Equilibrium under Ambiguity (EUA) for Belief Functions

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Abstract

We study Equilibria under Ambiguity (EUA) with optimism and pessimism as introduced in Eichberger and Kelsey (2014) for the case of beliefs modelled by belief functions. We show existence of equilibria for finite games with an arbitrary number of players and both general and specific ambiguity about the opponents' strategy choice. We illustrate by examples the potential of this approach to model behavior which cannot be obtained as a Nash equilibrium..

1 Introduction

From the beginning, the *theory of decision making under uncertainty* and the *theory of games* have been closely related. Luce and Raiffa (1957) open their chapter on *Individual Decision Making under Uncertainty* in their book *Games and Decisions*, with the following view of strategic interaction:

> "In a game the uncertainty is entirely to the unknown decisions of the other players, and, in the model, the degree of uncertainty is reduced through the assumption that each player knows the desires of the other players and the assumption that they will take whatever actions appear to gain their ends." (Chapter 13, p.275)

Already in 1921, Frank Knight distinguished situations according to the underlying type of uncertainty: (i) *"a priori probability",* (ii) *"statistical probability",* and (iii) *"estimates",* i.e., *situations where "there is no valid basis of any kind for classifying instances".* (Knight (1921), pp. 224/5). Frank Knight viewed uncertainty as the basis of economic institutions such as insurance companies, free enterprise, property, and management and control. When Luce and Raiffa (1957) wrote their view of game-theoretic analysis, formal analysis of decision making over lotteries by Neumann and Morgenstern (1944) and over acts by Savage (1954) had established the expected utility hypothesis as the dominant approach to decision making under uncertainty.

Following Ellsberg (1961)'s critique of the subjective expected utility paradigm, however, several approaches to decision making under uncertainty without the expected utility hypothesis were developed. Applying these more recent concepts to games, where uncertainty concerns the opponents' strategic choices, did raise some fundamental questions:

- How to model ambiguity about the opponents' strategies?
- What are adequate equilibrium notions?
- How to combine active randomization over one's own strategies (mixed strategies) with beliefs about the opponents' strategic choices?
- Do we need to optimism?

In the small literature which deals with the first two of these questions there is little discussion of the role of "mixed strategies" and little or no discussion of attitudes towards ambiguity other than "ambiguity aversion" or "pessimism".

In Eichberger and Kelsey (2014) a concept of *Equilibria under Ambiguity (EUA)* was introduced which allowed for players with optimistic as well as pessimistic attitudes towards ambiguity. Beliefs over the opponents' strategy choices were modelled by a convex capacity and preferences over payoffs by the Choquet integral of a JP-capacity, a special class of capacities studied by Jaffray and Philippe (1997). Since optimistic beliefs induce non-convex preferences over strategies, no general existence proof could be provided there.

In this paper, we present a general existence proof for finite games if beliefs are modelled by a belief function rather than a convex capacity. Belief functions, i.e., totally monotone capacities, are a special case of convex capacities. They do not restrict ambiguous beliefs substantially but have a simple Choquet integral and allow for a natural notion of "ambiguity". Both properties are useful in economic applications. In addition, belief functions allow us to link beliefs to the large literature in statistics which, following Dempster (1967) and Shafer (1976), establishes belief functions as the most consistent extension of probabilities.

For our existence proof, we will rely heavily on the fact that belief functions are characterized by their Möbius transform which is a probability distribution over the power set. This property opens up several other natural generalizations in the context of games. In particular, the well-known Möbius product provides us with a natural concept of "independence" among beliefs.

1.1 Related Literature

This paper draws on two sources of literature: (i) on the one hand, there is the literature on decision making under uncertainty, (ii) on the other hand, there is a limited literature on how to include notions of ambiguity in the game-theoretic context.

The first group deals with axiomatic foundations for various representations of preferences either over lotteries or over state-contingent outcomes. In this literature one finds careful discussions of the problem of how to separate ambiguity from ambiguity attitudes. The second group focuses on the degree of consistency between strategies chosen and beliefs about strategies of the opponents.

In the wake of von Neumann-Morgenstern's axiomatization, there have been many axiomatizations of expected utility *Subjective Expected Utility (SEU)* by Savage (1954), Anscombe and Aumann (1963), and, following the experimental critiques by Ellsberg (1961) and Kahneman and Tversky (1979), of other decision criteria under uncertainty, *Choquet Expected Utility (CEU)* by Schmeidler (1989), *Maxmin Expected Utility (MEU)* by Gilboa and Schmeidler (1989), *Multiple Prior Model with Optimism and Pessimism (α-MEU)* by Ghirardato, Maccheroni, and Marinacci (2004), *Smooth Model* by Klibanoff, Marinacci, and Mukerji (2005), and ....

There have been several attempts to include these new approaches to decision making under uncertainty in game-theoretic analysis: Dow and Werlang (1994): CEU, Klibanoff (1996): MEU, Lo (1996): MEU, Marinacci (2000a): CEU, Eichberger and Kelsey (2000)Eichberger and Kelsey (2013): CEU and α-MEU, Bade (2010), Grant, Meneghel & Tourky (2015), Hanany, Klibanoff & Mukerji (2015), and ....

2 Games and belief functions

Consider a finite game $\Gamma = (I, (S_i, u_i)_{i \in I})$ with a player set $I = \{1, 2, ..., n\}$ and, for each player $i \in I$, a finite strategy set S_i and a payoff function $u_i : S_1 \times ... \times S_n \to \mathbb{R}$. For notational convenience, denote by $\Sigma_i := 2^{|S_i|}$ the power set of S_i. Let $S_{-i} = \underset{j \neq i}{\times} S_j$ be the set of all strategy combinations for the opponents and denote by Σ_{-i} the power set of S_{-i}.

A probability distribution γ_i on Σ_{-i}, i.e., $\gamma_i \in \Delta(\Sigma_{-i})$, defines a *belief function* ϕ_i^γ as follows:

$$\phi_i^\gamma(E) \quad := \quad \sum_{A \subseteq E} \gamma_i(A) \quad \text{for all} \quad E \in \Sigma_{-i}.$$

The probability distribution γ_i is called Möbius transform of the capacity ϕ_i^γ.

Example 2.1 shows some well-known special cases of belief functions.

Example 2.1 *(i)* No ambiguity: *For any probability distribution $\pi_i \in \Delta(S_{-i})$, the Möbius transform*

$$\gamma_i(E) = \begin{cases} \pi_i(s_{-i}) & \text{for} \quad E = \{s_{-i}\} \\ 0 & \text{otherwise} \end{cases}$$

*defines the additive capacity (*probability distribution*)*

$$\phi_i^\gamma(E) = \sum_{s_{-i} \in E} \pi_i(s_{-i})$$

for all $E \in \Sigma_{-i}$.
(ii) Complete ambiguity*: The Möbius transform*

$$\gamma_i(E) = \begin{cases} 1 & \text{for} \quad E = S_{-i} \\ 0 & \text{otherwise} \end{cases}$$

defines the capacity of complete ignorance

$$\phi_i^\gamma(E) = \begin{cases} 1 & \textit{for} \quad E = S_{-i} \\ 0 & \textit{otherwise} \end{cases} .$$

(iii) Constant degree of ambiguity*: For any probability distribution* $\pi_i \in \Delta(S_{-i})$ *and any* $\varepsilon \in [0,1]$, *the Möbius transform*

$$\gamma_i(E) = \begin{cases} 0 & \textit{for} \quad E = \varnothing \\ \varepsilon\pi_i(s_{-i}) & \textit{for} \quad \varnothing \neq E \neq S_{-i} \\ 1-\varepsilon & \textit{for} \quad E = S_{-i} \end{cases}$$

defines the simple capacity *(or* ε-contamination*)*

$$\phi_i^\gamma(E) = \begin{cases} \varepsilon \sum\limits_{s_{-i}\in E} \pi_i(s_{-i}) & \textit{for} \quad E \neq S_{-i} \\ 1 & \textit{for} \quad E = S_{-i} \end{cases} .$$

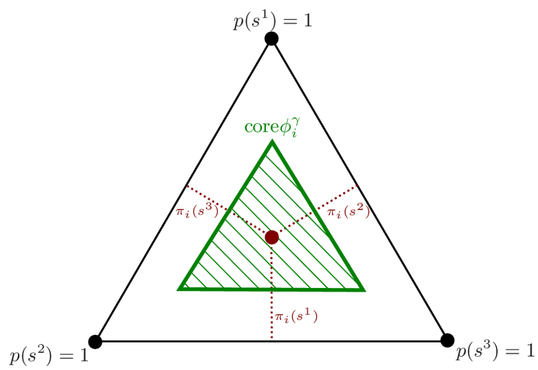
A belief function is a convex capacity and, hence, has a non-empty core:

$$\text{core}\,\phi_i^\gamma := \left\{p \in \Delta(S_i)| \quad p(E) \geq \phi_i^\gamma(E) \quad \text{for all} \quad E \in \Sigma_{-i} \right\}.$$

The core is the set of probability distributions which are consistent with the constraints imposed by the belief function. This property allows one to interpret a belief function as defining a multiple prior model.

The following diagram illustrates the core for the case of three strategies of the opponent.

Constant degree of ambiguity $\varepsilon \in [0,1]$ *for* $\pi_i \in \Delta(S_{-i})$



$\mathbf{S}_{-i} = \left\{s^1, s^2, s^3\right\}$

The area shaded in green represents the core of an ε-contamination, i.e., the set of probability distributions in the simplex $\Delta(S_{-i})$ which are compatible with the ε-contamination. For $\varepsilon = 0$, the case of complete uncertainty, the core equals the full simplex. For $\varepsilon = 1$ there is no ambiguity and the belief function coincides with the probability distribution π_i.

2.1 Payoffs under ambiguity

Ambiguity and ambiguity attitudes can be modelled by the Choquet integral of a JP-capacity. A JP-capacity[1] is the α-convex combination of a convex capacity μ with its dual capacity $\overline{\mu}$,

$$\nu^{JP}(\alpha, \mu) = \alpha\mu + (1-\alpha)\overline{\mu},$$

where the dual capacity $\overline{\mu}$ is defined as $\overline{\mu}(E) := 1 - \mu(S \backslash E)$ for any $E \subseteq S_{-i}$.

Following Schmeidler (1989) there is an extended literature which associates convex capacities with ambiguity. In this literature, the Choquet integral of a convex capacity reflects *ambiguity aversion* or *pessimism*, while the Choquet integral of the dual of a convex capacity represents an *optimistic* attitude towards the ambiguity reflected in the convex capacity. This interpretation of α as measuring ambiguity attitude and μ representing ambiguity is reflected in the multiple-prior representation of the Choquet integral of a JP-capacity. The Choquet expected payoff of a strategy $s_i \in S_i$ given beliefs $\nu_i^{JP}(\alpha_i, \mu_i)$ over the opponents' strategies in S_{-i} is[2]

$$\begin{aligned} V_i(s_i, \alpha_i, \mu_i) \quad : &= \int u_i\left(s_i, s_{-i}\right) d\nu_i^{JP}\left(s_{-i}|\alpha_i, \mu_i\right) \\ &= \alpha \min_{p \in \operatorname{core} \mu_i} \sum_{s_{-i} \in S_{-i}} u_i\left(s_i, s_{-i}\right) p(s_{-i}) \\ &\quad + (1-\alpha) \max_{p \in \operatorname{core} \mu_i} \sum_{s_{-i} \in S_{-i}} u_i\left(s_i, s_{-i}\right) p(s_{-i}). \end{aligned}$$

Hence, one can view the Choquet integral of a JP-capacity as an $\alpha-$MEU representation with respect to a set of priors given by $\operatorname{core} \mu_i$.

In this paper, we will suggest to model beliefs by a belief function ϕ_i^γ generated by the Möbius transform γ_i. We will argue in the next section that belief functions allow one also to model ambiguity in a very intuitive way.

For the JP-capacity $\nu_i^{JP}\left(\alpha_i, \phi_i^\gamma\right) := \alpha_i \phi_i^\gamma + (1-\alpha)\overline{\phi_i^\gamma}$ based upon the belief function ϕ_i^γ and $\alpha_i \in [0, 1]$ as pessimism parameter, one obtains a particularly intuitive Choquet integral.

Proposition 2.1 *Let γ_i be the Möbius transform of the belief function ϕ_i^γ, then*

$$\begin{aligned} V_i(s_i, \alpha_i, \phi_i^\gamma) \quad : &= \int u_i\left(s_i, s_{-i}\right) d\nu_i^{JP}\left(s_{-i}|\alpha_i, \phi_i^\gamma\right) \\ &= \sum_{E \subseteq S_{-i}} \gamma_i(E) \underbrace{\left[\alpha_i \min_{s_{-i} \in E} u_i\left(s_i, s_{-i}\right) + (1-\alpha_i) \max_{s_{-i} \in E} u_i\left(s_i, s_{-i}\right)\right]}_{:= V_i^\alpha(s_i, E)} \end{aligned}$$

[1] Jaffray and Philippe (1997) have introduced this type of capacity and studied it in great detail. Hence, we refer to it as *J(affray)P(hilippe)-capacity.*

[2] For a proof see, e.g., Eichberger and Kelsey (2014).

Proof. The result follows immediately from Proposition 1 in Jaffray and Philippe (1997) (p.175). ■

Proposition 2.1 shows that the Choquet integral of a belief function ϕ_i^γ is the weighted sum of the α-max-min payoffs $V_i^\alpha(s_i, E)$ over all events $E \subseteq S_{-i}$, weighted by the Möbius transform γ_i. It follows immediately that

$$\begin{aligned}
& V_i(s_i, \alpha_i, \phi_i^\gamma) \\
= \; & \alpha_i \left[\sum_{E \subseteq S_{-i}} \gamma_i(E) \min_{s_{-i} \in E} u_i(s_i, s_{-i}) \right] + (1-\alpha_i) \left[\sum_{E \subseteq S_{-i}} \gamma_i(E) \max_{s_{-i} \in E} u_i(s_i, s_{-i}) \right] \\
= \; & \alpha \min_{p \in \text{core}\, \phi_i^\gamma} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\, p(s_{-i}) + (1-\alpha) \max_{p \in \text{core}\, \phi_i^\gamma} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\, p(s_{-i}).
\end{aligned}$$

2.2 Ambiguity

The Möbius transform of a belief function is a probability distribution over all events in S_{-i}. If only singleton events $\{s_{-i}\}$ obtain positive weights then, by construction, the belief function is a probability distribution. In general, however, non-singleton events E in Σ_i may also obtain positive weights. Indeed, as the capacity of complete ignorance illustrates, all singleton events may get zero weight and the S_{-i} obtains a weight of one.

With belief functions, ambiguity of a decision maker can be modelled by positive values of the Möbius transform γ_i for *non-singleton* events. Positive weights of the Möbius transform γ_i for non-singleton events reflect information and concerns of the decision maker which cannot be factored in a single probability distribution. Gilboa and Schmeidler (1994) interpret positive weights for non-singleton events as "direct evidence" for the likelihood of some event which cannot be broken down to its subsets: *"One of the reasons one gets direct evidence for T but not for any subset thereof may be model misspecifications, i.e., that the states of the world included in the model do not exhaust the 'actual' ones". (p. 52).*

In the context of game theory where players form beliefs about their opponents' strategy choice it is assumed that players use their knowledge about the opponents' payoffs in order to deduce their optimal behavior. This information is, however, in general not sufficient to determine unambiguously the behavior of the opponents. In particular, in games with multiple Nash equilibria a player cannot predict the opponents' behavior from knowing the opponents' payoff and the assumption that the other players will also maximize their payoffs. As Luce and Raiffa (1957) (p. 275) put it *"the degree of uncertainty is reduced through the assumption that each player knows the desires of the other players and the assumption that they will take whatever actions appear to gain their ends",* but uncertainty can not be resolved in this way. Hence, it appears sensible to allow for other types of information in order to predict behavior in strategic interaction. Indeed, many game theorists, like Schelling (1960) in his famous book "Strategy of Conflict", appeal to custom and outside knowledge about a situation in order to obtain better predictions in games.

The following example is a simplified version of the minimum-effort games studied by Huyck, Battalio, and Beil (1990) in a series of experiments.

Example 2.2 (Minimum effort game) *There are two players, $I = \{1, 2\}$, who have to contribute an effort level e_i from the set $S = \{1, 2, 3\}$. Their payoffs are twice the minimum effort level chosen minus their contribution:*

$$p_i(e_1, e_2) = 2 \min\{e_1, e_2\} - e_i.$$

The following payoff matrix illustrates this game.

		Player 2		
		1	2	3
Player 1	1	1, 1	1, 0	1, −1
	2	0, 1	2, 2	2, 1
	3	−1, 1	1, 2	3, 3

This game has three Nash equilibria $\{(1, 1), (2, 2), (3, 3)\}$. Hence, even with complete information about payoffs, assuming mutual optimality will not suffice to predict behavior in this game[3]*. Equilibrium behavior is ambiguous and players may choose the equilibrium $(1, 1)$ simply because it guarantees a certain payoff of 1.*
On the other hand, realizing that equilibria can be strictly ranked by the Pareto principle and that the cost of failing to coordinate is increasing with the effort level, players may feel more ambiguity about the opponent playing 2 or 3, i.e., the event $\{2, 3\}$, than about the event $\{1, 2, 3\}$. Belief functions allow us to include such considerations in the ambiguity weights γ_i of the respective events, e.g., by assuming $\gamma_i(\{2, 3\}) > \gamma_i(\{1, 2, 3\})$.

Several concepts of a degree of ambiguity have been suggested in the literature (e.g., Eichberger and Kelsey (2014), Marinacci (2000b)). Most of these measures are built on the difference between the capacity and its dual,

$$\overline{\mu}(E) - \mu(E) = 1 - \mu(S_{-i} \backslash E) - \mu(E),$$

which measures deviation of the capacity value for an event and its complement from additivity. For a belief function ϕ_i^γ, one obtains easily

$$\overline{\phi}_i^\gamma(E) - \phi_i^\gamma(E) = \sum_{\{A: A \cap E \neq \varnothing \neq A \cap S_{-i} \backslash E\}} \gamma_i(A).$$

Hence, the degree of ambiguity with respect to an event E equals the sum of the weights γ_i for events which cannot be attributed either to the event E or to its complement.

[3] In fact, the experimental study of Huyck, Battalio, and Beil (1990) tries to find out whether players either have or can develop conventions which help them to coordinate on one of these equilibria.

In order to implement the idea of Luce and Raiffa (1957) that *"the degree of uncertainty is reduced through the assumption that each player knows the desires of the other players and the assumption that they will take whatever actions appear to gain their ends"*, we will assume that the weights of the Möbius transform γ_i of player i's belief function ϕ_i^γ which are put on non-singleton events in Σ_{-i} will be determined exogenously by outside information while the weights given to singleton events will be determined endogenously in an equilibrium under ambiguity.

For any set of events Σ_{-i}, denote by $S(\Sigma_{-i}) := \{E \in \Sigma_{-i} |\ |E| = 1\}$ the singleton subsets of Σ_{-i} and by $N(\Sigma_{-i}) := \{E \in \Sigma_{-i} |\ |E| \geq 2\}$ the set of non-singleton events. Ambiguity of a player about the opponents' strategy choice is reflected by the weights the Möbius transform γ_i puts on non-singleton events in $N(\Sigma_{-i})$. Denote by $\delta_i : N(\Sigma_{-i}) \to [0, 1]$ the vector of weights reflecting the ambiguity which Player i associates with the events in $N(\Sigma_{-i})$ and let $\overline{\delta}_i := \sum_{E \in N(\Sigma_i)} \delta_i(E)$ be the aggregate ambiguity about player i's strategy choice. W.l.o.g., we can assume $\overline{\delta}_i \leq 1$.

Special cases are $\overline{\delta}_i = 0$, i.e., no ambiguity, and $\overline{\delta}_i = 1$, e.g., for $\delta_i(S_{-i}) = 1$ when the player faces complete ambiguity. In the former case, $V_i(s_i, \alpha_i, \gamma(\delta_i))$ would be the expected utility with respect to $(\gamma_i(\{s_{-i}\})_{s_{-i} \in S_{-i}}$; in the latter case, it would be the Hurwicz criterion.

3 Equilibrium under Ambiguity

In order to determine the endogenous part of the belief function we will apply the notion of an Equilibrium under Ambiguity (EUA) suggested and analyzed in Eichberger and Kelsey (2014).

3.1 Best replies and Equilibrium under Ambiguity

For an arbitrary capacity μ_i on S_{-i} representing player i's beliefs, the best-reply correspondence of player i given beliefs μ_i, $R_i(\mu_i) = \operatorname{argmax}_{s_i \in S_i} V_i(s_i, \alpha_i, \mu_i)$ is well-defined. Following Dow and Werlang (1994), Marinacci (2000a) and Eichberger and Kelsey (2000) we describe an equilibrium in beliefs as a list of capacities $\widehat{\mu} = (\widehat{\mu}_1, ..., \widehat{\mu}_n)$ for which the support is a subset of best replies.

Definition 1 *An n-tuple of capacities $\widehat{\mu} = (\widehat{\mu}_1, ..., \widehat{\mu}_n)$ is an* Equilibrium Under Ambiguity (EUA) *if for all players, $i \in I$,*

$$\varnothing \neq \operatorname{supp} \widehat{\mu}_i \subseteq \underset{j \neq i}{\times} R_j(\widehat{\mu}_j).$$

For capacities which are additive and beliefs which are independent Definition 1 defines a Nash equilibrium.

A crucial aspect of this definition concerns the adequate definition of a support for a capacity. Several definitions for the support of a capacity have been suggested in the literature. In particular, Dow and Werlang (1994), Marinacci

(2000a) and Ryan (2002) suggest alternative notions. For convex capacities, the probabilities in the core of the capacity are a natural set of multiple priors. Hence, support notions for sets of probability distributions are also relevant in this context. Eichberger and Kelsey (2014) (Appendix A) provide a detailed comparison of these support notions and, in the light of these results, suggest as definition of support for a *convex capacity* μ the intersection of the supports of all probability distributions in the core of μ,

$$\operatorname{supp}\mu := \bigcap_{p \in \operatorname{core}\mu} \operatorname{supp} p.$$

In Dominiak and Eichberger (2015) we study the support notions for belief functions. Of particular relevance is the following result.

Lemma 2 (Dominiak and Eichberger (2015), Proposition 3.2) *Let ϕ_i^γ be a belief function and γ_i its Möbius transform, then $s_{-i} \in \bigcap_{p \in \operatorname{core}\phi_i^\gamma} \operatorname{supp} p$ if and only if $\gamma_i(\{s_{-i}\}) > 0$.*

3.2 Two-player games

We will study first two-player games where no correlation issues regarding the beliefs about the opponents' behavior exist. Given ambiguity weights $(\delta_1, \delta_2) \in [0,1]^{|N(\Sigma_2)|} \times [0,1]^{|N(\Sigma_1)|}$ and probability distributions $(\sigma_1, \sigma_2) \in \Delta(S_2) \times \Delta(S_1)$, for each player a belief function ϕ_i^γ can be defined by the Möbius transform $\widetilde{\gamma}_i(\sigma_i, \delta_i) \in \Delta(\Sigma_j)$ where

$$\widetilde{\gamma}_i(\sigma_i, \delta_i)(E) := \begin{cases} \delta_i(E) & \text{for} \quad E \in N(\Sigma_j) \\ (1-\overline{\delta}_i)\sigma_i(E) & \text{otherwise} \end{cases} .$$

Writing $V_i(s_i, \alpha_i, \sigma_i, \delta_i)$ instead of $V_i(s_i, \alpha_i, \phi_i^{\widetilde{\gamma}(\sigma_i,\delta_i)})$ in order to simplify notation, one can write the Choquet expected payoff of a strategy $s_i \in S_i$ as

$$\begin{aligned} V_i(s_i, \alpha_i, \sigma_i, \delta_i) &= \sum_{E \subseteq S_j} \widetilde{\gamma}_i(\sigma_i, \delta_i)(E) V_i^\alpha(s_i, E) \\ &= (1-\overline{\delta}_i) \sum_{s_j \in S_j} \sigma_i(s_j) V_i^\alpha(s_i, \{s_j\}) + \sum_{E \in N(\Sigma_j)} \delta_i(E) V_i^\alpha(s_i, E). \end{aligned}$$

For given (α_i, δ_i), $V_i(s_i, \alpha_i, \sigma_i, \delta_i)$ is a continuous function on $S_i \times \Delta(S_j)$.

We begin with some examples which illustrate the potential of EUA with belief functions for obtaining results which differ from those obtained without ambiguity.

Example 3.1 (coordination game) *Consider the following asymmetric coordination game with strategy sets $S_1 = \{u, d\}$ and $S_2 = \{l, r\}$:*

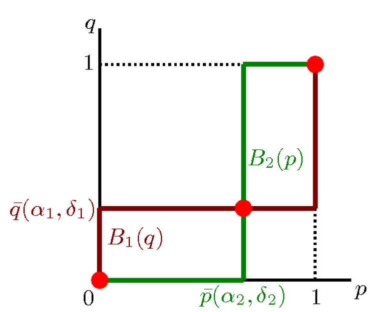
Player 2

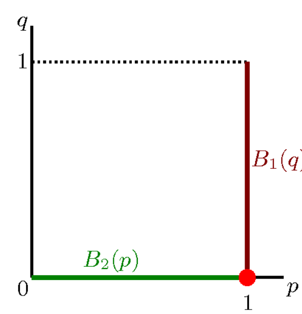
		l	r
Player 1	u	$1,2$	$1,1$
	d	$0,0$	$2,1$

In this game, there is a only one non-singleton event about which there may be ambiguity, $N(\Sigma_1) = \{S_1\}$ and $N(\Sigma_2) = \{S_2\}$, respectively. Hence, $(\delta_1, \delta_2) = (\delta_1(S_2), \delta_2(S_1)) = (\bar{\delta}_1, \bar{\delta}_2)$. Let $(\sigma_1(l), \sigma_1(r)) = (q, 1-q)$ and $(\sigma_2(u), \sigma_2(d)) = (p, 1-p)$. Choquet expected payoffs are easily computed as

$$\begin{aligned} V_1(u, \alpha_1, q, \delta_1) = 1 \quad &\text{and} \quad V_1(d, \alpha_1, q, \delta_1) = 2(1-\bar{\delta}_1)(1-q) + 2\bar{\delta}_1(1-\alpha_1), \\ V_2(r, \alpha_2, p, \delta_2) = 1 \quad &\text{and} \quad V_2(l, \alpha_2, p, \delta_2) = 2(1-\bar{\delta}_2)p + 2\bar{\delta}_2(1-\alpha_2). \end{aligned}$$

Maximizing the average payoff $W_1(q,p) = pV_1(u, \alpha_1, q, \delta_1) + (1-p)V_1(d, \alpha_1, q, \delta_1)$ with respect to p yields the correspondence of maximizers $B_1(p,q) = \{1\}$ for $q > \overline{q}(\alpha_1, \delta_1)$, $B_1(p,q) = \{0\}$ for $q < \overline{q}(\alpha_1, \delta_1)$, and $B_1(p,q) = [0,1]$ otherwise, where $\overline{q}(\alpha_1, \delta_1) := \frac{1}{2}\left(\frac{1-2\alpha_1\delta_1}{1-\delta_1}\right)$ denotes the critical value of q yielding indifference. Similarly, $W_2(q,p) = qV_2(l, \alpha_2, p, \delta_2) + (1-q)V_2(r, \alpha_2, p, \delta_2)$ and $B_2(p,q) = \{1\}$ for $p > \overline{p}(\alpha_2, \delta_2)$, $B_2(p,q) = \{0\}$ for $p < \overline{p}(\alpha_2, \delta_2)$, and $B_2(p,q) = [0,1]$ otherwise, where $\overline{p}(\alpha_2, \delta_2) := \frac{1}{2}\left(\frac{1-2(1-\alpha_2)\delta_2}{1-\delta_2}\right)$. The following diagram shows the correspondences of the maximizers.





For no ambiguity, $\delta_1 = \delta_2 = 0$, one obtains $\overline{q}(\alpha_1, 0) = \overline{p}(\alpha_2, 0) = \frac{1}{2}$ and there are three EUA $\{(\{u\}, \{l\}), (\{u,d\}, \{l,r\}), (\{d\}, \{r\})\}$ which correspond to the three Nash equilibria $\{(1,1), (\frac{1}{2}, \frac{1}{2}), (0,0)\}$. For little ambiguity, δ_1, δ_2 close to 0, the set of EUA remains the same as illustrated in the left panel of the diagram. The set of EUA changes if ambiguity increases. If players are sufficiently pessimistic, $\alpha_1, \alpha_2 > \frac{1}{2}$ and ambiguity increases $\delta_1, \delta_2 \to 1$, then strategies u and r, which guarantee the payoff of 1, will become dominant strategies leaving $(\{u\}, \{r\})$ as the unique EUA. This case is illustrated in the right-hand panel.

Example 3.1 shows that EUA can describe other types of behavior than Nash equilibria. For sufficient pessimism and sufficient ambiguity players will choose their safe strategies u and r, a behavior which appears quite sensible but cannot be modelled as a Nash equilibrium.

With two strategies as in Example 3.1 there can be only global ambiguity regarding all strategies of the opponent. The next example will resume the minimum effort game of Example 2.2 in order to show that EUA with belief functions can support behavior which depends on event-specific ambiguity.

		Player 2		
		1	2	3
Player 1	1	1, 1	1, 0	1, −1
	2	0, 1	2, 2	2, 1
	3	−1, 1	1, 2	3, 3

Example 3.2 (Example 2.2 resumed) *Assume that players are pure pessimists, $\alpha_1 = \alpha_2 = 1$, and symmetric in their perceived ambiguity, $\delta_1 = \delta_2 =: \delta$. Given these assumptions and using the fact that $\sigma_i(\{1\}) = 1 - \sigma_i(\{2\}) - \sigma_i(\{3\})$ and $\sum\limits_{E \in N(\Sigma_j)} \delta(E) = \overline{\delta}$, one obtains the following Choquet expected payoffs from choosing effort level e_i:*

$$\begin{aligned} V_i(e_i, 1, \sigma_i, \delta) &= \sum_{E \in \Sigma_j} \gamma_i(E) \min_{e_j \in E} u_i(e_i, e_j) \\ &= (1 - \overline{\delta}) \left[\sigma_i(\{1\}) u_i(e_i, 1) + \sigma_i(\{2\}) u_i(e_i, 2) + \sigma_i(\{3\}) u_i(e_i, 3) \right] \\ &\quad + \delta(\{1,2\}) \min_{e_j \in \{1,2\}} u_i(e_i, e_j) + \delta(\{1,3\}) \min_{e_j \in \{1,3\}} u_i(e_i, e_j) \\ &\quad + \delta(\{2,3\}) \min_{e_j \in \{2,3\}} u_i(e_i, e_j) + \delta(\{1,2,3\}) \min_{e_j \in \{1,2,3\}} u_i(e_i, e_j) \\ &= \begin{cases} 1 & \text{for} \quad e_i = 1 \\ 2(1 - \overline{\delta}) (\sigma_i(2) + \sigma_i(3)) + \delta(\{2,3\}) & \text{for} \quad e_i = 2 \\ (1 - \overline{\delta}) (2\sigma_i(2) + 4\sigma_i(3) - 1) + 2\delta(\{2,3\}) - \overline{\delta} & \text{for} \quad e_i = 3 \end{cases} . \end{aligned}$$

It is easily checked that there are three types of EUA:

1. *No ambiguity, $\overline{\delta} = 0$: There are three equilibria corresponding to the Nash equilibria in pure strategies:*

$$\begin{aligned} (\sigma_1(\{1\}), \sigma_2(\{1\})) &= (1,1), \\ (\sigma_1(\{2\}), \sigma_2(\{2\})) &= (1,1), \\ (\sigma_1(\{3\}), \sigma_2(\{3\})) &= (1,1). \end{aligned}$$

2. *If there is sufficient general ambiguity, $\delta(\{1,2,3\}) > \frac{1}{2}$, and no specific ambiguity for a particular event, i.e., $\delta(\{1,2\}) = \delta(\{1,3\}) = \delta(\{2,3\}) = 0$, then*

$$(\sigma_1(\{1\}), \sigma_2(\{1\})) = (1,1)$$

will be the unique *equilibrium under ambiguity.*

3. *For sufficient specific ambiguity, $\delta(\{23\}) > \frac{1}{2}$ and $\delta(\{1,2,3\}) = \delta(\{1,2\}) = \delta(\{1,3\}) = 0$, only*

$$(\sigma_1(\{2\}), \sigma_2(\{2\})) = (1,1)$$

will be an equilibrium under ambiguity.

In the experiments on behavior in the minimum-effort game which were conducted by Huyck, Battalio, and Beil (1990) players initially often chose intermediate levels of effort rather than the highest or lowest level. Such behavior can be modelled by an EUA where players experience more ambiguity about a particular event, here $\{2,3\}$ in the third case, than about other events. As argued before such specific ambiguity may be justified by the relative strength of the incentives to deviate from a strategy.

We conclude this section with a sketch of the proof of the general existence theorem provided in the next section.

For each player fix the parameters (α_i, δ_i), $i \in \{1,2\}$. The weighted averages of the Choquet expected payoffs

$$W_1(\sigma_1, \sigma_2) := \sum_{s_1 \in S_1} \sigma_2(s_1) V_1(s_1, \alpha_1, \sigma_1, \delta_1)$$

and

$$W_2(\sigma_1, \sigma_2) := \sum_{s_2 \in S_2} \sigma_1(s_2) V_2(s_2, \alpha_2, \sigma_2, \delta_2)$$

are linearly affine and, hence, continuous functions $W_i : \Delta(S_2) \times \Delta(S_1) \to \mathbb{R}$. Define the correspondence of maximizers $B_i(\sigma_1, \sigma_2)$ as

$$B_i(\sigma_1, \sigma_2) = \arg\max_{\sigma_j} W_i(\sigma_1, \sigma_2)$$

and let $B(\sigma_1, \sigma_2) := B_1(\sigma_1, \sigma_2) \times B_2(\sigma_1, \sigma_2)$ be the Cartesian product of these correspondences. By the maximum theorem B is a non-empty and uhc correspondence from $\Delta(S_2) \times \Delta(S_1)$ to $\Delta(S_2) \times \Delta(S_1)$, which is convex-valued since B is linear (σ_1, σ_2). Hence, by the Kakutani fixed-point theorem, there is $(\widehat{\sigma}_1, \widehat{\sigma}_2) \in \Delta(S_2) \times \Delta(S_1)$ such that

$$\begin{aligned} \widehat{\sigma}_2 &\in \arg\max_{\sigma_2} W_1(\widehat{\sigma}_1, \widehat{\sigma}_2), \\ \widehat{\sigma}_1 &\in \arg\max_{\sigma_1} W_2(\widehat{\sigma}_1, \widehat{\sigma}_2) \end{aligned}$$

holds. Since $W_i(\sigma_1, \sigma_2)$ is linear in σ_j, $\widehat{\sigma}_j \in \arg\max_{\sigma_j} W_i(\widehat{\sigma}_1, \widehat{\sigma}_2)$ implies $s_i \in \arg\max_{\sigma_j} W_i(\widehat{\sigma}_1, \widehat{\sigma}_2)$ for all s_i with $\widehat{\sigma}_j(s_i) > 0$, i.e., for all $s_i \in \operatorname{supp} \widehat{\sigma}_j$.

Thus we can conclude that $\operatorname{supp} \widehat{\sigma}_j \neq \varnothing$ and for all $s_i \in \operatorname{supp} \widehat{\sigma}_j$, $V_i(s_i, \alpha_i, \widehat{\sigma}_i, \delta_i) \geq V_i(s_i', \alpha_i, \widehat{\sigma}_i, \delta_i)$ for all $s_i' \in S_i$. By Lemma 2 we know that $\widetilde{\gamma}_i(\widehat{\sigma}_i, \delta_i)(\{s_j\}) = (1 - \delta_i)\widehat{\sigma}_i(\{s_j\}) > 0$ implies

$$s_j \in \bigcap_{p \in \operatorname{core} \phi_i^{\widetilde{\gamma}(\widehat{\sigma}_i, \delta_i)}} \operatorname{supp} p.$$

Hence, the belief functions $(\phi_1^{\widetilde{\gamma}(\widehat{\sigma}_1, \delta_1)}, \phi_2^{\widetilde{\gamma}(\widehat{\sigma}_2, \delta_2)})$ are an Equilibrium under Ambiguity (EUA),

$$\varnothing \neq \operatorname{supp} \phi_i^{\widetilde{\gamma}(\widehat{\sigma}_i, \delta_i)} \subseteq R_j(\phi_j^{\widetilde{\gamma}(\widehat{\sigma}_j, \delta_j)})$$

for $i, j \in \{1,2\}$, $i \neq j$.

4 The n-player case

In this section we want to prove existence of an EUA for the case of n players. Compared to the two-player case, the main additional problem encountered is the question of independence of beliefs about the opponents' strategies. In game theory without strategic uncertainty players are assumed to choose mixed strategies $\sigma = (\sigma_1, ..., \sigma_n) \in \underset{k=1}{\overset{n}{\times}} \Delta(S_k)$ independently. Hence, player i believes that the probability of facing a particular pure strategy combination of the opponents $\widetilde{s}_{-i} \in S_{-i}$ is $\prod_{k \in I,\ k \neq i} \sigma_k(\widetilde{s}_k)$, where $\widetilde{s}_k$ is the strategy of player k in the strategy combination $\widetilde{s}_{-i}$. Proceeding in this way assumes not only that players believe that opponents choose their mixed strategies independently but also that all players agree in their beliefs upon the probabilities with which a particular player i chooses pure strategies.

These strong assumptions about independence of and mutual agreement on beliefs of the opponents stand in strong contrast to the assumption of ambiguity about the opponents' strategy choice. Relaxing these assumptions would make it impossible to compare EUA with nash equilibrium when ambiguity vanishes. Equilibrium under Ambiguity with belief functions offers an intermediate way to deal with this problem: one can maintain independence and mutual agreement for singleton events, yet allow for ambiguity about this independence for non singleton events. In this way, one can approximate Nash equilibria for vanishing ambiguity. It is this intermediate approach we will choose in this section.

For each player $i \in I$, let $\delta_i : N(\Sigma_{-i}) \to [0, 1]$ be player i's ambiguity about the play of the opponents and let $\overline{\delta}_i := \sum_{E \in N(\Sigma_{-i})} \delta_i(E)$ be the aggregate ambiguity about the opponents' strategy choice. Given $\delta = (\delta_1, ..., \delta_n)$ and a *common vector of probability distributions* over players' pure strategies, $\sigma = (\sigma_1, ..., \sigma_n) \in \underset{i \in I}{\times} \Delta(S_i)$, define $\gamma_i(\sigma, \delta_i)$ on Σ_{-i} by

$$\gamma_i(\sigma, \delta_i)(E) := \begin{cases} \delta_i(E) & \text{for} \quad E \in N(\Sigma_{-i}) \\ (1 - \overline{\delta}_i) \prod_{k \in I,\ k \neq i} \sigma_k(\widetilde{s}_k) & \text{for} \quad \widetilde{s}_{-i} \in S_{-i} \end{cases} .$$

For $\overline{\delta}_i < 1$, $\gamma_i(\sigma, \delta_i) \in \Delta(\Sigma_{-i})$, i.e., $\gamma_i(\sigma, \delta_i)$ is the Möbius transform of a belief function $\phi_i^{\gamma_i(\sigma,\delta_i)}$, and one can define the Choquet expected payoff as in Proposition 2.1:

$$V_i(s_i, \alpha_i, \phi_i^{\gamma_i(\sigma,\delta_i)}) = \sum_{E \subseteq S_{-i}} \gamma_i(\sigma, \delta_i)(E) V_i^{\alpha}(s_i, E).$$

The Choquet expected payoff of a pure strategy can be seen as a continuous function of σ. Hence, we can apply well-known arguments to prove the following proposition.

Proposition 4.1 *Suppose $\overline{\delta}_i < 1$ for all $i \in I$. Then there is $\widehat{\sigma} = (\widehat{\sigma}_1, ..., \widehat{\sigma}_n) \in \times_{i \in I} \Delta(S_i)$ such that the belief functions $(\phi_1^{\gamma_1(\widehat{\sigma},\delta_1)}, ..., \phi_n^{\gamma_n(\widehat{\sigma},\delta_n)})$ defined by the Möbius transforms $(\gamma_1(\widehat{\sigma}, \delta_1), ..., \gamma_n(\widehat{\sigma}, \delta_n))$ are an EUA.*

Proof. See the appendix. ■

One can show that the endogenously derived beliefs of an EUA about the singletons $\widehat{\sigma}$ are a uhc correspondence of the exogenous ambiguity of all players $(\delta_1, .., \delta_n)$. Hence, any sequence $\widehat{\sigma}^{\nu}$ induced by a converging sequence $(\delta_1^{\nu}, .., \delta_n^{\nu}) \to (\delta_1^{o}, .., \delta_n^{o})$ will converge. Because the endogenous part of the players' belief functions $\phi_i^{\gamma_i(\sigma,\delta_i)}$ are based on the common independent beliefs σ, one obtains a Nash equilibrium when $\overline{\delta}_i \to 0$ for all players $i \in I$.

5 Concluding comments

In this paper, we have studied EUA for finite games when beliefs are represented by belief functions ϕ_i^{γ}. For belief functions, one can prove existence of an EUA for any degree of optimism and pessimism and arbitrary ambiguity given by the non-singleton events of the Möbius transform. The generality of the result allows for a wide range of applications in game theory and economics. We could also demonstrate by examples that EUA can characterize reasonable behavior which cannot be derived in a Nash equilibrium. The possibility to differentiate the ambiguity attached to specific sets of the opponents' strategies can also be used to model observe biases in strategic interaction which cannot be deduced from global ambiguity. EUA with belief functions, therefore, open the analysis of behavior under strategic uncertainty to many applications in theoretical and experimental economics.

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6 Appendix

6.1 Proof of Proposition 4.1

Proposition 4.1: Suppose $\overline{\delta}_i < 1$ for all $i \in I$. Then there is $\widehat{\sigma} = (\widehat{\sigma}_1, ..., \widehat{\sigma}_n) \in \underset{i \in I}{\times} \Delta(S_i)$ such that the belief functions $(\phi_1^{\gamma_1(\widehat{\sigma},\delta_1)}, ..., \phi_n^{\gamma_n(\widehat{\sigma},\delta_n)})$ defined by the Möbius transforms $(\gamma_1(\widehat{\sigma}, \delta_1), ..., \gamma_n(\widehat{\sigma}, \delta_n))$ are an EUA.
Proof. Fix $(\alpha_1, ..., \alpha_n)$ and $(\delta_1, ..., \delta_n)$. For $\sigma = (\sigma_1, ..., \sigma_n) \in \underset{i \in I}{\times} \Delta(S_i)$, define $\gamma_i(\sigma, \delta_i)$ on Σ_{-i} by

$$\gamma_i(\sigma, \delta_i)(E) := \begin{cases} \delta_i(E) & \text{for} \quad E \in N(\Sigma_{-i}) \\ (1 - \overline{\delta}_i) \prod\limits_{k \in I,\ k \neq i} \sigma_k(\widetilde{s}_k) & \text{for} \quad \widetilde{s}_{-i} \in S_{-i} \end{cases}.$$

By the premise $\overline{\delta}_i < 1$ for all $i \in I$. Hence, $\sum\limits_{E \in \Sigma_{-i}} \gamma_i(\sigma, \delta_i)(E) = 1$ and $\phi_i^{\gamma_i(\sigma,\delta_1)}$ is a belief function for all $I \in I$. By Proposition 2.1, the Choquet expected payoff of a pure strategy $s_i \in S_i$ is

$$V_i(s_i, \alpha_i, \phi_i^{\gamma_i(\sigma,\delta_i)}) = \sum_{E \in \Sigma_{-i}} \gamma_i(\sigma, \delta_i)(E) V_i^{\alpha}(s_i, E).$$

For all $i \in I$, define $W_i : \underset{i \in I}{\times} \Delta(S_i) \to \mathbb{R}$ as

$$W_i(\sigma) := \sum_{s_i \in S_i} \sigma_i(s_i) V_i(s_i, \alpha_i, \phi_i^{\gamma_i(\sigma,\delta_i)}).$$

$W_i(\sigma)$ is a continuous function on $\underset{i \in I}{\times} \Delta(S_i)$ and linear in σ_i. Let

$$B_i(\sigma) := \arg \max_{q_i \in \Delta(S_i)} W_i(\sigma)$$

be the correspondence of maximizers of $W_i(\sigma)$ with respect to σ_i. By the maximum theorem the correspondence $B_i : \Delta(S_1) \times ... \times \Delta(S_n) \to \Delta(S_i)$ is non-empty, uhc and convex-valued since $W_i(\sigma)$ is a continuous function on the compact set $\underset{i \in I}{\times} \Delta(S_i)$ and linear in σ_i. The Cartesian product of these correspondence $B(\sigma) := B_1(\sigma) \times ... \times B_n(\sigma)$ is a correspondence $B : \Delta(S_1) \times ... \times \Delta(S_n) \to \Delta(S_1) \times ... \times \Delta(S_n)$ which inherits the properties of its components. Hence, all conditions of Kakutani's fixed point theorem are satisfied and there exists $\widehat{\sigma} \in B(\widehat{\sigma})$.

We claim that the belief functions $(\phi_1^{\gamma_1(\widehat{\sigma},\delta_1)}, ..., \phi_n^{\gamma_n(\widehat{\sigma},\delta_n)})$ defined by the Möbius transforms $(\gamma_1(\widehat{\sigma}, \delta_1), ..., \gamma_n(\widehat{\sigma}, \delta_n))$ are an EUA.

Firstly, by Lemma 2, for all $i \in I$, $s_{-i} \in \operatorname{supp} \phi_i^{\gamma_i(\widehat{\sigma},\delta_i)}$ iff $\gamma_i(\widehat{\sigma}, \delta_i)(\{s_{-i}\}) > 0$. Since $\operatorname{supp} \widehat{\sigma}_k \neq \varnothing$ for all $k \in I$, $\operatorname{supp} \left(\prod\limits_{k \in I,\ k \neq i} \sigma_k \right) \neq \varnothing$, i.e., there exists $s_{-i} \in$

$\operatorname{supp}\left(\prod_{k\in I,\ k\neq i}\widehat{\sigma}_k\right)$ with $\prod_{k\in I,\ k\neq i}\widehat{\sigma}_k(s_{-i}) > 0$. Hence, there is $s_{-i} \in S_{-i}$ such that $\gamma_i(\widehat{\sigma},\delta_i)(\{s_{-i}\}) = (1-\overline{\delta}_i)\prod_{k\in I,\ k\neq i}\widehat{\sigma}_k(s_k) > 0$.

Secondly, take s_{-i} such that $\gamma_i(\widehat{\sigma},\delta_i)(\{s_{-i}\}) > 0$ and consider any component s_k of the strategy combination s_{-i}. Then $\widehat{\sigma}_k(s_k) > 0$ must be true. Since $\widehat{\sigma}_k \in B_k(\widehat{\sigma})$ and $W_{ik}(\sigma)$ is linear in σ_k, $\widehat{\sigma}_k(s_k) > 0$ implies

$$V_k(s_k,\alpha_k,\phi_k^{\gamma_k(\widehat{\sigma},\delta_k)}) \geq V_k(s'_k,\alpha_k,\phi_k^{\gamma_k(\widehat{\sigma},\delta_k)})$$

for all $s'_k \in S_k$. Hence, $s_k \in R_k(\phi_k^{\gamma_k(\widehat{\sigma},\delta_k)}) := \operatorname{argmax}_{s'_k\in S_k} V_k(s'_k,\alpha_k,\phi_k^{\gamma_k(\widehat{\sigma},\delta_k)})$ and

$$\varnothing \neq \operatorname{supp}\phi_i^{\gamma_i(\widehat{\sigma},\delta_i)} \subseteq \mathop{\times}_{k\neq i} R_k(\phi_k^{\gamma_k(\widehat{\sigma},\delta_k)}).$$

■