers always quit to firms paying a higher wage and so move efficiently from less to more productive firms. As a further implication of the cost structure assumed, endogenous firm size growth is consistent with Gibrat's law. The paper provides a complete characterization and establishes existence and uniqueness of the separating (non-steady state) equilibrium in the limiting case of equally productive firms. The existence of equilibrium with any finite number of firm types is also established. Finally, the model provides a coherent explanation

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of Danish manufacturing data on firm wage and labor productivity dispersion as well as the cross firm relationship between them.

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1 Introduction

The model studied in this paper is one in which employers set the wage paid in the tradition of Diamond (1971), Burdett and Judd (1983), Burdett and Mortensen (1998), Coles (2001) and Moscarini and Postel-Vinay (2010). It differs from these papers by introducing (i) recruiting behavior at a cost of the form estimated by Merz and Yashiv (2007), (ii) firm entry and exit, and (iii) firm specific productivity shocks. Its purpose is to identify a rich but tractable dynamic variant of the Burdett-Mortensen (BM) model that can be used for both macro policy applications and micro empirical analysis.

The framework developed contains several key contributions. First, we show that introducing a hiring margin into the BM model results in a surprisingly tractable structure. In the existing BM framework, wages are chosen both to attract and to retain employees and equilibrium wage dispersion arises in which the wage paid by a firm depends on its size. In contrast equilibrium wage and hiring strategies here depend only on firm productivity and the state of the aggregate economy. The resulting structure generates equilibrium dispersion in individual firm growth rates which, consistent with Gibrat's law, are size independent as documented in Haltiwanger et al. (2011). In particular more productive firms pay higher wages, enjoy positive expected growth, and so generally become larger. Low productivity firms instead decline because their low hire rate is not sufficient to replace employees quitting to better paying jobs.

In Moscarini and Postel-Vinay (2010), the existence of a (recursive rank-preserving) equilibrium in the BM framework requires a restriction on initial conditions. Specifically, because the wage strategy is size dependent in their model, higher paying firms must be larger initially to guarantee equilibrium. Unfortunately this condition is violated in real data because firms die and new start-up companies are typically small. The framework established here explicitly incorporates innovative start-up companies who are born small but (depending on realized productivity) can grow quickly over time. Conversely large existing firms may experience adverse productivity shocks and so enter periods of decline.

As a second key contribution, we suppose no future wage precommitment. Wages are determined in a model of asymmetric information where each firm's productivity $p \in [\underline{p}, \overline{p}]$, which is subject to shocks, is private information to the firm. As workers are long-lived, they care about the future expected income stream at any given employer. In this framework firm productivity is a persistent process: a high productivity firm is more likely than a low productivity firm to be highly productive tomorrow. As employees are more valuable to high productivity firms, a signalling equilibrium arises where more productive firms pay higher wages and, consequently, enjoy a lower quit rate. The lower quit rate occurs as employees believe the firm is not only highly productive today but is more likely to remain highly productive into the future and so will continue to pay high wages.

The equilibrium structure is thus not unlike an efficiency wage model of quit turnover (e.g. Weiss (1980)). Unlike a competitive economy where all firms pay the same wage (given equally productive workers), here high productivity firms pay higher wages to reduce the quit rate of its employees to better paying firms. Should a firm cut its wage, its employees believe the firm has experienced an adverse productivity shock. Given the fall in expected future earnings at this firm, this wage cut triggers a corresponding increase in employee quit rates.

Perhaps the central contribution of the paper, however, is the characterization of equilibrium labor market adjustment outside of steady state. The standard matching framework (e.g. Pissarides (2000)) determines wages via a Nash bargaining condition, so that wages depend only on the current state of the market s_t, and then describes dynamic (Markov) equilibria (e.g. Mortensen and Pissarides (1994)). In contrast equilibrium wages here are determined according to a signalling condition but this rule is also Markov, depending only on the current state s_t which determines the distribution of current firm values. The resulting structure not only generates equilibrium wage dispersion across employed workers, its infimum is pinned down by the value of home productivity b which ensures wages are not fully flexible over the cycle. Furthermore being a model of aggregate job creation (firm recruitment strategies) and of job-to-job transitions (via on-the-job search), it identifies a coherent, non-steady state framework of equilibrium wage formation and labor force adjustment. By focussing on Markov perfect (Bayesian) equilibria, the framework can be readily extended to a business cycle structure where the economy is itself subject to aggregate shocks.

Given the restrictions on primitives needed to guarantee the existence of bounded values for all agents in our model, we show that a unique separating equilibrium exists in the limiting case of equally productive firms. Formally, any equilibrium solution is isomorphic to the stable saddle path of an ordinary differential equation system that describes the adjustment dynamics of the value of a job-worker match and aggregate unemployment to their unique steady state values. In the case of firm heterogeneity with respect to productivity, we establish the existence of at least one separating equilibrium when the distribution of firm productivity limits to a finite number of firm types.

Menzio and Shi (2010) develop and study a recursive model of directed search that also allows for search on-the-job. In their paper, they suggest that directed search is a more useful approach for understanding labor market dynamics. They claim that models of random search in the Burdett-Mortensen tradition are intractable because the decision relevant state space is the evolving distribution of wages, which is of infinite dimension. Although the directed search model is arguably simpler in some respects, their principal objection to a random search model is not valid in the variant considered in the paper. Indeed, in the limiting case of equally productive firms, the relevant state variable is simply the aggregate level of unemployment, a scalar.

A troublesome implication of the original Burdett-Mortensen model for empirical implementation is that the equilibrium firm wage distribution is convex in the case of homogenous firms while in the data it has an interior mode. Although a unimodal distribution is possible when firms differ in labor productivity, Mortensen (2003) shows that model is not consistent with both the observed firm wage distribution and the distribution of firm productivity in Danish data. In the case of our model, the implied distribution of firm wages generally has an interior mode given the form of the roughly linear but decreasing wage-productivity profile observed in (Danish) data. Furthermore, the model is fully consistent with this shape under the plausible restriction that the productivity density over new entrants is decreasing and converges to zero.

2 The Model

Time is continuous. The labor market is populated by a unit measure of equally productive, risk neutral and immortal workers who discount the future at instantaneous rate r. Every worker is either unemployed or employed, earns a wage if employed, and the flow value of home production, $b \geq 0$, if not. There is also a measure of risk neutral, heterogeneous firms. Market output is produced by a matched worker and firm with a linear technology.

New firms enter at rate $\mu > 0$, continuing firms die at rate $\delta > 0$ so that the measure of firms is stationary and equal to μ/δ. At entry, the productivity of a new firm p is determined as a random draw from the c.d.f. $\Gamma_0(.)$. Continuing firms with productivity p are subject to a technology shock process characterized by a given arrival rate $\gamma \geq 0$ and a distribution of new values from c.d.f. $\Gamma_1(.|p)$. For ease of exposition, Γ_0, Γ_1 are continuous functions. As in Klette and Kortum (2004) and Lentz and Mortensen (2008), one can think of the entry flow as firms with new products and the exit flow as firms that are destroyed because their product is no longer in demand.

Given the above productivity and turnover processes, it is a straightforward algebraic exercise to compute the stationary distribution of firm productivity $\Phi(p)$. It is convenient, however, to instead rank firms by their productivity; i.e. a firm with productivity p is equivalently described as having rank $x \in [0,1]$ solving $x = \Phi(p)$. The inverse function $p(x) = \Phi^{-1}(x)$ then identifies the productivity of a firm with rank x. For the main part, we assume $p(.)$ is a strictly increasing function with $p(0) > b$ and denote $p(1) = \overline{p}$. Define $\widehat{\Gamma}_0(x) = \Gamma_0(p(x))$ and $\widehat{\Gamma}_1(.|x) = \Gamma_1(.|p(x))$ which thus describe the above productivity processes but in rank space $x \in [0,1]$. Throughout we require first order stochastic dominance in $\widehat{\Gamma}_1(.|x)$, so that higher productivity firms x are more likely to remain more productive into the future. Let $[0, \overline{x}(x)]$ denote the support of $\widehat{\Gamma}_1(.|x)$ which we assume is connected and that $\lim_{x \to 0^+} \overline{x}(x) = 0$ so that productivity rank $x = 0$ is an absorbing state [till firm death].

Each firm is characterized by (x, n, s) where x summarizes its productivity rank (with corresponding productivity $p = p(x)$), n is the (integer) number of employees and s represents the aggregate market state. Throughout we only consider Markov Perfect (Bayesian) equilibria where the market state process s_t is Markov and known to all agents. As all agents are small, each takes this process as given. Below we shall establish that the payoff relevant state s_t at date t is the distribution function $N_t(.)$ describing the total number of workers employed at firms with rank no greater than x. In equilibrium $N_t(.)$ evolves according to a simple first order differential equation.

There is asymmetric information at the firm level: each firm knows its productivity type x but its employees do not. Given the history of observed wages at this firm, each employee generates beliefs on the firm's current type x and so computes $W(.)$ denoting the expected value of employment at this firm.

New firms enter with a single worker, the innovator. Once a new firm enters, the innovator sells the firm to risk neutral investors for its value and reverts to his/her role as a worker. Each firm faces costs of expanding its labour force. If a firm with n employees decides to recruit an additional worker at rate H, then the cost of recruitment is $nc(H/n)$ where H/n is the recruitment effort required per employee in vetting job applicants and training new hires. Assume $c(.)$ is increasing and strictly convex with $c'(0) = c(0) = 0$.

Recruitment is random in that any hire is a random draw from the set of workers with expected lifetime value less than W where W denotes the expected lifetime payoff of a worker at the hiring firm. This also implies workers quit a firm if they receive an outside offer with (perceived) value strictly greater than current W. We let $\lambda(s)$ denote the arrival rate of (outside) job offers in aggregate state s and $\lambda(s)F(W,s)$ denote the arrival rate of such offers with value no greater than W. Finally at rate μ each worker, whether employed or unemployed, conceives a new business idea and so has the opportunity to start-up a new firm. We assume the worker always chooses to accept the opportunity and so μ describes the entry rate of new firms.[1]

2.1 Firm Size Invariance.

Firms in this paper signal their productivity x through their choice of wage w. In BM, more productive firms pay higher wages to attract and to retain more employees than do less productive firms. The same insight applies here: higher productivity firms have a greater willingness to pay a higher wage to reduce employee quit rates. In the following we identify a separating equilibrium in which each firm (x, n, s) uses an optimal wage strategy $w = w(x, n, s)$ which is strictly increasing in x. Assuming workers observe the number of employees at the firm n and the market state s, then the current wage paid fully reveals the firm's type x. In what follows, however, we shall focus entirely on optimal strategies that are also firm size invariant. Such an equilibrium has the following critical properties: (i) the firm's optimal wage strategy does not depend on firm size, and so (ii) optimal worker quit strategies do not depend on firm size.

The restriction to firm size invariance is most useful. Of course it may be that a firm size invariant equilibrium does not exist (e.g. BM, Coles (2001),

[1] This restriction is made for simplicity. Were it not so, then the entry decision is endogenous to the process under study. Adding this complication is both realistic and worth pursuing but goes beyond the scope of this paper.

Moscarini and Postel-Vinay (2010)). The critical difference here is that firms have an additional policy choice - to recruit new employees with effort H. As developed in Coles and Mortensen (2011) - though in a world of symmetric information and reputation effects - equilibrium finds the wage strategies are indeed firm size independent, depending only on the firm's productivity x. For ease of exposition we simply anticipate this result.

3 A Separating Equilibrium.

The following identifies a separating equilibrium in which $w(x, s)$ describes the optimal wage strategy of firm (x, n, s) which is independent of firm size n and is strictly increasing in x. In any such equilibrium, let $\widehat{x}(w, s)$ denote the worker's belief on the firm's type x given wage announcement w in aggregate state s. Of course a separating equilibrium requires $\widehat{x}$ solves $w = w(\widehat{x}, s)$. Let $W(x, s)$ denote the worker's expected value of employment at firm (x, n, s), given belief $\widehat{x} = x$.

We start with some standard observations. First note that if a firm pays wage $w = b$, it is not optimal for its employees to quit into unemployment - by remaining employed each worker retains the option of remaining employed at his/her current employer which has positive value (the firm may possibly increase its wage tomorrow while the worker can always quit tomorrow if needs be). Assuming workers do not quit into unemployment if indifferent to doing so yields two key simplifications:

(S1) any firm with $n \geq 1$ must make strictly positive profit (as $p(x) > b$ and the firm can always post wage $w = b$);

(S2) any equilibrium wage announcement $w(x, s)$ by firm (x, n, s) must yield employment value $W(\widehat{x}(w, s), s)$ at least as large as the value of unemployment, denoted as $V_u(s)$.[2] Thus all unemployed workers will accept the first job offer received.

As previously described, outside job offers arrive at rate $\lambda = \lambda(s)$ where $F(W, s)$ is the fraction of job offers in state s which offer employment value no greater than W. With no recall, the employee's optimal quit rate at a firm (believed to be) $\widehat{x}$ is then $q(\widehat{x}, s) = \lambda(s)[1 - F(W(\widehat{x}, s), s)]$ which does not depend on firm size. Given this quit structure, consider now optimal firm behavior.

[2] $W < V_u$ generates zero profit as all employees quit into unemployment, and this strategy is then dominated by posting $w = b$.

3.1 Firm Optimality.

Because individual workers are hired and quit sequentially, the number of employees in a continuing firm is a stochastic process. Indeed, the size of a firm, denoted by n, is a birth-death process with an absorbing state that occurs when the firm dies. That is over any sufficiently short time period of length $dt > 0$, the firm's labor force size is an integer that can only transit from the value n to $n+1$ if a worker is hired, from n to $n-1$ if a worker quits, or to zero if the firm loses its market. The transition rates for these three events are respectively the hire frequency $H(x,n,s)$, the quit frequency $nq(\widehat{x},s)$ and the destruction frequency δ.

Suppose firm (x,n,s) posts wage w, recruits new employees at rate $h = H/n$ and employees infer the firm is type $x = \widehat{x}(w,s)$. Firm (x,n,s) thus chooses w,h to solve the Bellman equation:

$$(r+\delta)\Pi(x,n,s) = \max_{w,h\geq 0} \left\langle \begin{array}{c} n[p(x)-w] - nc(h) \\ +nh\left[\Pi(x,n+1,s) - \Pi(x,n,s)\right] \\ +n[\mu + q(\widehat{x}(w,s),s)]\left[\Pi(x,n-1,s) - \Pi(x,n,s)\right] \\ +\gamma\int_0^1 \left[\Pi(z,n,s) - \Pi(x,n,s)\right] d\widehat{\Gamma_1}(z|x) + \frac{\partial \Pi}{\partial t} \end{array} \right\rangle$$

where

$$q(\widehat{x},s) = \lambda(s)[1 - F(W(\widehat{x},s),s)].$$

In words the flow value of the firm equals its flow profit less hiring costs plus the capital gains associated with (i) a successful hire ($n \to n+1$) (ii) the loss of an employee through a quit ($n \to n-1$), and (iii) a firm specific productivity shock with new draw $z \sim \widehat{\Gamma_1}(.|x)$. The last term captures the effect on $\Pi(.)$ through the non-steady state evolution of $s = s_t$. As the quit rate $q(.)$ is firm size invariant, it is immediate the solution to this Bellman equation is $\Pi(x,n,s) = nv(x,s)$ where $v(x,s)$, the value of each employee in firm x, solves:

$$(r+\delta+\gamma+\mu)v(x,s) = \max_{w,h\geq 0} \left\langle \begin{array}{c} p(x) - w - q(\widehat{x}(w,s),s)v(x,s) + hv(x,s) - c(h) \\ +\gamma\int_0^1 v(z,s)d\widehat{\Gamma_1}(z|x) + \frac{\partial v}{\partial t} \end{array} \right\rangle. \tag{1}$$

The following tranversality condition is also necessary for a solution to this dynamic programming problem:

$$\lim_{t\to\infty} e^{-rt}v(x,s_t) = 0 \tag{2}$$

3.2 Worker Optimality.

Consider firm (x, n, s) which adopts the equilibrium wage strategy $w = w(x, s)$. As an employee correctly infers firm type $x = \widehat{x}(w, s)$ then, in a separating equilibrium, the worker's expected lifetime payoff given employment at firm x is:

$$\begin{aligned} rW(x,s) &= w(x,s) + \delta[V_u(s) - W(x,s)] \\ &+\gamma \int_0^1 [W(z,s) - W(x,s)]\, d\widehat{\Gamma_1}(z|x) \\ &+\lambda(s) \int_x^1 [W(z,s) - W(x,s)] d\widehat{F}(z,s) \\ &+\mu \int_0^1 [v(z,s) + W(z,s) - W(x,s)]\, d\widehat{\Gamma_0}(z) + \frac{\partial W}{\partial t}. \end{aligned} \tag{3}$$

In other words, the flow value of employment is equal to the wage income plus the expected capital gains associated with the possibility of firm destruction, a firm specific productivity shock, being offered a better job elsewhere, creating a business start-up and capital gains as the state variable s evolves outside of steady state. Note this payoff does not depend on the quit strategies of colleagues as the wage paid does not depend on firm size.

Given that $\widehat{\Gamma}_1(.|x)$ is stochastically increasing in x and that a separating equilibrium requires $w(x, s)$ is strictly increasing in x, it follows that the expected value of employment at firm x, $W(x, s)$, is strictly increasing in x. Proposition 1 now establishes a standard result.

Proposition 1. In a separating equilibrium, $W(0, s_t) = V_u(s_t)$ for almost all t.

Proof: Strictly positive profit for firm $x = 0$ implies $W(0, s) \geq V_u(s)$ for all s. To establish the equality holds, we use a contradiction argument: Suppose instead $V_u(s_t) < W(0, s_t)$ over some non-empty time period $t \in [t_0, t_1)$. Thus throughout this time interval, being employed at the least productive firm is strictly preferred to being unemployed. Suppose at any date $t \in [t_0, t_1)$, firm $x = 0$ deviates and pays wage $w = w(0, s_t) - \varepsilon$ where $\varepsilon > 0$. Given this deviation, workers update their beliefs on the firm's type $\widehat{x}$ and choose a correspondingly optimal quit strategy. The worst case scenario, however, is that they believe the firm is type $\widehat{x} = 0$ and so anticipate employment value $W(0, s_\tau) > V_u(s_\tau)$ for all $\tau \in (t, t_1)$ in the subgame. As this deviating wage is expected to be paid only for an instant it has an arbitrarily small impact on worker payoffs and so employees at this firm do not quit into unemployment, though each will quit to any outside offer (as $\widehat{x} = 0$ and $w < w(0, s_t)$). This quit strategy, however, is the same turnover strategy were firm $x = 0$ to pay $w = w(0, s_t)$. This contradicts equilibrium as firm $x = 0$ can thus profitably deviate by announcing $w = w(0, s_t)) - \varepsilon$ while $t \in [t_0, t_1)$. This completes the proof of Proposition 1.

An immediate corollary to Proposition 1 is that a separating equilibrium implies

$$w(0, s) = b. \tag{4}$$

This follows as, given all job offers are acceptable, the value of being unemployed in a separating equilibrium is:

$$\begin{aligned} rV_u(s) &= b + \mu \int_0^1 [v(z,s) + W(z,s) - V_u(s)]\, d\widehat{\Gamma_0}(z) \\ &\quad + \lambda(s) \int_0^1 [W(z,s) - V_u(s)] d\widehat{F}(z,s) + \frac{\partial V_u}{\partial t}. \end{aligned} \tag{5}$$

Putting $x = 0$ in (3), using (5) and noting that productivity state $x = 0$ is absorbing ($\widehat{\Gamma_1}(0|0) = 1$) then yields (4). As a separating equilibrium requires $w(.)$ is strictly increasing in x, $w(0, s) = b$ thus describes the lowest wage paid in the market.

3.3 The Value of an Employee.

Proposition 4 below determines the wage outcome in a separating equilibrium. Its derivation relies on the value of an employee $v(.)$ being increasing in x and bounded for any state s. This section formally establishes this result.

Equation (6) below identifies an upper bound for $v(.)$. Assumption 1 is a restriction on fundamental parameters which ensures this bound exists.

Assumption 1: A positive solution for $\overline{v}$ exists to

$$\overline{v} = \frac{\overline{p} - b + \max_h\{h\overline{v} - c(h)\}}{\delta}. \tag{6}$$

For any $v \geq 0$, define the hire function

$$h^*(v) = \arg\max_{h \geq 0}[hv - c(h)]. \tag{7}$$

The assumed properties of $c(.)$ ensure $h^*(v)$ is unique, non-negative, strictly increasing and differentiable for all $v \geq 0$. By establishing that the highest productivity firms do not grow too quickly, Proposition 2 ensures the ergodic distribution of firm sizes is well-defined.

Proposition 2. $h^*(\overline{v}) \leq \delta$.

Proof. By the Envelope Theorem, the right hand side of equation (6) is an increasing, convex function of $\overline{v}$ with slope $h^*(\overline{v})/\delta$. As the right hand side is also strictly positive at $\overline{v} = 0$ then, given a positive solution exists for $\overline{v}$, it satisfies $h^*(\overline{v}) \leq \delta$. ■

The Bellman equation (1) implies the optimal recruitment strategy of firm (x, s) is

$$h(x, s) = h^*(v(x, s)). \tag{8}$$

Using Assumption 1, we now obtain the following crucial result.

Proposition 3. The value of an employee $v(x, s)$ is increasing in x and bounded above by $\overline{v}$ in every state s.

Proof. The forward solution to (1) that satisfies the transversality condition (2) along any arbitrary future time path for the state $\{s_t\}_0^\infty$ is the fixed point of the following transformation

$$\begin{aligned}(Tv)(x, s_0) &= \int_0^\infty \max_{w_t, h_t \geq 0} \left\langle p(x) - w_t + h_t v(x, s_t) - c(h_t) + \gamma \int_0^1 v(z, s_t) d\widehat{\Gamma_1}(z|x) \right\rangle \\ &\quad \times \exp\left(- \int_0^t (r + \delta + \gamma + \mu + q(\widehat{x}(w_z, s_z), s_z)) dz \right) dt.\end{aligned}$$

As $q(\widehat{x}(w_z, s_z), s_z)) \geq 0$ in general and $w_t \geq b$ by Proposition 1, it follows that

$$\begin{aligned}(Tv)(x, s_0) &\leq \int_0^\infty \max_{h_t \geq 0} \langle p(x) - b + h_t \overline{v} - c(h_t) + \gamma \overline{v} \rangle \, e^{-(r+\delta+\gamma+\mu)t} dt \\ &\leq \frac{\max_{h \geq 0} \langle \overline{p} - b + h\overline{v} - c(h) + \gamma \overline{v} \rangle}{r + \delta + \gamma + \mu} \\ &= \frac{\delta + \gamma}{r + \delta + \gamma + \mu} \overline{v} < \overline{v}\end{aligned}$$

for any $v(x, s) \leq \overline{v}$. Because $p(x)$ is increasing in x and $\widehat{\Gamma_1}(.|x)$ is stochastically increasing in x, $(Tv)(x, s_t)$ is increasing in x if $v(x, s)$ is increasing in x. Thus the transformation T maps the set of uniformly bounded functions that are increasing in x into itself. Further, the transformation T is increasing and

$$\begin{aligned}
T(v(x,s_0)+k) &= v(x,s_0)+|k|\int_0^\infty (h^*(v)+\gamma) \\
&\quad \times \exp\left(-\int_0^t (r+\delta+\gamma+\mu+q(\widehat{x}(w_z,s_z),s_z))dz\right) dt \\
&\leq v(x,s_0)+|k|\int_0^\infty (h^*(\overline{v})+\gamma)e^{-(r+\delta+\gamma+\mu)t}dt \\
&\leq v(x,s_0)+\frac{\delta+\gamma}{r+\delta+\gamma+\mu}|k| \text{ for all } s_0
\end{aligned}$$

because $q(\widehat{x}(w_z,s_z),s_z)) \geq 0$ and $h^*(\overline{v}) \geq h^*(v(x,s))$ for any $v(x,s) \leq \overline{v}$. In short, the map satisfies Blackwell's condition for a contraction map which thus guarantees that a unique fixed point exists in the set of bounded functions increasing in x. This completes the proof of Proposition 3.

Armed with this result we can now fully characterise the strategies of firms and workers in a separating equilibrium.

3.4 Equilibrium Wage and Quit Strategies.

The Bellman equation (1) implies the optimal wage strategy minimizes the sum of the wage bill and turnover costs. Formally,

$$w(x,s) = \arg\min_w [w + q(\widehat{x},s)v(x,s)] \tag{9}$$

where $\widehat{x} = \widehat{x}(w,s)$. Characterizing the solution to (9) requires first characterising the equilibrium quit rate function $q(.)$.

Define $\widehat{F}(x,s)$ as the fraction of job offers made by firms with type no greater than x in aggregate state s. As a separating equilibrium implies $W = W(x,s)$ is strictly increasing in x, it follows that $\widehat{F}(x,s) = F(W(x,s),s)$. By now determining $\lambda(s)$ and $\widehat{F}(x,s)$, the equilibrium quit rate function is given by $q(x,s) = \lambda(s)[1-\widehat{F}(x,s)]$ where $x = \widehat{x}$ describes the worker's (degenerate) belief on the firm's type.

In state $s = s_t$ at date t, let $G_t(W)$ denote the total number of workers in the economy with value no greater than W. As job offers are random then, to hire at rate $H = nh$ while offering a wage which yields expected employment value W, the firm must make job offers at rate $H/G_t(W)$ (as an offer is only accepted with probability G_t). But $W(.)$ strictly increasing in x implies

$G_t(W(x,s)) = U_t + N_t(x) \equiv \widehat{G}_t(x)$, where recall $N_t(x)$ is the measure of workers employed at firms of productivity rank x or less and $U_t = 1 - N_t(1)$ is the measure of workers who are unemployed. Thus a firm (x, n, s_t) which recruits at optimal rate $h(x, s_t)$ makes job offers at rate $nh(x, s_t)/\widehat{G}_t(x)$. Given there is a unit mass of workers and letting $n_t(x)dx = d\widehat{G}_t(x)$ denote the employment density over productivity rank at date t, aggregating job offer rates across all firms implies the arrival rate of a job offer to any given worker is

$$\lambda(s_t) = \int_0^1 \frac{n_t(z)h(z,s_t)}{\widehat{G}_t(z)}dz = \int_0^1 \frac{h(z,s_t)d\widehat{G}_t(z)}{\widehat{G}_t(z)} = \int_0^1 \frac{h(z,s_t)dN_t(z)}{U+N_t(z)}. \quad (10)$$

Furthermore the arrival rate of offers from firms with type greater than x is

$$\lambda(s_t)[1 - \widehat{F}(x,s_t)] = \int_x^1 \frac{n_t(z)h(z,s_t)}{\widehat{G}_t(z)}dz = \int_x^1 \frac{h(z,s_t)dN_t(z)}{U+N_t(z)} \quad (11)$$

Hence a worker who believes he/she is employed at a firm with productivity $\widehat{x}$ has quit rate

$$q(\widehat{x},s_t) = \int_{\widehat{x}}^1 \frac{h(z,s_t)dN_t(z)}{U+N_t(z)}. \quad (12)$$

We are now in a position to describe the equilibrium wage strategy of firm (x,n,s). Using (12) in equation (9), the optimal wage strategy solves:

$$w(x,s) = \arg\min_w \left[w + v(x,s)\int_{\widehat{x}(w,s)}^1 \frac{h(z,s)dN(z)}{U+N(z)}\right] \quad (13)$$

where $N(.) \equiv N_t(.)$ in state $s = s_t$. Consider $x \in (0,1)$ and, for ease of exposition, assume $\widehat{x}$ is differentiable. The necessary first order condition for optimality is:

$$1 - v(x,s)\frac{h(\widehat{x},s)N'(\widehat{x})}{U+N(\widehat{x})}\frac{\partial \widehat{x}}{\partial w} = 0. \quad (14)$$

By marginally increasing the wage w, the firm marginally increases its employees' beliefs $\widehat{x}$ about its type, which marginally reduces their quit rates. As $v(x,s)$ describes the retention value of each employee, optimality ensures the marginal return to the lower quit rate equals the cost to paying each employee a marginally higher wage. We now identify the equilibrium wage function.

Proposition 4. For given s, a separating equilibrium implies the wage strategy $w(.)$ is the solution to the differential equation:

$$\frac{\partial w}{\partial x} = \frac{v(x,s)h(x,s)N'(x)}{U + N(x)} \text{ for all } x \in [0,1] \tag{15}$$

with initial value $w(0,s) = b$.

Proof: A separating equilibrium requires that the optimal wage w solving the first order condition (14) must yield a wage function $w = w(x,s)$ whose inverse function corresponds to $\widehat{x}(w,s) = x$. Using these restrictions in (14) establishes (15).

To show the solution to the necessary condition for optimal $w(.)$ describes a maximum for each firm (x,s), we have to verify the second order condition holds. Thus consider firm x which instead announces wage $w' = w(x',s)$ where $x' \in (x,1]$. As w' satisfies (15) and $v(x',s) > v(x,s)$ by Proposition 3, the marginal cost to announcing wage $w' > w$ for firm x is

$$\begin{aligned}
\frac{\partial}{\partial w}(w + q(\widehat{x},s)v(x,s))|_{w=w'} &= 1 - v(x,s)\frac{h(\widehat{x}',s),s)\widehat{G}_t'(\widehat{x}')}{\widehat{G}_t(\widehat{x}')}\frac{\partial \widehat{x}'}{\partial w} \\
&= 1 - \frac{v(x,s)}{v(x',s)} > 0.
\end{aligned}$$

Hence for any $x' \in (x,1]$, announcing wage $w' > w(x,s)$ increases the total cost of labor to firm x. The same argument establishes that for any $x' \in [0,x)$, the marginal cost to announcing wage $w' = w(x',s) < w$ for firm x is always negative. Thus announcing wage $w = w(x,s)$ is more profitable than announcing any other wage $w' = w(x',s)$ for $x' \in [0,1]$.

Suppose instead the firm announces wage $w < w(0,s) = b$. To ensure this is not a profitable deviation, assume its employees believe $\widehat{x} = 0$ when $w < b$. As they anticipate wage $w = b$ at this firm in the entire future [$x = 0$ is an absorbing state] they quit into unemployment. As this outcome yields zero profit, no firm announces wage $w < b$.

Finally suppose the firm announces $w > w(1,s)$. In that case assume its employees believe $\widehat{x} = 1$ and, given those beliefs and resulting quit turnover, announcing wage $w(1,s)$ then strictly dominates paying the higher deviating wage. Hence the optimal wage announcement of any firm $x \in [0,1]$ is identified as the solution to the differential equation (15) with initial value $w(0,s) = b$. This completes the proof of Proposition 4.

The economic intuition underlying the result is simply that higher productivity firms enjoy higher employee values $v(.)$ and so are willing to pay marginally more for a reduced quit rate. Equilibrium has an auction structure where for each type x, a too low wage bid yields a costly higher quit rate, while a higher wage bid is not economic as the reduction in quit rate is too small.

3.5 Formal Definition of a Separating Equilibrium.

Fix a rank $x \in [0,1]$ and consider the number $N_t(x)$ of employed workers in firms with type no greater than x. Equilibrium turnover implies $N_t(.)$ evolves according to:

$$\begin{aligned} \dot{N}_t(x) &= \lambda(s_t)\widehat{F}(x,s_t)U_t + \mu U_t \widehat{\Gamma_0}(x) + \gamma \int_x^1 \widehat{\Gamma_1}(x|z)dN_t(z) \\ &\quad - \left(\delta + \lambda(s_t)[1-\widehat{F}(x,s_t)] + \mu[1-\widehat{\Gamma_0}(x)]\right) N_t(x) - \gamma \int_0^x \left[1-\widehat{\Gamma_1}(x|z)\right] dN_t(z) \\ &= \left(\int_0^x \frac{h(z,s_t)dN_t(z)}{U_t+N_t(z)} + \mu\widehat{\Gamma_0}(x)\right) U_t + \gamma \int_0^1 \widehat{\Gamma_1}(x|z)dN_t(z) \\ &\quad - \left(\delta + \int_x^1 \frac{h(z,s_t)dN_t(z)}{U_t+N_t(z)} + \mu[1-\widehat{\Gamma_0}(x)]\right) N_t(x) - \gamma N_t(x) \end{aligned} \tag{16}$$

by (11) where the dot refers to the time derivative $\partial N_t/\partial t$ and unemployment $U_t = 1 - N_t(1)$. The inflow includes those unemployed who become employed at a firm no greater than x either because they are unemployed and find a job with such a firm or start-up such a new firm, plus those employed at firms with $z \geq x$ but which are hit by an adverse shock $x' \leq x$. The outflow includes job destruction due to firm death, quits to start new firms, and worker departures to more productive firms plus the employment of the firm flow that experience a sufficiently favorable productivity shock. We now formally define a separating equilibrium where $s_t = N_t(.)$ is the aggregate state variable.

Definition: Given state $s = N(.)$, a *separating equilibrium* is a wage policy function, hire rate policy, and equilibrium quit rate such that
(i) $w(x,N(.)) = b + \int_0^x \frac{v(z,N(.))h(z,N(.))dN(z)}{U+N(z)}$;
(ii) $h(x,N(.)) = h^*(v(x,N(.)))$;

(iii) $q(x, N(.)) = \int_x^1 \frac{h(z,N(.))dN(z)}{U+N(z)}$.

Along the equilibrium path, $v_t(x) \equiv v(x; N_t(.))$ and $N_t(.)$ are solutions to the system of ordinary differential equations composed of equation (16) together with

$$(r+\delta+\gamma+\mu)v_t(x) - \dot{v}_t(x) = \left\langle \begin{array}{c} p(x) - w(x, N(.)) - c(h^*(v_t(x))) \\ + \left[h^*(v_t(x)) - q(x, N_t(.))\right] v_t(x) + \gamma \int_0^1 v_t(z) d\widehat{\Gamma_1}(z|x) \end{array} \right\rangle. \tag{17}$$

Furthermore an equilibrium solution is consistent with the initial distribution of employment $N(.)$ and the transversality condition

$$\lim_{t\to\infty} v_t(x)e^{-rt} = 0 \ \forall x \in [0, 1].$$

4 Homogenous Firms.

Although it is true that the market state $N_t(.)$ is of infinite dimension in the general case, it need not be so in practice. In this section we fully characterize the unique separating equilibrium in the limiting case of homogenous firms.

In the homogenous firm case, we suppose $p(x)$ is (arbitrarily close to) $\overline{p}$ for all x. With (limiting) equal productivity, incentive compatibility implies $v(x; N)$ cannot depend on x. Let $v_t = v(x; N_t)$ denote the value of an employee in each firm in the limiting case. Optimal recruitment effort $h_t = h^*(v_t)$ is thus also independent of x.

Putting $x = 0$ in (17) implies:

$$(r + \delta + \mu + \lambda_t)v_t - \dot{v}_t = \left\langle \overline{p} - b + \max_h \{hv_t - c(h)\} \right\rangle$$

As the definition of equilibrium further implies job offer arrival rate:

$$\lambda_t = \int_0^1 \frac{h(z, N_t(.))dN(z)}{U + N(z)} = -h^*(v_t) \ln U_t, \tag{18}$$

this differential equation for v_t reduces to:

$$\dot{v}_t = (r + \delta + \mu - h^*(v_t) \ln U_t)v_t - \left(\overline{p} - b + \max_{h\geq 0} \left[hv_t - c(h)\right]\right) \tag{19}$$

which depends only on v_t and the unemployment rate U_t. The equilibrium unemployment dynamics are

$$\dot{U}_t = \delta(1 - U_t) - \left[\mu - h^*(v_t) \ln U_t\right] U_t. \tag{20}$$

where the first term describes the inflow [job loss through firm destruction] and the second is outflow through either firm creation or job creation. Note then that employee value v_t and unemployment U_t evolve according to the pair of autonomous differential equations (19) and (20).

Thus for the limiting case of homogenous firms, we can restrict the aggregate state vector to $s_t = U_t$ which is a scalar. The solution of interest, $v = v(U)$, solves the differential equation

$$\frac{dv}{dU} = \frac{\dot{v}}{\dot{U}} = \frac{(r + \delta + \mu - h^*(v) \ln U)\, v - (\overline{p} - b + \max_{h \geq 0} \{hv - c(h)\})}{\delta - [\delta + \mu - h(v) \ln U]\, U}.$$

It is well known that a unique continuous solution exists to this equation for all $U \in [0, 1]$ if and only if the ODE system composed of (19) and (20) has a unique steady state solution and the steady state is a saddle point. Indeed, the branch of the saddle path that converges to the steady state for every initial value of aggregate unemployment describes the equilibrium value of $v(.)$. Below we prove that these necessary and sufficient conditions hold.

Any steady state solution is the (U, v) pair defined by the pair of equations

$$\delta - (\mu + \delta)U = -h^*(v)U \ln U \tag{21}$$

$$(r + \delta + \mu - h^*(v) \ln U)\, v = \overline{p} - b + \max_{h \geq 0} \{hv - c(h)\} \tag{22}$$

We first show there exists a single solution pair (v, U) to these equations.

Equation (21) describes the $\dot{U} = 0$ locus drawn in Figure 1 below. The LHS of (21) is zero at $U = \frac{\delta}{\delta+\mu} < 1$ and decreases at the constant rate $\delta + \mu$. For any $v > 0$, the RHS is positive and strictly concave in U for $U \in (0, 1)$. Hence a unique, positive value of U strictly less than $\delta/(\mu + \delta)$ exists for every positive value of v. As $h^*(.)$ is an increasing function, it follows that U decreases as v increases along the locus with limiting properties $U \to \delta/(\mu+\delta)$ as $v \to 0$ and $U \to 0$ as $v \to \infty$.

Equation (22) describes the $\dot{v} = 0$ locus in Figure 1. The RHS does not depend on U, is strictly positive at $v = 0$ and, for $v \in [0, \overline{v}]$, the Envelope Theorem implies it is a strictly increasing function of v with slope $h^*(v) < \delta$ [Proposition 2]. The LHS is instead zero at $v = 0$ and is a strictly increasing function of v with slope strictly greater than $r + \mu + \delta$. Thus if a solution exists to equation (22) it must be unique. Note further that at $U = 1$, the unique solution for v satisfies $v = v_1 < \overline{v}$. As the LHS is decreasing in U, it

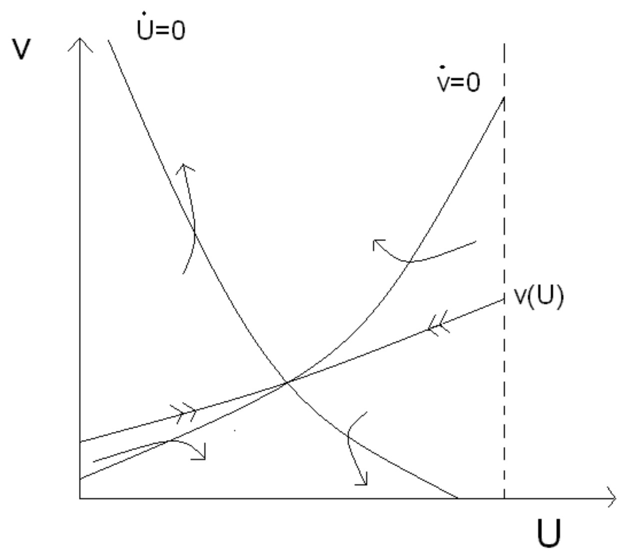

Figure 1: Phase Diagram (v,U)

follows that a solution for $v \in [0, \overline{v}]$ exists for all $U \in [0, 1]$ where v increases as U increases with limiting properties $v \to 0$ as $U \to 0$ and $v = v_1 < \overline{v}$ at $U = 1$. Continuity now implies a unique steady state solution for the pair (v, U) exists and steady state $U \in [0, \delta/(\mu + \delta)]$.

The dynamics implied by the ODE system composed of (19) and (20) are illustrated by its phase diagram portrayed in Figure 1. The intersection of the two singular curves is a saddle point that attracts a unique converging saddle path from any initial value of U. Finally, because the growth rate in v on the unstable path above the saddle path must eventually exceed the rate of interest, while the unstable path below the steady state ultimately yields zero v (which contradicts optimal firm behavior), the stable path represents the only separating equilibrium. This argument thus establishes Theorem 1.

Theorem 1 *A unique separating equilibrium exists in the limiting case of equally productive firms. Further the equilibrium value of an employee $v(U)$ increases with unemployment.*

Equilibrium behaviors depend on the interaction between the value of an employee (which stimulates greater recruitment effort by firms) and the arrival rate of outside offers λ_t. Note at the steady state the value of an employee is given by

$$v = \frac{\overline{p} - b + \max_h\{hv - c(h)\}}{r + \delta + \mu + \lambda} \tag{23}$$

which depends on the arrival rate of outside offers λ (the only endogenous object). (23) determines steady state $v = v^*(\lambda)$ where the higher the arrival rate of outside offers, the lower the value of an employee $v^*(\lambda)$. This quit propensity in turn depends on the recruitment effort of competing firms as

$$\lambda = -h^*(v(U)) \ln U. \tag{24}$$

At steady state U, given by equation (21), it is possible to show λ implied by (24) is an increasing function of v: the higher the value of an employee, the greater the recruitment rate of competing firms and thus the higher arrival rate of outside offers. This interaction between the value of an employee and competing firm recruitment strategies ensure a unique steady state.

The non-steady state dynamics are interesting. Suppose there is a one-off employment shake-out which increases unemployment above its steady state level. Theorem 1 implies the value of an employee $v = v(U)$ increases which, in turn, increases firm recruitment rates $h = h^*(v(U))$. At first sight this seems empirically unlikely - that hiring rates are counter-cyclical (increasing with unemployment). It should be noted, however, that this response is necessary for the stability of the economy: if recruitment rates were to fall as unemployment increases, then unemployment would continue to increase. It is particularly interesting, then, that Yashiv (2011) finds empirically that the hiring rate (H/N) in the U.S. is indeed countercyclical in this sense. The model's corresponding implication for the cyclicality of gross hiring flows $H = h^*(v(U))[1 - U]$ is, however, ambiguous.

Note that any common and unanticipated positive shock to the productivity of a match $\overline{p}$ shifts up the $\dot{v} = 0$ curve in Figure 1. The result is an increase in the steady state value of an employee (v) and a decrease in unemployment (U) as in the canonical search and matching model. Along the adjustment path, the equilibrium value of v jumps up initially and adjusts slowly downward along the path converging to the new steady state value. This implies quit turnover also jumps up to a favorable aggregate productivity shock: firms increase their recruitment effort and workers in low rank firms are more likely to receive a preferred outside offer. The initially large increase in job-to-job turnover gradually falls, however, as the economy converges to the new steady state.

It is straightforward to back out equilibrium micro-behavior. The differential equation (15) for equilibrium wages simplifies to

$$\frac{\partial w(x, N_t(.))}{\partial x} = h^*(v_t)v_t \frac{N_t'(x)}{U + N_t(x)},$$

which, given initial value $w(0, N_t(.)) = b$, yields

$$w(x, N_t(.)) = b + h^*(v_t)v_t \ln\left(\frac{U_t + N_t(x)}{U_t}\right)$$

where $v_t = v(U_t)$. This expression describes equilibrium wage dispersion in the limiting case of homogenous firms. Specifically, $w(.)$ is increasing in x, where $w(0; s) = b$ is the lowest wage paid. Wage dispersion arises as hiring is costly and firms offer different wages to reduce their employee quit rates. As in BM, the wages offered are ranked by productivity x where higher ranked firms pay higher wages and enjoy lower quit rates. Unlike BM, however, there is no simple correlation between wages and firm size.

The equilibrium quit rate from firm $(x, N_t(.))$ is

$$q(x, N_t(.)) = -h^*(v_t) \ln[U_t + N_t(x)] \tag{25}$$

which is decreasing in x, being $-h^*(v(U_t)) \ln U_t$ at $x = 0$ (the bottom rank firm) and zero at $x = 1$. Note a firm's equilibrium quit rate depends directly on the level of unemployment. This occurs as firms are more likely to recruit from the pool of unemployed workers the larger is that pool.

The expected growth rate of employment depends only on whether or not unemployment U exceeds its steady state value. There is, however, dispersion in individual firm growth rates: a rank x firm enjoys expected growth rate $h^*(v_t)\left[1 + \ln[U + N_t(x)]\right]$. Consistent with Gibrat's law, a firm's growth rate is independent of its size n but depends critically on its productivity rank x (which is subject to shocks) and the level of unemployment. High productivity (rank) firms pay high wages and attract workers both from the unemployment pool and from low wage firms. Such firms grow over time, while low rank firms contract. Firm size $n(x, t)$ thus evolves according to a geometric Markov process where firms with x satisfying $U + N(x) > 1/e \simeq 0.37$ have positive expected growth rates. Thus if unemployment exceeds 37% this condition implies all existing firms have positive expected growth rates. Finally note that currently large firms must typically have existed for a longer time, have enjoyed higher than average growth rates, and, consequently, have been more productive.

5 Heterogeneous Firms.

This section generalizes the analysis to a finite number of firm types. Let p_i represent the productivity of firms of type $i = 1, .., I$; i.e. $p(x) = p_i$ for all $x \in (x_{i-1}, x_i] \subseteq [0, 1]$ where the set $(x_{i-1}, x_i]$ represents the firms of type i and $x_0 = 0$, $x_I = 1$. As the value of an employee is the same for all firms of the same type, let $v_i(N(.)) = v(x_i; N(.))$ for $x \in (x_{i-1}, x_i]$, $i = 1, 2, ...I$, denote the value of an employee in type i firms in aggregate state $N(.)$. $\mathbf{v} = (v_1, v_2, ..., v_I)$ denotes the corresponding vector of employee values. Let $N_i = N(x_i)$ denote the number of workers employed in firms of type i or less and $\mathbf{N} = (N_1, N_2, ..., N_I)$ denotes the corresponding vector. Note unemployment $U = 1 - N_I$. Let $w_i = w(x_i; N(.))$ denote the wage paid by firm $x = x_i$. Conditional on firm type j receiving a productivity shock, let θ_{jk} denote the probability its type becomes k. Assume the θ_{jk} are consistent with first order stochastic dominance and $\theta_{11} = 1$ [the lowest productivity state is an absorbing state (till firm death)].

Proposition 5. A separating equilibrium implies w_i are defined recursively by

$$w_i = w_{i-1} + v_i h^*(v_i) \ln \frac{1 - N_I + N_i}{1 - N_I + N_{i-1}}$$

with $w_0 = b$. The value of a type i firm solves:

$$\dot{v}_i = (r+\delta+\gamma+\mu)v_i - \left\langle \begin{array}{c} p_i - b + \max_{h \geq 0} \{hv_i - c(h\} + \gamma \sum_{j=1}^{I} \theta_{ij} v_j \\ - \sum_{j=1}^{i} v_j h^*(v_j) \ln \frac{1-N_I+N_j}{1-N_I+N_{j-1}} \\ -v_i \sum_{j=i+1}^{I} h^*(v_j) \ln \frac{1-N_I+N_j}{1-N_I+N_{j-1}} \end{array} \right\rangle \quad (26)$$

Proof. In any separating equilibrium, (11) implies

$$\lambda[1 - \widehat{F}(x_i, N(.)] = \int_{x_i}^{1} \frac{h(z, N(.))dN(z)}{1 - N_I + N(z)} = \sum_{j=i+1}^{I} \left[h^*(v_j) \ln \frac{1 - N_I + N_j}{1 - N_I + N_{j-1}} \right]. \quad (27)$$

Now consider type i firms. For $x \in (x_{i-1}, x_i]$ such firms have productivity p_i. As each type i firm has the same value then, to ensure equal profit, the equilibrium wage equation has to satisfy

$$w(x, N(.)) + \lambda[1 - \widehat{F}(x, N(.))]v_i = w_{i-1} + v_i \sum_{j=i}^{I} \left[h^*(v_j) \ln \frac{1 - N_I + N_j}{1 - N_I + N_{j-1}} \right]$$

for all such x. Putting $x = x_i$ and using (27) yields the stated recursion for w_i. As this recursion implies

$$w_i = b + \sum_{j=1}^{i} \left[v_j h^*(v_j) \ln \frac{1 - N_I + N_j}{1 - N_I + N_{j-1}} \right],$$

the differential equation for v_i follows by putting $x = x_i$ in equation (17) in the definition of equilibrium. This completes the proof of Proposition 5.

Using equation (16), it follows the N_i evolve according to:

$$\begin{aligned} \dot{N}_i = & \sum_{j=1}^{i} \left[h^*(v_j) \ln \frac{1 - N_I + N_j}{1 - N_I + N_{j-1}} \right] [1 - N_I] + \mu \Gamma_{0i} + \gamma \sum_{j=1}^{I} \theta_{ij} N_j \\ & - \left(\delta + \mu + \gamma + \sum_{j=i+1}^{I} \left[h^*(v_j) \ln \frac{1 - N_I + N_j}{1 - N_I + N_{j-1}} \right] \right) N_i \end{aligned} \tag{28}$$

where Γ_{0i} is the probability that a new firm is initially of type i or less.

Theorem 2 *With a finite number of firm types, a separating equilibrium exists if initial unemployment is positive; i.e.* $U_0 = 1 - N_{I0} > 0$.

The equilibrium values are represented by a stationary real valued vector function $\mathbf{v}(\mathbf{N}) = (v_1(\mathbf{N}), ..., v_I(\mathbf{N}))$ where $\mathbf{N} = (N_1, ..., N_I)$ which is a particular solution to the differential equation system compose of (26) and (28) consistent with the arbitrary initial distribution of workers over types $\mathbf{N}_0$ and the transversality condition $\lim_{t \to \infty} v_i e^{-rt} = 0$, $i = 1, ..., I$. Define $\mathbf{v}(\mathbf{N})_\Delta$ as the fixed point of the following familiar forward recursion in discrete time

$$(M\mathbf{v})_i(\mathbf{N})_\Delta = \frac{\left\langle \begin{array}{c} p_i - b - \sum_{j=1}^{i} v_j(\mathbf{N}') h^*(v_j(\mathbf{N}')) \ln \frac{1 - N_I' + N_j'}{1 - N_I' + N_{j-1}'} \\ + \max_{h \geq 0} \{ h v_i(\mathbf{N}') - c(h) \} + \gamma \sum_{j=1}^{I} \theta_{ij} v_j(\mathbf{N}') \end{array} \right\rangle \Delta + v_i(\mathbf{N}')}{1 + \left(r + \delta + \gamma + \mu + \sum_{j=i+1}^{I} h^*(v_j(\mathbf{N}')) \ln \frac{1 - N_I' + N_j'}{1 - N_I' + N_{j-1}'} \right) \Delta}$$

where $\Delta > 0$ indexes the length of a "period" and next period $\mathbf{N}'$ is given by

$$\begin{aligned} N_i' &= \sum_{j=0}^{i}\left[h^*(v_j)\Delta\ln\frac{1-N_I+N_j}{1-N_I+N_{j-1}}\right][1-N_I]+\mu\Delta\Gamma_{0i}+\gamma\Delta\sum_{j=1}^{I}\theta_{ij}N_j \\ &\quad+\left[1-\left(\delta+\mu+\gamma+\sum_{j=i+1}^{I}\left[h^*(v_j)\ln\frac{1-N_I+N_j}{1-N_I+N_{j-1}}\right]\right)\Delta\right]N_i \end{aligned}$$

$i = 1, ..., I$. Note that $N_i' < 1$ if $N_I < 1$ which implies that $\mathbf{N}_t < 1$ for all t if $U_0 = 1 - N_{01} < 1$.

As $\lim_{\Delta\to 0}\left[(M\mathbf{v})_i(\mathbf{N})_\Delta - v_i(\mathbf{N})\right]/\Delta = -\dot{v}_i$ and $\lim_{\Delta\to 0}\left[N_i' - N_i\right]/\Delta = \dot{N}_i$, the $\lim_{\Delta\to 0}\mathbf{v}(\mathbf{N})_\Delta = \mathbf{v}(\mathbf{N})$ is an equilibrium vector of value functions. Our strategy is to show that $\mathbf{v}(\mathbf{N})_\Delta$ exists for every small $\Delta > 0$. As we demonstrate that it lies in a compact metric space, every sequence $\{\mathbf{v}(\mathbf{N})_\Delta\}_{\Delta\to 0}$, has a convergent subsequence in the supnorm.

First, we establish that the transform M maps bounded functions into bounded function under Assumption 1 and $p_i > b$. Namely, for any $\mathbf{v}(\mathbf{N}) \leq (\overline{v}, ..., \overline{v})$ where $\overline{v}$ is the scalar defined by equation (6),

$$\begin{aligned} (M\mathbf{v})_i(\mathbf{N}) &\leq \frac{\left[p_i - b + \max_{h\geq 0}\{hv_i(\mathbf{N}') - c(h)\} + \gamma\sum_{j=1}^{I}\theta_{ij}v_j(\mathbf{N}')\right]\Delta + v_i}{1+(r+\delta+\gamma+\mu)\Delta} \\ &\leq \frac{\left[\overline{p} - b + \max_{h\geq 0}\{h\overline{v} - c(h)\} + \gamma\overline{v}\right]\Delta + \overline{v}}{1+(r+\delta+\gamma+\mu)\Delta} \\ &= \frac{(\delta\overline{v}+\gamma\overline{v})\Delta + \overline{v}}{1+(r+\delta+\gamma+\mu)\Delta} < \overline{v} \end{aligned}$$

by substitution from equation (6). Further, as $p_i > p_{i-1}$ one can easily show that $v_i(\mathbf{N}') > v_{i-1}(\mathbf{N}')$ implies $(M\mathbf{v})_i(\mathbf{N}) > (M\mathbf{v})_{i-1}(\mathbf{N})$ as in the proof to Proposition 2. Finally, since $p_1 > b$, $Mv_1(\mathbf{N}) > 0$ if $v_1(\mathbf{N}') \geq 0$. Thus, $Mv(\mathbf{N}) > 0$ for any $v(\mathbf{N}) \geq 0$.

As $h^*(v)$ is a differentiable function with bounded derivatives on $(0, \overline{v}]$, equation (7) and the derivatives of $\ln\frac{1-N_I'+N_j'}{1-N_I'+N_{j-1}'}$ are bounded for all $N_i \leq N_I < 1$, the continuous transformation M maps the set of bounded, positive, differentiable, and Lipschitz continuous function $\mathbf{v}(\mathbf{N})_\Delta$ into itself. As this set is a compact metric space under the supnorm, at least one fixed point with these properties exists by Schauder's Fixed Point Theorem for every $\Delta > 0$.

Finally, consider any infinite sequence $\{\mathbf{v}(\mathbf{N})_\Delta\}$ with $\Delta \to 0$. As every element is a bounded real vector function a subsequence that converges in the supnorm exists and this limit, say $\mathbf{v}(\mathbf{N})$, satisfies all the equilibrium conditions by construction. This comment completes the proof.

6 Wage and Productivity Dispersion

The aim of this section is to derive conditions under which the model generates wage and productivity dispersion which is consistent with matched employer-employee data such as that available for Danish manufacturing. The empirical employment weighted distributions of the average hourly firm wages paid (annual wage bill divided by employment measured in annual standard hours worked) and hourly labor productivity (annual value added per standard hour worked) for four different Danish manufacturing industries are illustrated by the two solid lines in Figure 2.[3] Note that the general shapes of the distributions are quite similar across industries. In all four cases, average firm wage dispersion is characterized by a distribution with single interior mode and some upper tail skew but less than the distributions of labor productivity.[4] Figure 3 presents the cross firm wage-productivity relationship in each of the four industries where the solid line represents the nonparametric regression point estimate and the shaded area is the 90% confidence interval. Obviously, there is a strong positive relationship between the two, as our theory predicts. Further, the profile is roughly linear over most of the mass of the productivity distribution but with diminishing slope that tends to zero in the extreme right tail.[5] In this section we demonstrate that the formal model can provide a coherent explanation for these general features of the data.

We focus on steady state so that unemployment and the distribution of employment across firms are consistent with firm and worker turnover. We also abstract from the idiosyncratic shock to productivity by setting $\gamma = 0$.

[3] The data described in this secition is documented by and the graphs illustrating the data can be found in Bagger, Christensen, and Mortensen (2011).

[4] Bagger et al. (2011) show that the same shapes characterize firm wage distributions in non-manufacturing as well.

[5] Although the point estimates suggest a negative slope near the upper support, there is not enough data in the region to make that inference.

We motivate this restriction by noting that firm productivity is quite persistent and that there is a strong positive correlation between the average wage paid and firm size in firm data. Our model need not generate either correlation if γ is very large. Specifically as all start-up firms are initially small, any currently large firm must have enjoyed high growth rates in the past. If γ were large so that firm productivity is not very persistent, then the predicted correlation between current wages paid and firm size is correspondingly small. Conversely, if γ is sufficiently small, then large firms remain highly productive for long period, thus yielding the observed positive correlation between firm size and wage paid. In steady state with $\gamma = 0$, (16) implies $N(x)$ satisfies:

$$\lambda\widehat{F}(x)U + \mu U\widehat{\Gamma_0}(x) = \left(\delta + \lambda[1 - \widehat{F}(x)] + \mu[1 - \widehat{\Gamma_0}(x)]\right) N(x) \tag{29}$$

where, by (11), the quit rate is

$$\lambda[1 - \widehat{F}(x)] = \int_x^1 \frac{h(z)dN(z)}{U + N(z)}. \tag{30}$$

The Bellman equation (1) and the Envelope Theorem imply

$$v'(x) = \frac{p'(x)}{r + \delta + \mu - h^*(v(x)) + \lambda[1 - \widehat{F}(x)]}, \tag{31}$$

while the wage equation solves

$$w'(x) = h^*(v(x))v(x)\frac{N'(x)}{U + N(x)}. \tag{32}$$

The aim is to determine whether these restrictions are consistent with he empirical observations summarized in Figures 2 and 3

Figure 3 describes the empirical wage-firm productivity relationship $\widetilde{w}(p) = w(x)$ where $x = \Phi(p)$. The slope is identified in the model as

$$\frac{d\widetilde{w}}{dp} = \frac{w'(x)}{p'(x)} \text{ for } x = \Phi(p) \in [0, 1].$$

Differentiating (29) with respect to x and simplifying yields

$$N'(x) = \frac{\mu\Gamma_0'(p)[U + N(x)]}{\delta - h^*(v(x)) + \int_x^1 \frac{h^*(v(z))dN(z)}{U+N(z)} + \mu[1 - \Gamma_0(p)]} p'(x).$$

Using this and (32) then implies

$$\frac{d\widetilde{w}}{dp} = \left(\frac{\mu h^*(v(x))v(x)}{\delta - h^*(v(x)) + \int_x^1 \frac{h^*(v(z))dN(z)}{U+N(z)} + \mu[1-\Gamma_0(p)]} \right) \Gamma_0'(p). \tag{33}$$

where $x = \Phi(p)$, a c.d.f.. Clearly $\widetilde{w}(.)$ is an increasing function whose slope is the product of two positive terms. The first term is increasing in p as $v(.)$ and $h^*(v(.))$ are both increasing functions of x. The second term describes the productivity p.d.f. over new start-ups. This analysis establishes Proposition 6.

Proposition 6. In any steady state with $\gamma = 0$, the wage-productivity profile $\widetilde{w}(p)$ is concave and tends to zero as $p \to \overline{p}$ only if $\Gamma_0'(.)$ is strictly decreasing in p and has a long right tail in the sense that $\lim_{p \to \overline{p}} \Gamma_0'(p) = 0$.

Now consider the distribution of wages paid across workers. Define $\digamma(.)$ by $\digamma(w(x)) = U+N(x)$ as the fraction of workers who are either unemployed or employed at a wage no greater than w. Differentiating with respect to x and using (32) yields

$$\digamma'(w(x)) = \frac{N'(x)}{w'(x)} = \frac{\digamma(w(x))}{h^*(v(x))v(x)}.$$

Differentiating again with respect to x and simplifying:

$$\frac{\digamma''(w(x))}{\digamma'(w(x))} = \left(\frac{1}{h^*(v(x))v(x)} \right) \left[1 - \frac{v'(x)}{w'(x)} \frac{\partial}{\partial v} \left[h^*(v)v \right] \right].$$

Using (31) to substitute out $v'(x)$, letting $\eta = \frac{v}{h^*}\frac{dh^*}{dv}$ denote the elasticity of the optimal hire rate with respect to the value of an employee yields

$$\digamma''(w(x)) = \frac{\digamma'(w(x))}{h^*(v(x))v(x)} \left[1 - \frac{h^*(v)[1+\eta]}{r+\delta+\mu-h^*(v(x))+\lambda[1-\widehat{F}(x)]} \frac{p'(x)}{w'(x)} \right], \tag{34}$$

where (33) describes $d\widetilde{w}/dp = w'(x)/p'(x)$. The bracketed term determines whether the density of wages paid is increasing or decreasing. If $d\widetilde{w}/dp$ decreases with p [as implied by the data] then the bracketed term is strictly decreasing in x and so any interior mode, if it exists, must be unique. We thus obtain the following proposition.

Proposition 7. In any steady state with $\gamma = 0$, the steady state distribution of wages paid, $\digamma(.)$, has at least one interior local mode if (i) $\Gamma_0'(\underline{p})$ is sufficiently large and (ii) $\Gamma_0'(p) \to 0$ as $p \to \overline{p}$. Furthermore there is a unique interior mode if (iii) $c(h)$ is a power function and (iv) $d\widetilde{w}/dp$ is decreasing in p.

Proof. Using (33) to substitute out $w'(x)/p'(x)$ in (34) it follows that $F'' > 0$ if and only if

$$\Gamma_0'(p) > \frac{[1+\eta]\left[\delta - h^*(v(x)) + \int_x^1 \frac{h^*(v(z))dN(z)}{U+N(z)} + \mu[1-\Gamma_0(p)]\right]}{\mu v(x)\left[r + \delta + \mu - h^*(v(x)) + \lambda[1-\widehat{F}(x)]\right]}$$

where $x = \Phi(p)$. Thus $\Gamma_0'(\underline{p})$ sufficiently large ensures $F'' > 0$ for p small enough. Furthermore $\Gamma_0'(p) \to 0$ as $p \to \overline{p}$ ensures $F'' < 0$ for p large enough and so the mode must be interior. Restriction (iii) ensures η does not depend on x. If (iii)-(iv) also hold, then the term in the square brackets on the RHS of (34) is strictly decreasing and so implies a unique mode. ■

Propositions 6 and 7 suggest the key to explaining the shapes of the empirical wage distributions $F(w)$ and the wage/productivity profiles $\widetilde{w}(p)$ is a distribution of productivity $\Gamma_0(p)$ across new start-ups which has a decreasing density over most of its support. Thus most new start-ups suffer low productivity draws and struggle to grow. Conversely a relatively small number of start-ups enjoy high productivity draws and grow quickly over time. Note this restriction is also consistent with the unimodal employment weighted distribution of productivity, $\widetilde{N}(p)$, as illustrated in Figure 2. As $\widetilde{N}(p) = N(\Phi(p))$, the above implies

$$\frac{d\widetilde{N}}{dp} = \frac{N'(x)}{p'(x)} = \left(\frac{\mu[U+N(x)]}{\delta - h^*(v(x)) + \int_x^1 \frac{h^*(v(z))dN(z)}{U+N(z)} + \mu[1-\Gamma_0(p)]}\right)\Gamma_0'(p),$$

with $x = \Phi(p)$. As the first term, which is the average number of workers employed by a firm of productivity p, is increasing in $x = \Phi(p)$, the distribution $\widetilde{N}(p)$ has an interior mode as long as $\Gamma_0'(.)$ does not fall too quickly at $p = \underline{p}$ and $\Gamma_0'(p) \to 0$ as $p \to \overline{p}$.

7 Conclusion.

We have shown the introduction of a hiring margin into the matching framework with on-the-job search yields a surprisingly rich and tractable equilibrium setting in a model with firm heterogeneity in productivity. We have fully characterized and established the existence of Markov perfect (Bayesian) equilibria in non-steady state economies where firms have private information on their own productivity. The environment considered is particularly rich. There is turnover of firms with new start-up companies replacing existing firms that suffer firm destruction shocks. There is labor turnover where, in equilibrium, workers quit less productive firms to take employment in more productive firms. Equilibrium wage dispersion arises as more productive firms are willing to pay a higher wage to reduce their employee's quit rates. Furthermore, firm growth rates are size independent where higher productivity firms pay higher wages, enjoy low quit rates and recruit more new employees. Hence, sufficiently high productivity firm have a positive expected growth rate. The structure also allows for firm specific productivity shocks, so that previously successful firms may ultimately decline should they receive a sufficiently unfavorable sequence of productivity draws. Finally, the model provides a coherent explanation for the properties of firm wage and productivity distributions as well as the cross section relationship between them.

The characterization of equilibrium is particularly simple in the limiting case of equally productive firms. Even though the distribution of firm sizes is infinitely dimensional, equilibrium aggregate behavior depends only on the level of unemployment. A particularly useful insight is that the value of a firm is increasing in the level of unemployment. This occurs as, with higher unemployment, firms are less likely to poach each others' employees. As greater employee value generates greater recruitment effort by firms, the non-steady state dynamics of the economy are intrinsically stable. This result appears consistent with the U.S. business cycle where Yashiv (2011) finds the aggregate hiring rate (H/N) does indeed covary positively with unemployment.

This new, rich, and tractable framework opens up several important directions for future research. The equally productive firms case is important as equilibrium dichotomizes into (i) macroeconomic behavior where, depending only on the level of unemployment U, equilibrium determines gross job creation rates and (ii) microeconomic behavior where wages and quit turnover at the firm level depends on a (possibly transitory) firm fixed effect x, the collective recruitment effort of firms (determined in the macroequilibrium) and the distribution of firm sizes which itself evolves endogenously over time.

Given the Markov structure of the model, it is clear it will generalize to a framework where aggregate productivity and job destruction parameter evolve according to a stochastic Markov process. The extension is interesting not only because firms use optimal wage setting strategies, rather than Nash bargaining, but also because the insights of Coles and Moghaddasi (2011) suggest this framework will fit the business cycle volatility and persistence data as described in Shimer (2005). Indeed the model will automatically generate procyclical quit turnover: high aggregate productivity will increase firm hiring rates, thus increasing worker quits from the lower end of the productivity distribution. Furthermore periods of high unemployment will have lower quit rates as newly available jobs are more likely to be filled by the unemployed.

An important distinction between this paper and the BM approach is that in the latter framework the wage has two functions: a higher wage both attracts new employees and retains existing ones. Here instead, the hiring margin is fully targeted by the firm's recruitment strategy, leaving wages to target only the quit margin. The properties of the resulting equilibrium wage structure is correspondingly different. Specifically, the (steady state) density of wages paid is unimodal given the shape of the firm wage-productivity profile observed in Danish data and that shape is consistent with the model under plausible restrictions on the form of the distribution of productivity of entering firms. Furthermore, the model's equilibrium dynamics addresses wage distribution evolution over the cycle, an important topic for future empirical research.

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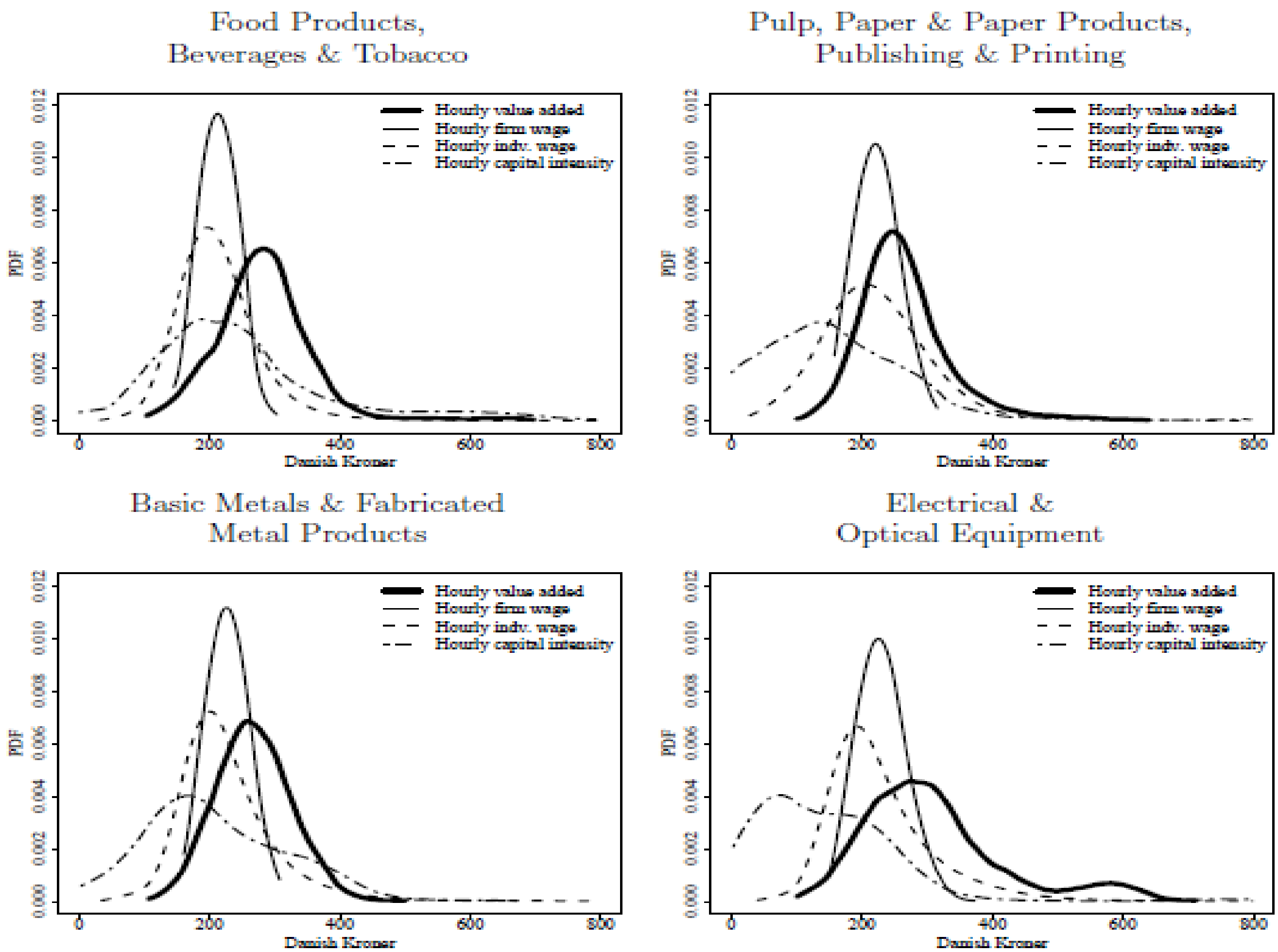
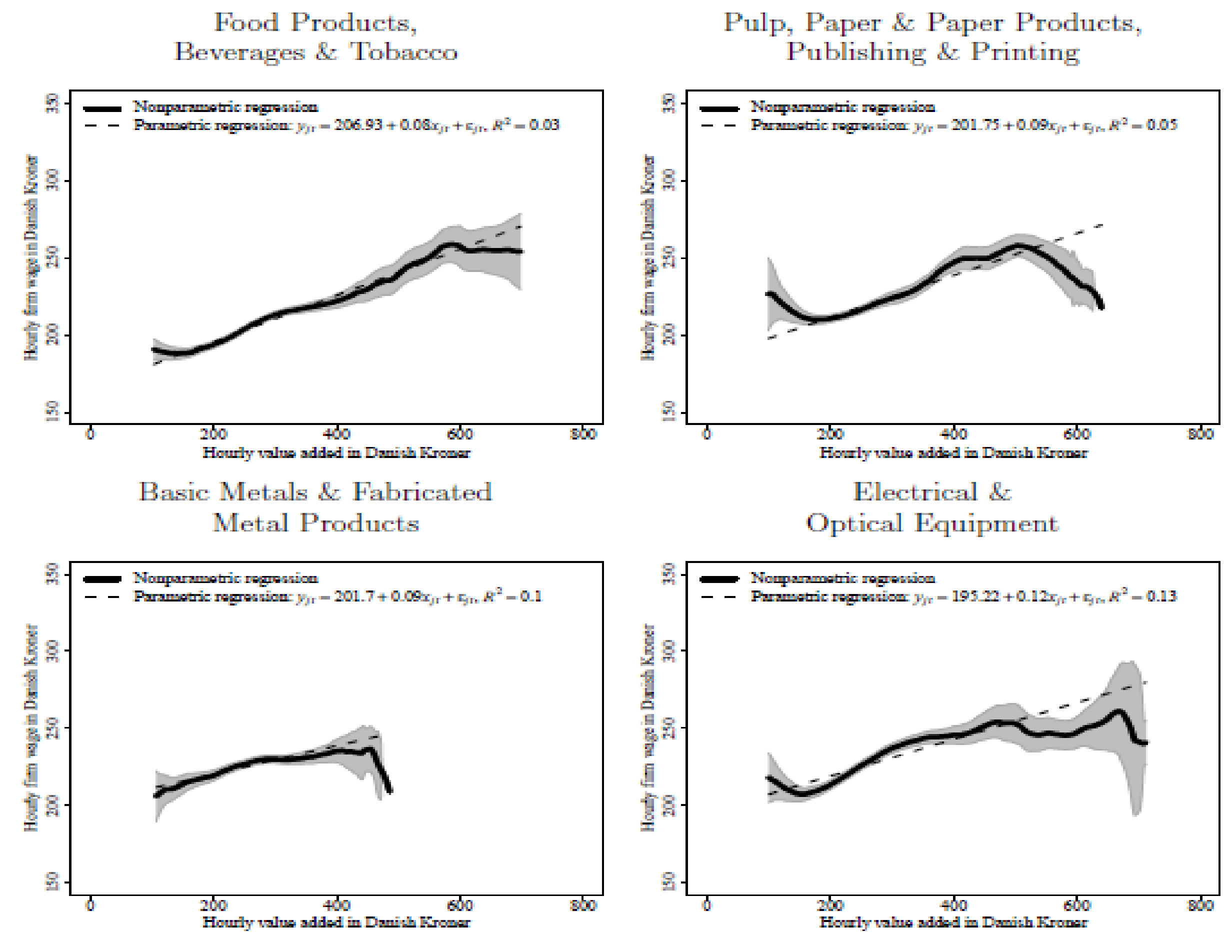


Note: The densities of hourly value added, firm-level average wages and individual wages are estimated using the Epanechnikov kernel with a bandwidth of 25 (Danish Kroner). The density of hourly capital intensity is estimated using the Epanechnikov kernel with a bandwidth of 35 (Danish Kroner).

Figure 2: Danish Manufacturing Wage and Productivity Distributions, Source: Bagger et al. (2011)



Note: The nonparametric regressions are estimated using the Epanechnikov kernel with a bandwidth of 25 (Danish Kroner).

Figure 3: Wage vs Labor Productivity in Danish Manufacturing Industries, Source: Bagger et al. (2011)