Warrant Exercise and Bond Conversion in Large Trader Economies

by

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Abstract

It is well known that the sequential (premature) exercise of American-type warrants *may* be advantageous for large warrantholders, even in the absence of regular dividends, because using exercise proceeds to expand the firm's scale increases the riskiness of an equity share. We show that for realistic interest rate levels even large warrantholders are better off not to exercise prematurely. This result, however, does not justify in general the simplifying restriction that warrants or convertible securities are valued as if exercised as a block. We show that the option to exercise only a *fraction* of the outstanding convertibles at the maturity date (*partial exercise option*) has a positive value if and only if the firm has debt in its capital structure and there is at least one large warrantholder. Moreover, we show that there is not only a gain from hoarding American-type warrants but also a gain from hoarding European-type warrants in the presence of at least two large warrantholders.

Key words: Warrants, Convertible Bonds, Large Trader, Sequential Exercise, Partial Exercise Option

JEL: C72, G12, G32

Contents

Warrants, unlike call options, are issued by companies and when exercised new shares are created with the exercise proceeds increasing the firm's assets. Because of this, there is some dilution of equity and dividend when warrants are exercised. Therefore the value accruing to one warrantholder is not independent of what other warrantholders do. Under certain conditions, the premature exercise of a warrant can increase the value of the warrants that remain outstanding, because using exercise proceeds to expand the firm's scale increases the riskiness of an equity share. Emanuel (1983), Constantinides (1984) and Constantinides and Rosenthal (1984) demonstrate the potential advantage of a sequential exercise strategy assuming a firm without (senior) debt. All these papers compare a sequential exercise strategy with an exercise strategy, called block exercise, where all warrantholders completely exercise their warrants simultaneously or not at all. Emanuel (1983) studies the monopolistic case, whereas Constantinides and Rosenthal (1984) focus on pricetaking warrantholders. Constantinides (1984) shows that the warrant price in a competitive equilibrium is smaller than or equal to the warrant price under the block exercise constraint, if all projects of the firm have a zero net present value and the firm pays dividends and coupons. In the absence of dividend payments, Cox and Rubinstein (1985) and Ingersoll (1987) demonstrate that a sequential exercise policy is never optimal for a pricetaker, while it can be beneficial to a monopoly warrantholder. Spatt and Sterbenz (1988) generalize this result to oligopoly warrantholders and show that there are reinvestment policies of the firm for which sequential exercise is not beneficial to non-pricetaking warrantholders. Their analysis helps to justify the frequent simplifying restriction that warrants or convertible securities are valued as if exercised as a block. Articles on warrant valuation which rely on the reasonableness of block exercise include Ingersoll (1977), Brennan and Schwartz (1977, 1980), Schulz and Trautmann (1994), and Crouhy and Galai (1994).

Unfortunately, the analysis of Spatt and Sterbenz (1988) (like that of Emanuel, 1983, and Constantinides, 1984) is also restricted to a firm without senior debt in its capital structure. However, the *existence of senior debt* causes a *positive value* for the option to exercise only a fraction of the outstanding warrants at maturity in *large trader economies*. For competitive markets, Bühler and Koziol (2002) have demonstrated that allowing senior debt in the capital structure causes a partial conversion of convertible bonds to be optimal. This result is primarily driven by a wealth transfer from the stockholders to the (senior) debtholders. Both the values of common stock and the values of the senior debt can differ for block conversion as well as partial conversion. However, the value of the convertible bond is never below the corresponding value in the block conversion case (and above only in case of premature exercise due to dividend payments). Koziol (2003) extends these results for convertible bonds with conversion strategies in a monopoly while Koziol (2006) examines exercise strategies for warrants in a competitive market.

This paper extends the analysis of Koziol (2006) to large trader economies. We present and compare exercise strategies and the corresponding warrant values for a competitive economy (with only pricetakers), an economy with one large trader and a competitive fringe, an economy consisting only of two large traders, and a monopoly. We show that for realistic interest rate levels it is not optimal to exercise long-lived warrants sequentially, if the firm uses the exercise proceeds to rescale its investment. Therefore, it turns out that from a theoretical perspective the potential advantage of sequential exercise strategies is not the main obstacle against the use of the block exercise assumption. The latter assumption, however, is questionable on the ground that it may be advantageous not to exercise all warrants if they finish in the money. It turns out that partial exercise strategies — compared to block exercise strategies — are beneficial to all warrantholders if and only if at least one warrantholder is a non-pricetaker. The warrant values increase with the concentration of the warrant ownership distribution in the economy. Moreover, we show that there is not only a gain from hoarding American-type warrants caused by the sequential exercise option but also from hoarding European-type warrants due to the partial exercise option if there are at least two non-pricetaking warrantholders.

The partial exercise option of warrants has the same value as the partial conversion option of convertible bonds in case of European-type convertibles. In the absence of dividend payments and coupon payments the value of American-type convertible bonds equals the value of European-type convertible bonds since converting such bonds prematurely does not change the firm's value. Therefore, we analyse the value of the partial exercise option in case of warrants and compare it later on with the more special case of convertible bonds.

The paper is organized as follows: In Section 1 we specify the model and define the different exercise policies. Section 2 looks at the partial exercise policies of European-type warrants and compares the warrant prices with block exercise constraint to the ones without it. Section 3 examines the optimality of sequential exercise strategies of American-type warrants under the firm policy that the exercise proceeds are used to rescale the firm's investment. Section 4 summarises the results in case of convertible bonds. Section 5 concludes the paper. All technical proofs are given in Appendix A.

1 Model

We consider a firm with value V_t at time t following a Geometric Brownian Motion.[1] The firm is financed by issuing equity, warrants and debt and pays no regular

[1] However, most of the examples are given in a binomial setting.

dividends. Exercise proceeds are used to rescale the firm's investment. Furthermore we assume throughout the paper that there are no taxes or transaction costs, and no arbitrage opportunities in the project market. At $t = 0$ the warrantholders know the firm value V_0 and the parameters of the log-normal distribution of V_t at maturity T. The risk neutral probability measure is denoted by Q.

1.1 Capital structure

At time 0 the firm's equity consists of N outstanding shares and n warrants with maturity T and strike price K. Every warrant entitles its owner to get one share of common stock when exercising the warrant at times 0 and T (American-type warrant) or only at maturity (European-type warrrant). Senior debt is issued in the form of a zero coupon bond with a common face value of F and maturity T_D with $0 < T < T_D$. At $t \in [0, T_D]$ we denote the price of one stock by S_t, one warrant by W_t and the debt by D_t. According to Modigliani and Miller (1958) we assume that the firm value is equal to the value of all shares, all warrants, and total debt:[2]

$$V_t \;=\; NS_t + nW_t + D_t \qquad \text{for all } t \in [0, T)\,.$$

We denote the exercise policy of the warrantholders by $m \in [0, n]$ and the firm value immediately before time t by V_{t^-}. $\overline{S}_t$ denotes the total value of common stock. After the maturity of the warrants the firm value is

$$V_t = (N + m)S_t + D_t = \overline{S}_t + D_t \qquad \text{for all } t \in [T, T_D)\,.$$

If at time T_D the firm value is less than the face value of the debt (i.e. $V_{T_D} \leq F$), a default occurs and the stocks get worthless, i.e. $S_{T_D} = 0$ and $D_{T_D} = V_{T_D}$. Otherwise the common stock equals the firm value minus the face value of the debt, so we get the equation

$$V_{T_D} = \overline{S}_{T_D} + \min\{F; V_{T_D}\} \qquad \text{for all } t \in [T, T_D)\,. \tag{1}$$

If at time t the warrantholders exercise m warrants, the firm value increases to $V_t = V_{t^-} + mK$ and the firm uses the exercise proceeds mK to *rescale the firm's investment*. According to equation (1) the value of the total common stock $\overline{S}_t \equiv \overline{S}_t(V_t)$ equals the value of a call option on the firm value V_t with maturity T_D and strike price F at time $t \in [T, T_D)$. Since V_t follows a geometric Brownian motion, $\overline{S}_t$ behaves similarly as the *Black/Scholes*-value of a European call option does, where the firm value includes the exercise proceeds. For all $V \in \mathbb{R}_+$ we have $\Delta_T(V) = \partial\overline{S}_T(V)/\partial V \in (0, 1)$ and $\Gamma_T(V) = \partial^2\overline{S}_T(V)/\partial V^2 \geq 0$.

[2]This representation assumes that in $t = 0$ no warrant is exercised. Otherwise if m_0 warrants are exercised in $t = 0$ the number of stocks increases to $N + m_0$, the number of warrants decreases to $n - m_0$, and the firm value increases to $(V_0 + m_0K)V_t/V_0$.

1.2 Warrantholders and their payoff functions

The set of the warrantholders is denoted by I and P is a measure on I. Every warrantholder $i \in I$ holds n_i warrants with $\int_I n_i dP = n$. Furthermore, we assume that warrantholders do not own shares of common stock of the firm and that every warrantholder knows the number of warrants of each other warrantholder (complete information on the distribution of warrant ownership).[3]

European-type warrants

In the case of European-type warrants the set of strategies of warrantholder $i \in I$ are all possible exercise policies $m_i \in [0, n_i]$ at time T. The number of warrants exercised by all warrantholders is $m = \int_I m_i dP \in [0, n]$, while m_{-i} denotes the number of warrants exercised by all warrantholders except i with $m = m_i P(\{i\}) + m_{-i}$. We call warrantholder $i \in I$ a *pricetaker* if $P(\{i\}) = 0$, because the asset prices are independent of his trading and exercise policy (the latter does not affect the number of warrants exercised and therefore the asset prices). The payoff of warrantholder i is defined as the exercise value of warrants exercised by warrantholder i, i.e.[4]

$$\pi_i(m_i, m_{-i}, V_{T^-}) = m_i \left(\frac{\overline{S}_T(V_T)}{N + m} - K \right) .$$

As the payoff function of each pricetaking warrantholder i is a function which is *linear* in the number of warrants exercised by himself, his payoff function is maximised at $m_i = 0$ or $m_i = n_i$. Only if we have $\overline{S}_T(V_T) - (N + m)K = 0$, every exercise policy of i maximises his payoff.

In contrast to a pricetaking warrantholder we call warrantholder $A \in I$ with $P(\{A\}) = 1$ a *non-pricetaker*. His exercise policy influences the asset prices, in particular the stock price $\overline{S}_T(V_T)$ under all reinvestment policies of the firm, and his payoff function is defined by

$$\pi_A(m_A, m_{-A}, V_{T^-}) = m_A \left(\frac{\overline{S}_T(V_T)}{N + m_A + m_{-A}} - K \right) .$$

[3]If the warrantholders own shares of the common stock, the analysis follows the same lines, but the results are more complex. For the sake of simplicity we make the simplifying assumption. Incomplete information remains open for future research.

[4]Since rational warrantholders will choose $m_i = 0$ if $\overline{S}_T(V_T)/(N+m) - K < 0$, it is not necessary to denote the exercise value of one warrant by the positive part of this function.

American-type warrants

In case of American-type warrants we assume that at time $t = 0$ the warrantholders have two options: either they exercise warrants or they sell warrants.[5] We denote the sequential exercise strategy with m (the number of warrants exercised in $t = 0$), since the exercise strategy in $t = T$ is well known by the behavior of pricetakers. So when exercising $m \in [0, n]$ warrants (with immediate sales of the new stocks) and selling the remaining $n - m$ warrants to pricetakers, the payoff function of a pricetaking warrantholder $i \in I$ is

$$\pi_i^a(m_i, m, V_0) = m_i \left(S_0(V_0 + mK) - K\right) + (n_i - m_i) W_0(V_0 + mK)\,, \tag{2}$$

where V_0 and m denote the firm value at time $t = 0$ and the total number of warrants exercised at time $t = 0$, respectively, and S_0 and W_0 are the stock price and the warrant price in $t = 0$, if at the warrants' maturity date all warrantholders are pricetakers. The corresponding payoff function of a non-pricetaking warrantholder $A \in I$ reads now as follows (please recall that $S_0(V_0 + mK)$ and $W_0(V_0 + mK)$ depend on the exercise policy m_A):

$$\pi_A^a(m_A, m_{-A}, V_0) = m_A \left(S_0(V_0 + mK) - K\right) + (n_A - m_A) W_0(V_0 + mK)\,. \tag{3}$$

1.3 Block exercise, partial exercise and sequential exercise

Stock prices rationally reflect anticipation of the number of warrants exercised and the assumed use of the exercise proceeds. We distinguish between three kinds of exercise policies:

Definition 1 *Warrantholders follow a so-called* block exercise *strategy if the number of warrants exercised at the maturity date is given by*

$$m \;=\; \begin{cases} 0 & \text{for} \quad \frac{1}{N+n}\overline{S}_T(V_T) \;\in\; [0, K) \\ n & \text{for} \quad \frac{1}{N+n}\overline{S}_T(V_T) \;\in\; [K, \infty)\,. \end{cases}$$

Otherwise the warrantholders follow a so-called partial exercise *strategy at the maturity date, or they follow a so-called* sequential exercise *strategy if they exercise American-type warrants before maturity.*

[5] As it is well known, holders of American-type warrants have usually at every trading date three options: they can exercise, sell or hold the warrants. For the sake of tractability, we do not consider the latter option and assume that all non-pricetaker exit the warrant market at time $t = 0$. This simplified framework avoids a time-consuming numerical analysis to calculate the current values of stocks and warrants in dependence of the market structure. Furthermore, this is tantamount to consider only the warrantholders' real wealth in the spirit of Jarrow (1992).

Definition 2 *The* partial exercise option *is the option to follow a partial exercise strategy instead of a block exercise strategy. The* sequential exercise option *is the option to follow a sequential exercise strategy instead of a block exercise strategy.*

The value of a partial exercise option is the difference between the warrant price with partial exercise and the warrant price under the block exercise constraint. The value of a sequential exercise option is the difference between the warrant price with sequential exercise and the warrant price under the block exercise constraint.

We model the warrantholders' exercise behavior as a noncooperative game and consider a Nash equilibrium as an optimal exercise strategy for the warrantholders. The noncooperative game is defined by the set of warrantholders, the exercise policies as the strategies sets, and the payoff functions. While Constantinides (1984) and other authors analyse a zero-sum game between the warrantholders and the stockholders (as passive players), our game is not zero-sum (like in Bühler and Koziol (2002) and Koziol (2003, 2006)), because there is a wealth transfer from the stockholders and the warrantholders to the debtholders by the exercise of a warrant.

Definition 3 *In case of European-type warrants the exercise strategy $(m_i^*)_{i\in I}$ in time $t = T$ is a* Nash equilibrium *if for every warrantholder $i \in I$*

$$\pi_i(m_i^*, m_{-i}^*, V_{T^-}) \geq \pi_i(m_i, m_{-i}^*, V_{T^-}) \qquad \textit{holds for all } m_i \in [0, n_i]\,.$$

In case of American-type warrants the exercise strategy $(m_i^*)_{i\in I}$ in time $t = 0$ is a Nash equilibrium *if for every warrantholder $i \in I$*

$$\pi_i^a(m_i^*, m_{-i}^*, V_0) \geq \pi_i^a(m_i, m_{-i}^*, V_0) \qquad \textit{holds for all } m_i \in [0, n_i]\,.$$

In a Nash equilibrium each warrantholder takes the other warrantholders' exercise policy as given and maximises his payoff. We call a Nash equilibrium an optimal exercise strategy and we show that optimal exercise strategies always exist, although the latter may not be unique (e.g. if all warrantholders are pricetakers, the optimal exercise strategy is not unique). Nonetheless, the stock price and warrant price is unique for all optimal exercise strategies. So the value of a partial exercise option and a sequential exercise option is well defined.

2 Partial exercise of European-type warrants

We start our analysis with a useful lemma. It characterizes properties of a critical number of warrants exercised, the stock price as a function of the number of warrants exercised and the optimal exercise policy of a non-pricetaker, if the firm uses the warrant exercise proceeds to rescale the firm's project:

Lemma 1 *Let $\widehat{m} \geq 0$ denote the total number of warrants exercised such that the stock price equals the strike price, $\overline{S}_T(V_{T^-} + \widehat{m}K)/(N + \widehat{m}) = K$. Then the following three statements hold:*

(a) There exists at most one $\widehat{m} \geq 0$. For all $m < \widehat{m}$ the stock price is above the strike price and for all $m > \widehat{m}$ the stock price is below the strike price.

(b) The stock price as a function of the total number of warrants exercised is strictly decreasing and convex as long as the stock price is above the warrants' strike price.

(c) The payoff function of warrantholder A is quasi-concave with respect to the number of warrants exercised in the range $[0, \widehat{m} - m_{-A})$.[6]

The proof is given in Appendix A. We use statement (c) of Lemma 1 in the following way: If the marginal payoff $\partial \pi_A / \partial m_A = 0$ is zero at $m_A^* \in [0, \widehat{m} - m_{-A})$ then m_A^* maximises the payoff function (with $\widehat{m} = \infty$, if no critical number of warrants exercised $\widehat{m}$ exists). This statement is also proved in Appendix A.

2.1 Exercise policies in a competitive economy

In a competitive economy every warrantholder is a pricetaker. For the sake of consistency the measure of all pricetakers must be positive, e.g. $P(I) = 1$, whereas each single warrantholder has a measure of zero. From the linearity of all warrantholders' payoff functions we get directly the optimal exercise policy for all warrantholders:

Proposition 1 *If all warrantholders are pricetakers, then the following exercise strategy is a Nash equilibrium:*

$$m_i^* = \begin{cases} 0 & \text{for} \quad V_{T^-} \in [0, \underline{V}) \\ x_i^* & \text{for} \quad V_{T^-} \in [\underline{V}, \overline{V}) \\ n_i & \text{for} \quad V_{T^-} \in [\overline{V}, \infty) \end{cases} \qquad \text{for all } i \in I\,,$$

where $\underline{V}$ and $\overline{V}$ fulfill $\overline{S}_T(\underline{V})/N = K$ and $\overline{S}_T(\overline{V} + nK)/(N + n) = K$, respectively, and $x^* = \int_I x_i^* dP$ solves the equation

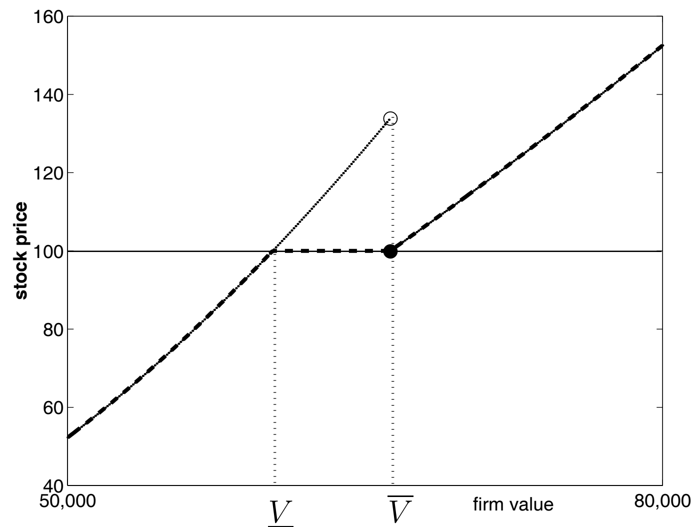
$$\frac{1}{N + x^*} \overline{S}_T \Big(V_{T^-} + x^* K \Big) = K\,. \tag{4}$$

[6] A function is called quasi-concave if the set of points for which the function takes on values greater than or equal to some arbitrary value comprises a convex set (Silberberg and Sun, 2001, p.139).

If the firm has no senior debt in its capital structure (i.e. $F = 0$), we get $\underline{V} = \overline{V}$ and the block exercise strategy is optimal. Furthermore, the optimal exercise strategy in Proposition 1 is not unique: Although equation (4) has a unique solution x^*, any exercise strategy $(x_i^*)_{i \in I}$ with $x^* = \int_I x_i^* dP$ is a Nash equilibrium.

Figure 1: **Stock price in a competitive economy**

The figure shows the stock price as a function of the firm value at time T in a competitive economy (dashed line) and under the block exercise constraint (dotted line). We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$. The critical firm values are $\underline{V} = 60,330.53$ and $\overline{V} = 66,258.47$.



According to this proposition, for $V_{T^-} \leq \underline{V}$ and $V_{T^-} \geq \overline{V}$ the optimal partial exercise policy equals the block exercise strategy

$$m_i = \begin{cases} 0 & \text{for} \quad V_{T^-} \in [0, \overline{V}) \\ n_i & \text{for} \quad V_{T^-} \in [\overline{V}, \infty) \end{cases} \quad \text{for all } i \in I\,.$$

If $V_{T^-} \in (\underline{V}, \overline{V})$, the stock price under the block exercise strategy is higher than the strike price according to Lemma 1, whereas in a competitive economy the warrantholders exercise so many warrants that the stock price equals the strike price in a Nash equilibrium (see Figure 1). If the stock price is higher than the strike price, warrantholders can increase their payoff by exercising more warrants. Nevertheless, the warrant price equals zero in both cases: Under the optimal partial exercise strategy the stock price equals the strike price, so that the warrantholders make no profit by exercising warrants, and under the block exercise strategy no warrant is exercised.

In Example 1 we (1) illustrate the optimal exercise policy in a competitive market, and (2) compare this policy to the optimal exercise policy in a monopolistic market. This example emphasizes the need for analysing the optimal exercise policies in large trader economies.

Example 1 We assume that in the interval $[T, T_D]$ the firm value V follows a simple binomial process where the firm value can increase or decrease by 50 % rather than a Geometric Brownian Motion. Furthermore, we assume an interest rate of zero percent such that the risk neutral probability for an increase or decrease of the firm value equals 0.5, respectively.

We assume that the firm has issued $N = 100$ shares of the common stock, $n = 100$ warrants with a strike price of $K = 100$ and a zero coupon bond with a face value of $F = 53,950$. At time T^- the firm value equals $V_{T^-} = 50,000$. Then for the firm value V_{T_D} the following two realisations are possible:

$$V_T = 50,000 + 100m \nearrow \quad \begin{aligned} V^u_{T_D} &= 75,000 + 150m \\ S^u_{T_D} &= \tfrac{1}{N+m}[V^u_{T_D} - F]^+ \\ &= \tfrac{1}{100+m}(21,050 + 150m) \end{aligned}$$

$$V_T = 50,000 + 100m \searrow \quad \begin{aligned} V^d_{T_D} &= 25,000 + 50m \\ S^d_{T_D} &= \tfrac{1}{N+m}[V^d_{T_D} - F]^+ = 0 \end{aligned}$$

At time T the stock price equals $S_T = 0.5 \cdot S^u_{T_D} = (10,525 + 75m)/(100 + m)$. According to Proposition 1 the optimal exercise policies in a competitive economy can be calculated by solving $S_T = K$. This results in $m^* = 21$, a stock price of $S_T = 100$ and an exercise value of the warrants of $W_T = 0$. If more warrants were exercised, the exercise value of the warrants would be negative, and if less warrants were exercised the exercise value of the warrants would be positive and every single pricetaking warrantholder would be better off exercising more warrants.

Now we assume that one monopolistic warrantholder A owns all warrants, so his payoff function and its first derivative with respect to the number of warrants exercised satisfy

$$\pi_A(m_A, V_{T^-}) \quad = \quad m_A \left(\frac{1}{100 + m_A}(10,525 + 75m_A) - K \right),$$

$$\frac{\partial}{\partial m_A}\pi_A(m_A, V_{T^-}) \;=\; \frac{1}{(100+m_A)^2}\left(-25m_A^2 - 5{,}000m_A + 52{,}500\right) ,$$

respectively. Warrantholder A maximises his payoff by exercising $m_A^* = 10$ warrants. Then the stock price equals $S_T = 102.5$ and the exercise value of the warrants equals $W_T = 2.5$.

2.2 Exercise policies in large trader economies

Example 1 demonstrates that the exercise value in a monopoly can differ from the exercise value in a competitive economy. This section compares the exercise policies of two other large trader economies with the one in the competitive economy. First we assume one non-pricetaking warrantholder and a competitive fringe and then an economy consisting of exactly two large traders.

Exercise policies when one non-pricetaker exists

First we look at a market structure with exactly one large warrantholder $A \in I$. Again, let P be the measure on the set of warrantholders with $P(\{A\}) = 1$ and $P(\{i\}) = 0$ for all $i \in I, i \neq A$. Non-pricetaker A owns $n_A \in (0, n]$ warrants and the pricetaking warrantholders $n_{-A} < n$. Please note that the monopoly is a special case of this economy with $n_A = n$ and $n_{-A} = 0$. The number of warrants exercised by all pricetakers is denoted by $m_{-A}^* = \int_{I\setminus\{A\}} m_i^* dP$ so that the total number of warrants exercised satisfies $m^* = m_A^* + m_{-A}^*$.

Proposition 2 *(a) In the presence of one non-pricetaker the following strategy is a Nash equilibrium:*

$$(m_A^*, m_{-A}^*) \;=\; \begin{cases} (0,0) & \text{for} \quad V_{T^-} \in [0, \underline{V}) \\ (0, x_{-A}^*) & \text{for} \quad V_{T^-} \in [\underline{V}, \underline{V}_A) \\ (x_A^*, n_{-A}) & \text{for} \quad V_{T^-} \in [\underline{V}_A, \overline{V}_A) \\ (n_A, n_{-A}) & \text{for} \quad V_{T^-} \in [\overline{V}_A, \infty) \end{cases}$$

where $\underline{V}$ *solves* $\overline{S}_T(\underline{V}) = NK$, $\underline{V}_A$ *solves* $\overline{S}_T(\underline{V}_A + n_{-A}K) = (N + n_{-A})K$ *and* $\overline{V}_A$ *solves* $\partial\pi_A(n_A, n_{-A}, \overline{V}_A)/\partial m_A = 0$. *The exercise policies* x_{-A}^*, x_A^* *are the solutions of*

$$\begin{aligned} \frac{1}{N + x_{-A}^*}\overline{S}_T\Big(V_{T^-} + x_{-A}^* K\Big) \;&=\; K \\ \frac{\partial}{\partial m_A}\pi_A(x_A^*, n_{-A}, V_{T^-}) \;&=\; 0\,, \end{aligned}$$

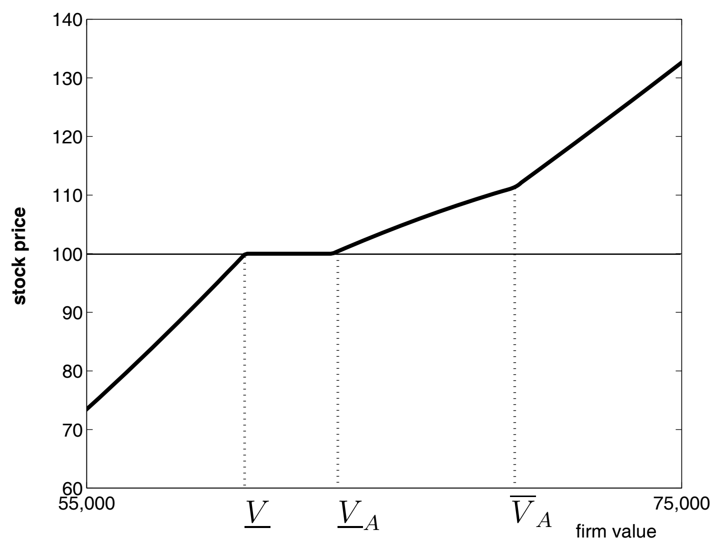
respectively.

(b) *Let m^* be the optimal number of warrants exercised in a competitive market. Then for all $V_{T^-} \in (\underline{V}_A, \overline{V}_A)$ we have $m^*_A + m^*_{-A} < m^*$ and for all $V_{T^-} \notin (\underline{V}_A, \overline{V}_A)$ we have $m^*_A + m^*_{-A} = m^*$.*

The proof is given in Appendix A. Without senior debt in the firm's capital structure we get $\underline{V} = \underline{V}_A = \overline{V}_A$ and the block exercise strategy is optimal.

Figure 2: **Stock price in an economy with one large trader**

The figure shows the stock price as a function of the firm value at time T in an economy with one large trader and a competitive fringe. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$, $n_{-A} = 40$ and $K = 100$. The critical firm values are $\underline{V} = 60,330.53$, $\underline{V}_A = 63,225.29$ and $\overline{V}_A = 69,372.27$.



According to part (b) of Proposition 2 less warrants are exercised in the presence of one large trader than in a competitive economy and therefore — according to Lemma 1 — the stock price is higher in the presence of one large trader. The optimal exercise policy of the pricetakers is to exercise all their warrants if the stock price exceeds the strike price. Although all pricetakers would benefit if they exercise less warrants, every warrantholder wants to be a free rider and exercises as many warrants as possible without incuring a loss. Therefore the stock price can only be above the strike price if all pricetakers exercise all their warrants. This holds for all firm values $V_{T^-} > \underline{V}_A$ (see Figure 2).

In contrast to a pricetaker, a non-pricetaker can increase the stock price through his exercise policy, increasing the exercise value of the warrants, and increasing his payoff. If $V_{T^-} \in (\underline{V}_A, \overline{V}_A)$ the non-pricetaker is better off when exercising less warrants than pricetakers would in a competitive economy. The higher exercise value of the warrants exercised compensates the lower number of warrants exercised.

Exercise policies when two non-pricetakers exist

We now assume a market structure with two non-pricetaking warrantholders and without a competitive fringe.[7] The two non-pricetakers b and B (i.e. $I = \{b, B\}$) own n_b and n_B warrants with $n_b + n_B = n$ where $n_b \leq n_B$. The optimal exercise policies of the two non-pricetakers are given in Proposition 3. Furthermore we compare the optimal exercise policy in this economy with the optimal exercise policy in an economy with only one large trader and a competitive fringe and in a monopoly.

Proposition 3 *(a) In the presence of two non-pricetakers, the following strategy is a Nash equilibrium:*

$$(m_b^*, m_B^*) = \begin{cases} (0,0) & \text{for} \quad V_{T^-} \in [0, \underline{V}) \\ (x^*, x^*) & \text{for} \quad V_{T^-} \in [\underline{V}, \overline{V}_b) \\ (n_b, x_B^*) & \text{for} \quad V_{T^-} \in [\overline{V}_b, \overline{V}_B) \\ (n_b, n_B) & \text{for} \quad V_{T^-} \in [\overline{V}_B, \infty) \end{cases}$$

where $\underline{V}$ solves $\overline{S}_T(\underline{V}) = NK$, $\overline{V}_b$ solves $\partial\pi_b(n_b, n_b, \overline{V}_b)/\partial m_b = 0$ and $\overline{V}_B$ solves $\partial\pi_B(n_B, n_b, \overline{V}_B)/\partial m_B = 0$ and x^* and x_B^* solve the equations

$$\begin{aligned} \frac{\partial}{\partial m_b}\pi_b(x^*, x^*, V_{T^-}) &= 0 \\ \frac{\partial}{\partial m_B}\pi_B(x_B^*, n_b, V_{T^-}) &= 0\,, \end{aligned}$$

respectively.

(b) Let (m_A^*, m_{-A}^*) be the optimal exercise strategy in the presence of one non-pricetaker and $n_{-A} = n_b$. For all $V_{T^-} \in (\underline{V}, \overline{V}_b)$ we have $m_b^* = m_B^* < m_{-A}^*$, $m_b^* = m_B^* > m_A^*$ and $m_b^* + m_B^* < m_A^* + m_{-A}^*$ and for all $V_{T^-} \notin (\underline{V}, \overline{V}_b)$ we have $m_b^* + m_B^* = m_A^* + m_{-A}^*$.

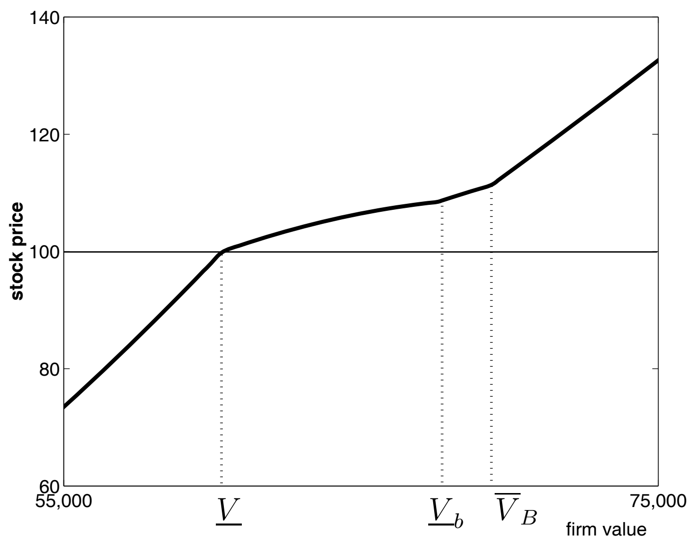
[7] Without loss of generality we can omit a competitive fringe, since a large trader will only exercise some warrants if the pricetaking warrantholders exercise all their warrants — see Proposition 2.

(c) Let $n_A = n$ and m_A^* be the optimal exercise policy in a monopoly. For all $V_{T^-} \in (\underline{V}, \overline{V}_A)$ we have $m_A^* < m_b^* + m_B^*$ and for all $V_{T^-} \notin (\underline{V}, \overline{V}_A)$ we have $m_A^* = m_b^* + m_B^*$.

The proof is given in Appendix A. Without senior debt in the firm's capital structure we get $\underline{V} = \overline{V}_b = \overline{V}_B$ and the block exercise strategy is optimal. Proposition 3 can be generalised to a market structure with any number of non-pricetakers and in combination with Proposition 2 to a market structure with any number of non-pricetakers and a competitive fringe.

Figure 3: **Stock price in an economy with two large traders**

The figure shows the stock price as a function of the firm value at time T in an economy with two large traders. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$, $n_b = 40$ and $K = 100$. The critical firm values are $\underline{V} = 60,330.53$, $\overline{V}_b = 67,581.81$ and $\overline{V}_B = 69,372.27$.



Surprisingly, warrantholder B exercises as many warrants as warrantholder b if $V_{T^-} \in [\underline{V}, \overline{V}_b)$, although he owns more warrants. This is due to the fact that the payoff function of a non-pricetaker does not depend on the total number of warrants he holds. So if an optimal exercise policy is an inner solution for one warrantholder, the same exercise policy is optimal for another (non-pricetaking) warrantholder even if he holds a different number of warrants.

Since all non-pricetakers exercise their warrants strategically, two non-pricetakers exercise less warrants than one non-pricetaker plus a competitive fringe if the latter owns as many warrants as one of the two non-pricetakers. Thus the stock price and the warrant price are higher. On the other hand, if only one monopoly warrantholder exists, his payoff must be at least as high as the added payoff of two non-pricetakers. For some firm values a monopoly warrantholder can increase the stock price, the exercise value of the warrants and his payoff by exercising less warrants than in the situation with two non-pricetaking warrantholders. Since the competitive economy is one extreme with the lowest stock price, the monopoly is the other extreme with the highest stock price.

2.3 Comparison of exercise policies

Figure 4 illustrates the differences of optimal exercise policies and their corresponding exercise values due to four different market structures. According to the figure in panel A, 100% of the outstanding warrants will be exercised in a competitive market at the critical firm value $\overline{V} = 66,258.47$ (the same percentage as with the block exercise strategy) while only a percentage between 40 and 66 will be exercised in the three large trader economies for the same firm value. The figure in panel B confirms, first of all, the well-known fact that there is no difference between warrant values in a competitive economy and a block exercise-constrained economy although the optimal exercise strategy in a competitive market deviates from the block exercise strategy. Moreover, this figure demonstrates that an increasing concentration of the warrant ownership distribution may lead to substantially higher exercise values of the outstanding warrants.

Statement (b) of Proposition 2 and statements (b) and (c) of Proposition 3 result in the following

Proposition 4 *The partial exercise option has a positive value if and only if the firm has senior debt in its capital structure and there is at least one non-pricetaking warrantholder.*

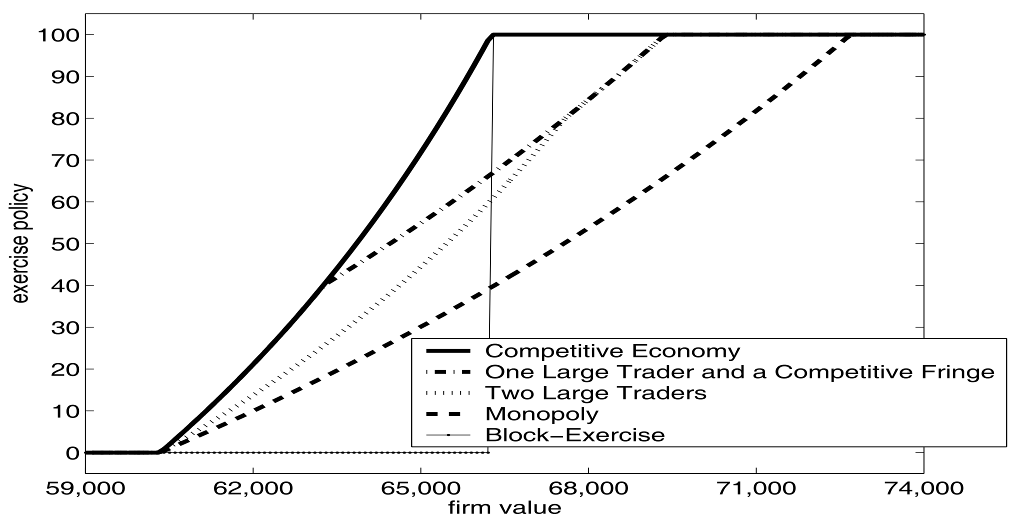
As the warrant price in a competitive market equals the warrant price under the block exercise constraint, we will compare this price to the warrant price in the presence of a monopoly warrantholder to see the maximum price impact of the partial exercise option. Figure 5 illustrates the absolute and the relative price differences in example.

Since at maturity the prices differ only if $V_{T^-} \in (\underline{V}, \overline{V}_A)$, the price difference decreases as the probability $Q(\{V_{T^-} \in (\underline{V}, \overline{V}_A)\})$ decreases. This is shown in Figure 5 Panel A. On the other hand a warrant price in the presence of a non-pricetaker

Figure 4: **Exercise policies and exercise values**

The figure shows the exercise rate of all players as a function of the firm value and the exercise value of a warrant as a function of the firm value at time T. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$, $n_{-A} = n_b = 40$ and $K = 100$. The critical firm values are $\underline{V} = 60,330.53$ and $\overline{V} = 66,258.47$.

Panel A: **Optimal exercise policies**



Panel B: **Exercise values of European-type warrants**

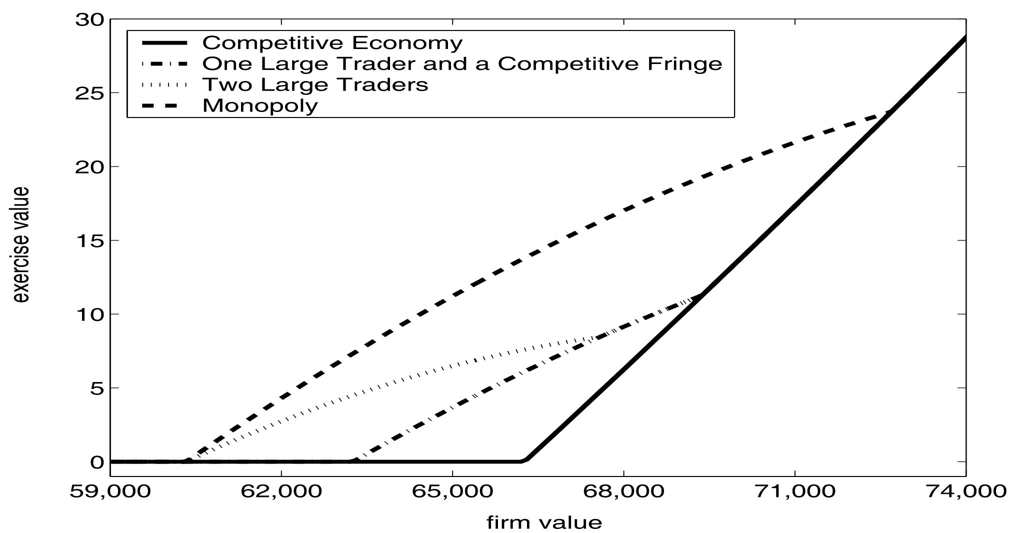
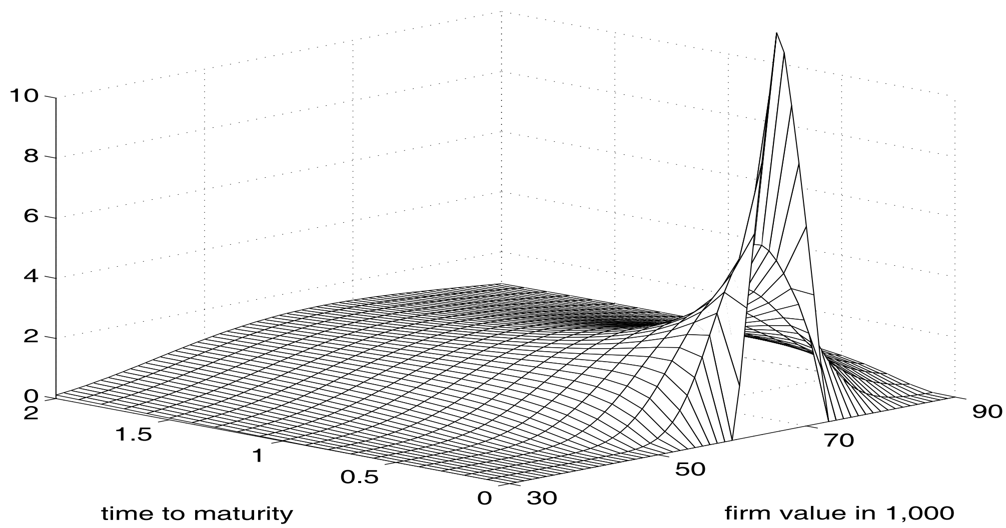


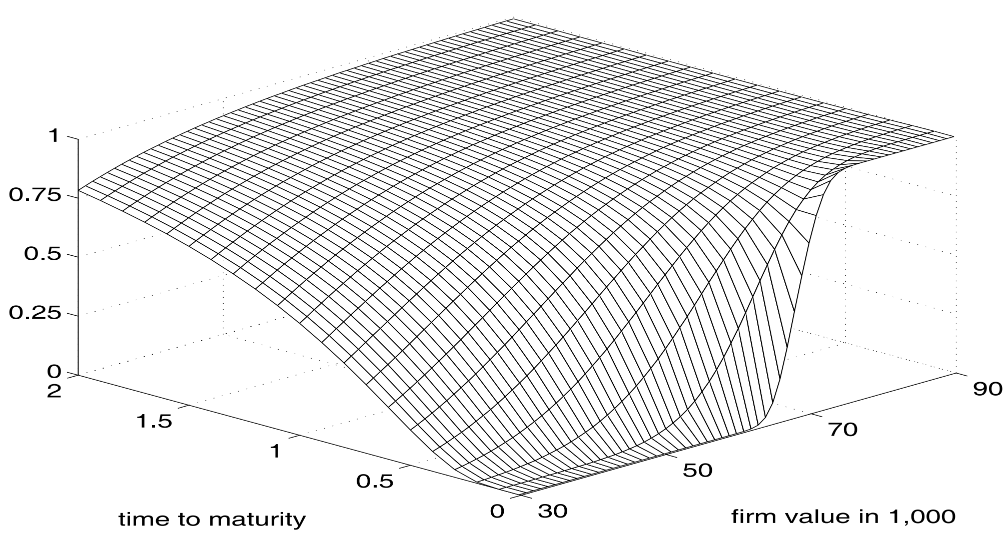
Figure 5: **Monopoly versus block exercise constraint**

The figures show the absolute and relative differences between warrant prices under the block exercise constraint and in a monopoly market. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$. The critical firm value is $\underline{V}_T = 60,330.53$.

Panel A: Absolute price difference $\left(W^{mono} - W^{block}\right)$



Panel B: Relative price difference $\left(W^{block}/W^{mono}\right)$



is strictly positive at time T if $V_{T^-} \geq \underline{V}$, whereas a warrant price under the block exercise constraint is strictly positive, if $V_{T^-} \geq \overline{V}$. Therefore the warrant price in the presence of a monopoly warrantholder W_t^{mono} is bounded by

$$W_t^{mono} \geq e^{-r(T-t)} E_Q \left[W_T^{mono} \cdot 1_{\{V_{T^-} \in (\underline{V}, \overline{V})\}} \right] + W_t^{block}$$

where W_t^{block} is the warrant price under the block exercise constraint. So if the warrant is out-of-the-money under the block exercise constraint ($W_t^{block} \approx 0$) and the probability $Q(\{V_{T^-} \in (\underline{V}, \overline{V})\})$ is sufficiently high, the warrant price in the presence of a monopoly warrantholder is higher than the warrant price under the block exercise constraint. This is shown in Figure 5 Panel B.

2.4 Gains from hoarding European-type warrants

We now approach the question of how a warrantholder can arise a monopoly position or how a firm can eliminate the gains from hoarding warrants. Ingersoll (1987) and Spatt and Sterbenz (1988) answered this question for a firm *without* debt in its capital structure and outstanding *American-type* warrants: The advantage from hoarding American-type warrants is that sequential exercise policies can be beneficial for large traders. Extending the quoted literature (see also Cox and Rubinstein, 1985) we show that there is also a gain from hoarding *European-type* warrants because also partial exercise strategies can be optimal in large traders economies. Unfortunately only large traders will sell their warrants to a potential monopolist so that a large trader economy equals an economy with one large trader and a competitive fringe.

A non-pricetaker (or a potential non-pricetaker) cannot buy a sufficient number of warrants from pricetakers. An offer from the non-pricetaker to buy a certain number of warrants is always rejected by the pricetakers for the following reason: the offered price is smaller than the present value of a warrant if the offer when accepted does not lead to a negative net-present-value for the non-pricetaker (recall that a non-pricetaker would only exercise a fraction of the warrants he could buy). This is due to the fact that the pricetaker's decision has no impact on the stock price and therefore on the warrants' exercise value. So every pricetaker wants to be a free rider, and therefore no pricetaker will sell his warrants to the non-pricetaker. Also no non-pricetaker will sell his warrants to pricetakers, because the present value of a warrant will decrease if he does.

In the presence of *two* non-pricetakers, one non-pricetaker will always sell his warrants to the other, as they will both profit from the additional value due to the merger of their position (see statement (c) in propositon 3). Unfortunately, this argumentation does not hold in the presence of three or more non-pricetakers, since also non-pricetakers like to be free riders. This is shown in Example 2:

Example 2 As in Example 1 we assume that the firm value follows a binomial process where the firm value can increase or decrease by 25%. At time T^- the firm value equals $V_{T^-} = 60,000$. Furthermore, we assume an interest rate of 5% so that the risk neutral probability for an increase of the firm value equals $q = ((1+0.05) - 0.75)/(1.25 - 0.75) = 0.6$. The firm has issued debt with a face value of $54,000$, 100 stocks, and 100 warrants with a strike price of 100. Each of the warrantholders A, B, C holds 20 warrants while the remaining warrants are held by pricetakers (the pricetakers' payoffs are considered as one entity). We compute the optimal exercise policies in this example with the algorithm given in Appendix B. Without any trade warrantholders' payoffs are as follows:

	All pricetakers	A	B	C
Exercise policy	40.00 of 40	7.41 of 20	7.41 of 20	7.41 of 20
Stock price	101.37	101.37	101.37	101.37
Payoff	54.71	10.13	10.13	10.13

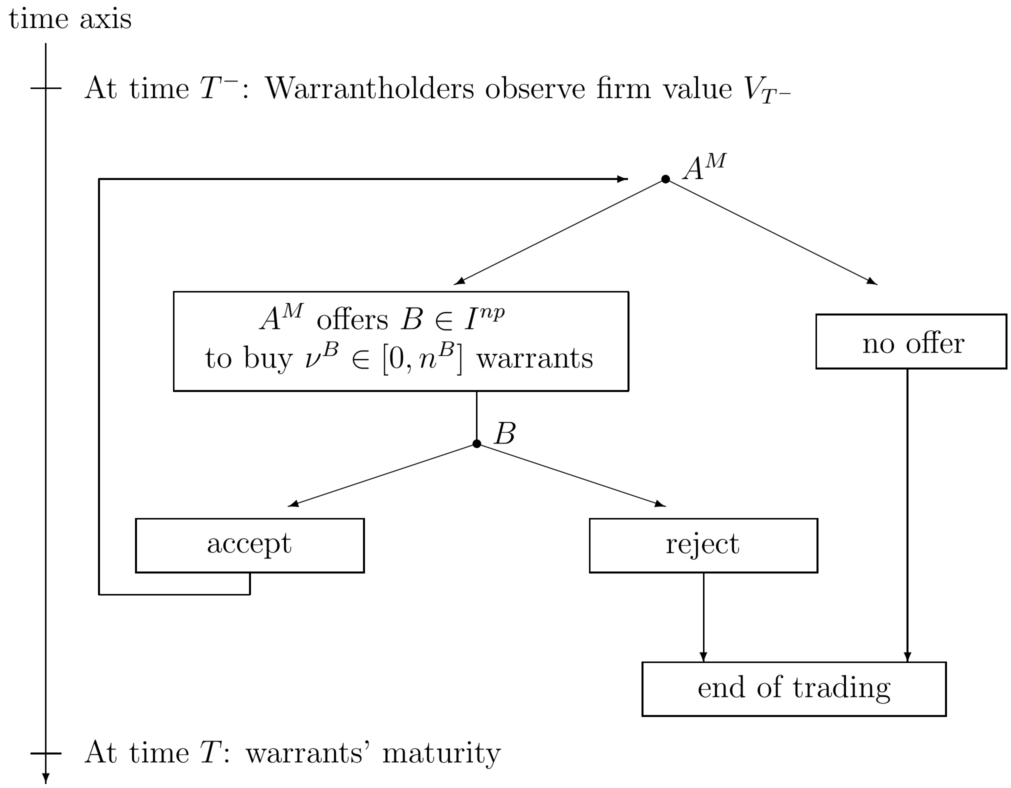
Selling their warrants to warrantholder A for a price of 13.85 for each warrant position (this is one third of the new payoff of warrantholder A, if he owns 60 warrants — see the next table) would increase the payoff value of warrantholders B and C.

	All pricetakers	A	B	C
Exercise policy	40.00 of 40	14.27 of 60		
Stock price	102.91	102.91		
Payoff	116.53	41.57		
		-27.70	13.85	13.85

However, selling the warrants to A is not optimal for warrantholders B and C, respectively. For example, if warrantholder B sells his warrants, warrantholder C is better off if he does not sell to A:

	All pricetakers	A	B	C
Exercise policy	40.00 of 40	9.79 of 40		9.79 of 20
Stock price	101.87	101.87		101.87
Payoff	74.62	18.26		18.26
		-13.85	13.85	

Figure 6: **Trading in a Large Trader Economy**



This example shows that we have to model hoarding as a non-cooperative game. In the following analysis we assume a game structure as illustrated by Figure 6 — using the following conventions: $I^{np} = \{A \in I | P(\{A\}) > 0\}$ denotes the set of non-pricetakers and $A^M \in I^{np}$ denotes the potential monopolist. We assume that the potential monopolist $A^M \in I^{np}$ makes his first offer immediately before maturity of the warrants (i.e. at time T^-) and such that he and all other warrantholders know the corresponding firm value V_{T^-}. He has the right to offer one of the non-pricetakers to buy all his warrants or only a fraction of his warrants for a fixed price. If the non-pricetaker accepts the offer, the potential monopolist can make a further offer to another non-pricetaker. If the non-pricetaker rejects the offer, trading stops (this assumption guarantees that the potential monopolist makes only acceptable offers). Trading also stops if the potential monopolist does not want to make an offer or if he has bought all outstanding warrants. After the trading activities are finished all warrantholders exercise some, all or no warrants and get the exercise value of the warrants they exercised. Under this trading structure we get the following result:[8]

[8]In our framework the potential monopolist can make take-it-or-leave-it-offers. Our results keep

Proposition 5 *If the trading of warrants is organized like in Figure 6, one non-pricetaker buys all warrants from the other non-pricetakers. If all warrantholders are non-pricetakers, the exercise value of European-type warrants in a large trader economy equals the exercise value in a monopoly.*

Proof: In a first step the potential monopolist A^M offers the non-pricetakers to buy all warrants which the non-pricetakers do not exercise in the original warrant distribution. The non-pricetakers could give up this warrants to A^M without any remuneration since otherwise these warrants would expire worthlessly.

After these trades all warrantholders (except A^M) exercise all their warrants if trading stops. If the potential monopolist buys some warrants and trading stops, again all warrantholders (except A^M) exercise all their warrants, but warrantholder A^M has the chance to increase his payoff by exercising less warrants. So the potential monopolist can pay more than the original exercise value when buying further warrants. Thus in the second step the potential monopolist buys the warrants of the other non-pricetakers successively, increasing the exercise value of the warrants. The trading stops if the potential monopolist has bought all warrants of all large traders, i.e. there is only one large trader in the economy and a competitive fringe.

□

Example 3 We assume the same parameters as in Example 2. Again we refer for the computation of the optimal exercise policies to the algorithm given in Appendix B. Following the proof of Proposition 5 warrantholder A buys 12 warrants of warrantholder B and 12 warrants of warrantholder C for a price of zero, since the payoffs remain unchanged.

	All pricetakers	A	B	C
Exercise policy	40.00 of 40	7.41 of 44	7.41 of 8	7.41 of 8
Stock price	101.37	101.37	101.37	101.37
Payoff	54.71	10.13 - 0	10.13 + 0	10.13 + 0

Now warrantholder A can buy the warrants of B for a price between 10.13 and 11.64, since B will sell his warrants only if the price is higher than his payoff (10.13), and A will only buy warrants if his new payoff (see the next table) minus the price is higher than his original payoff ($21.77-10.13 = 11.64$). Warrantholder C acts like a pricetaker before and after the trade: He exercises

the same if we shift the bargaining power from the potential monopolist to the non-pricetakers.

(nearly) all his warrants. So A can maximize his payoffs without a wealth transfer to another warrantholder (C and the pricetakers are shareholders).

We assume that warrantholder A offers to buy the warrants of B for a price of 11.

	All pricetakers	A	B	C
Exercise policy	40.00	10.62		8.00
	of 40	of 52		of 8
Stock price	102.05	102.05		102.05
Payoff	81.99	21.77		16.40
		-11.00	11.00	

In the last step warrantholder A also buys the remaining warrants of C for a price between 16.40 and 19.80. We assume a price of 17.

	All pricetakers	A	B	C
Exercise policy	40.00	14.27		
	of 40	of 60		
Stock price	102.91	102.91		
Payoff	116.53	41.57		
		-11.00	11.00	
		-17.00		17.00

In sum, Proposition 5 shows that the warrants of non-pricetakers will be finally (i.e. at the warrants' maturity) held by just *one* non-pricetaker. So, in an informationally efficient market the current warrant price will reflect the fact that there is only one large warrantholder just before maturity.[9] Therefore, the warrant price for pricetakers is unique under all initial market structures, as long as the non-pricetakers do not trade with pricetakers. The condition that all non-pricetakers eventually sell their warrants to one large warrantholder just before maturity T must only hold for the range of firm values where a partial exercise is beneficial (compared to the block exercise) for the warrantholders.

If we assume that warrants are indivisible (and if the number of the warrants is finite), every warrantholder is a non-pricetaker and sells his warrants to the potential monopolist. Then all warrantholders behave as if there was a monopoly market in T. Surprisingly, the warrant price depends on whether pricetaking warrantholders who hold at least some warrants have a positive measure or not. The reason is that if the measure of pricetaking warrantholders is positive the number of pricetakers is infinite, since per definition a single pricetaker has a measure of zero. A potential

[9] Recall that we have assumed that all warrantholders know the number of warrants held by non-pricetakers.

monopolist can buy warrants from infinitely many warrantholders only by a public offer, but then every single pricetaker wants to be a free rider.[10]

In sum, it turns out that warrantholders have a gain from hoarding European-type warrants in a large trader economy if the firm has issued additional debt.[11] More precisely, *all* warrantholders have a gain if one warrantholder hoards warrants. The reason for the gain of hoarding warrants is that an increasing concentration of the warrant ownership distribution leads to an increasing value of the *partial exercise option*. This extends the existing warrant literature, focusing only on the value of the *sequential exercise option* of American-type warrants (the option to exercise a fraction of the outstanding warrants prematurely).

3 Sequential exercise of American-type warrants

Emanuel (1983) and Constantinides (1984) emphasize the potential advantage of sequential exercise strategies by warrantholders, even absent regular dividend payments. Cox and Rubinstein (1985), Ingersoll (1987) and Spatt and Sterbenz (1988) illustrate the potential optimality of sequential exercise based upon differing assumptions about the firm's policy regarding the use of warrant exercise proceeds and about the distribution of warrant ownership. All these examples disregard straight debt in the capital structure of the firm which is, however, considered in the following analysis. Without additional debt a wealth transfer from the stockholders to the warrantholders is possible when exercising warrants sequentially. The following analysis shows that in a model with additional debt the situation is more complex: The value of the debt can both increase and decrease due to the exercise of a warrant. Example 4 illustrates a wealth transfer from the debtholder to the stockholders and warrantholders.

Example 4 We assume that the firm value follows a binomial process with two periods starting in $t = 0$ and $t = T$. In each period the firm value can increase by 27% or decrease by 25%. The interest rate equals $r = 1\%$ so that the risk neutral probability for an increase of the firm value is $q = 0.5$. The current firm value equals $V_0 = 160,000$. Furthermore, we assume that the firm has issued a zero coupon bond with a face value of 110,000, 100 stocks and 100 warrants with a strike price of $K = 100$ and we assume that the firm

[10] Proposition 5 has also another consequence: If warrants are priced under the assumption of one large warrantholder, a pricetaking warrantholder can hedge his portfolio with shares of the common stock and risk-free bonds, as the warrant can be duplicated by these securities.

[11] Of course, warrantholders also have a gain from hoarding *American-type* warrants in a large trader economy, see Spatt and Sterbenz (1988).

pays no dividends. A algorithm for the computation of the optimal exercise policies is given in Appendix B.

- In a competitive economy the optimal exercise policy for every warrantholder is to exercise no warrants.
- In an economy where one large trader owns 33 warrants and a competitive fringe owns the remaining warrants the optimal exercise policies for every warrantholder is to exercise no warrants.
- In an economy where one large trader owns 66 warrants and a competitive fringe owns the remaining warrants the optimal exercise policy for the large trader is to exercise all his warrants, whereas the pricetaking warrantholders exercise no warrants.
- A warrantholder with monopoly power will exercise $n = 100$ warrants.

	Competitive economy	One large trader ($n_A = 33$)	One large trader ($n_A = 66$)	Monopoly
Stock price	325.66	325.66	327.50	328,45
Warrant price	226.65	226.65	228.49	229,44
Debt value	104,769.14	104,769.14	104,465.86	104,309.63

In the foregoing example the assumed interest rate of $r = 1\%$ was mainly responsible for the optimality of a sequential exercise strategy, because exercising warrants prematurely is only beneficial if the interest rate is low. If we assume an interest rate of $r = 4\%$ a sequential exercise strategy is never optimal. Most examples of the related literature (e.g., Ingersoll, 1987, and Spatt and Sterbenz, 1988, proof of theorem 3) even assume an interest rate of $r = 0\%$. This leads to the question: Under which conditions is a sequential exercise beneficial to warrantholders?

It is well known that a rational pricetaker will never exercise a warrant before maturity in the absence of dividend payments. Now we consider a non-pricetaking warrantholder A holding $n_A \in (0, n]$ warrants and a competitive fringe holding $n_{-A} = n - n_A$ warrants. The payoff function of warrantholder A is defined by equation (3).

Proposition 6 *In the absence of dividend payments the sequential exercise option has zero value if the interest rate satisfies*

$$r \geq \frac{1}{T} \ln \left(\frac{N + n_A}{N} \right) . \tag{5}$$

The proof is given in Appendix A.

Please note that the lower bound of Proposition 6 does not depend on the firm value V_0, the distribution of the firm value process and the debt characterisitics. Of course, this lower bound represents a tradeoff between the sharpness of the bound and the simplicity of its calculation. Nonetheless this bound is good enough to show that a sequential exercise policy is only optimal for warrants whose time to maturity is short.

The lower bound is plotted in Panel A of Figure 7. Please note that we do not need any information about the firm's capital structure except the maturity of the warrants, the non-pricetakers number of warrants and the number of stocks outstanding. Panel A of Figure 7 confirms that for relevant maturities of warrants and ownership concentration (measured by the ratio n_A/N) sequential exercise is not optimal for (non-pricetaking) warrantholders. If the non-pricetaking warrantholder owns $n_A = 10$ warrants with maturity $T = 10$ and $N = 100$ stocks are outstanding, the non-pricetaker do not exercise any warrant if the interest rate is above 1%.

Unfortunately, if the non-pricetaking warrantholder A holds many warrants whose time to maturity is short, Proposition 6 is not very useful. In this case we refer to Lemma A.2 presenting a more precise lower bound on interest rate levels preventing sequential exercise. Panel B of Figure 7 shows for the same parameters (we assume that the firm value follows a Geometric Brownian Motion with volatility $\sigma \leq 0.25$) that a non-pricetaking warrantholder will not exercise his warrants if the interest rate is above 4%. Furthermore, for $n_A = 20$ and $T = 1$ non-pricetaker A does not exercise any warrant if the interest rate is above 1.8%. However, both lower bounds increases with a decreasing time to maturity T. Nonetheless, large warrantholders cannot increase their payoff substantially exercising short-lived warrants. According to Lemma A.2 the upper bound of the marginal payoff of one more exercised warrant goes to zero if the time to maturity goes to zero.

Proposition 6 justifies the assumption that warrants are not exercised prematurely if the exercise proceeds are used to expand the firm's investment. This result holds also for alternative reinvestment strategies, like those analysed in Spatt and Sterbenz (1988): reinvestment in riskless zero-coupon bonds or repurchase of shares plus issuance of new warrants.

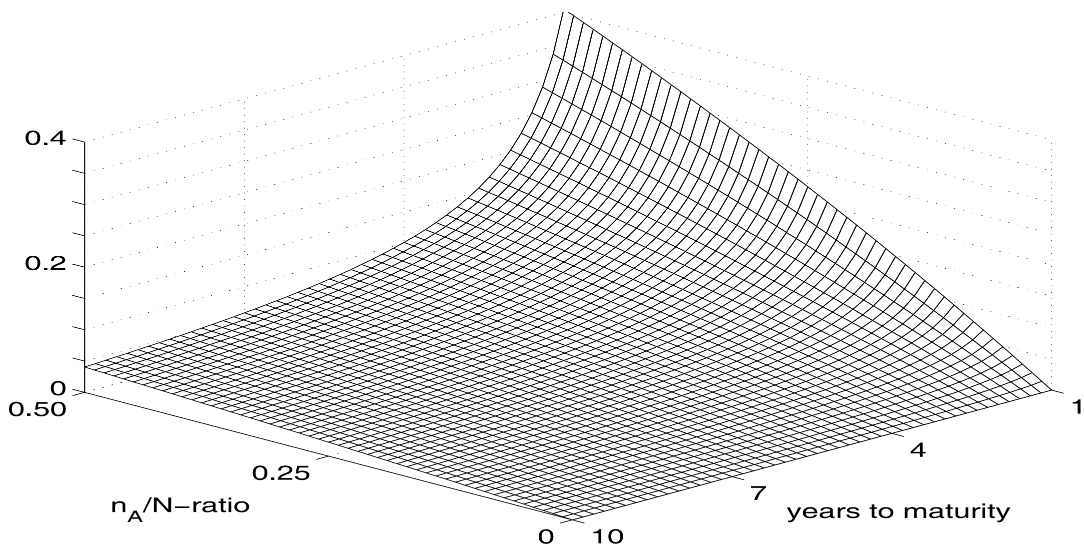
4 Convertible Bonds

In this section we assume a firm financed by issuing equity, debt and convertible bonds which pays no regular coupons. Again at time 0 the equity is split into N

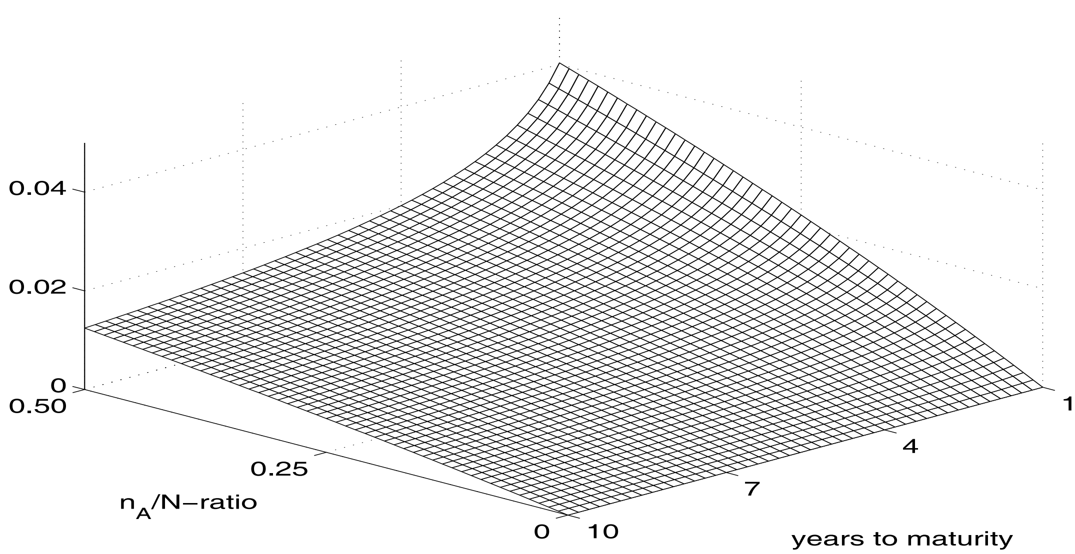
Figure 7: **Lower bounds on interest rate levels preventing sequential exercise**

The figures shows lower bounds for the interest rate. For any interest rate above these lower bounds a sequential exercise policy is not optimal. We assume in panel B an asset return volatility of $\sigma \leq 0.25$.

Panel A: **Lower bound according to Proposition 6**



Panel B: **Lower bound according to Lemma A.2**



outstanding shares and n convertible bonds. Every convertible allows for a conversion into one stock. If there is no conversion, each convertible bond pays K at time T as long as the firm value is sufficiently high for the redemption. If the firm value is not high enough to cover the redemption payment, the firm is liquidated and the firm value is distributed without bankruptcy costs among the holders of convertible bonds in proportion to their holdings. This bankruptcy rule implies that the additional debt is subordinated in accordance to Bühler and Koziol (2002). The debt has a common face value F and maturity T_D with $0 < T < T_D$.

European-type Convertible Bonds

Accordings to the bankruptcy rule the payoff function of a pricetaking holder of European-type convertible bonds is defined by

$$\pi_i(m_i, m_{-i}, V_{T^-}) = \frac{m_i}{N+m}\overline{S}_T\Big(V_{T^-} - (n-m)K\Big) + (n_i - m_i)\min\left\{\frac{V_{T^-}}{n-m}, K\right\} ,$$

where $\overline{S}_T(V_T)$ equals zero, if V_T is negative. Since a default can occur at time T, the redemption value of a non-converted bond, i.e. $\min\{V_{T^-}/(n-m), K\}$, is risky. If $V_{T^-} \leq (n-m)K$ the payoff function collapses to $\pi_i(m_i, m_{-i}, V_{T^-}) = (n_i - m_i)V_{T^-}/(n-m)$ and the conversion of a bond can never be the optimal strategy for a holder of convertible bonds. Otherwise, if $V_{T^-} > (n-m)K$ the payoff function in case of convertibles is similar to the payoff function in case of warrants with the difference of two constants. The payoff function of a non-pricetaker is defined by

$$\begin{aligned} \pi_A(m_A, m_{-A}, V_{T^-}) &= \frac{m_A}{N + m_A + m_{-A}}\overline{S}_T\Big(V_{T^-} - (n - m_A - m_{-A})K\Big) \\ &\quad + (n_A - m_A)\min\left\{\frac{V_{T^-}}{n - m_A - m_{-A}}, K\right\} . \end{aligned}$$

Like a pricetaker a non-pricetaker does not convert a bond if $V_{T^-} \leq (n - m_A - m_{-A})K$. So for European-type convertible bonds the optimal conversion strategy in large trader economies and competitive economies, respectively, is similar to the optimal exercise strategy in case of warrants.

American-type Convertible Bonds

In contrast to the premature exercise of American-type warrants the premature conversion of bonds does not change the total value of the firm. Therefore in the absence of dividend payments and coupon payments it turns out that it is not advantageous to convert the bonds before maturity.

We denote the price of a convertible bond by W_t at time t. As in the case of warrants we assume that convertible bonds that are not converted at $t = 0$ are sold to pricetakers. The payoff function of the pricetaking holder of convertible bonds i is defined by

$$\pi_i^a(m_i, m, V_0) = m_i S_0(V_0) + (n_i - m_i) W_0(V_0)$$

and the payoff function of the non-pricetaking holder of convertible bonds A is defined by

$$\pi_A^a(m_A, m_{-A}, V_0) = m_A S_0(V_0) + (n_A - m_A) W_0(V_0)\,.$$

Please note that the premature conversion of bonds does not change the firm value V_0, but it changes the value of a share of the common stock and the value of a warrant. In analogy to Proposition 6 we have the following

Proposition 7 *The option to exercise convertible bonds sequentially never has a positive value if the firm pays neither coupons nor dividends.*

Proof: Since the stock price and the bond price only depend on the number of bonds converted at maturity, the holders of convertible bonds are not worse off converting their bonds in $t = T$ instead of converting their bonds in $t = 0$. □

Without dividend and coupon payments it is never beneficial for the holder of convertible bonds (pricetaking or non-pricetaking) to convert his bonds sequentially in contrast to the optimal exercise policy of warrantholders: the optimal conversion strategy of an American-type convertible bond is the same as the optimal conversion strategy of an European-type convertible bond. The only reason for a premature conversion of bonds is a high dividend payment. This result does not change if we assume that the convertible bond pays regular coupons.

5 Conclusion

This paper investigates the impact large traders have on the optimal exercise strategies for convertibles and their corresponding market values. As distinguished from the existing literature, our analysis considers a firm that issues (additional) senior debt besides shares of common stock and warrants. We present exercise strategies and the corresponding warrant values for three different large traders economies and compare them to the corresponding results in a competitive economy.

We find that the number of warrants exercised is inversely related to the warrants' ownership concentration. Substantial price differences occur with respect to the competitive situation if short-lived warrants are deep out-of-the-money.

We show that a sequential exercise can only be beneficial to a non-pricetaking warrantholder if the interest rate is below a critical lower bound. However, for a realistic parameter setting the interest rate is above the lower bound and a premature exercise of long-lived warrants is not beneficial. Hence, it turns out that from a theoretical perspective the potential advantage of sequential exercise strategies is not the main obstacle against the use of the block exercise condition in the absence of dividend payments. The latter condition is however questionable on the ground that it may be advantageous not to exercise all warrants if they finish in the money (partial exercise option). It turns out that in case of senior debt outstanding this option has a positive value if and only if one or more warrantholders are non-pricetakers. This option value increases with the concentration of the warrant ownership distribution in the economy and leads to a gain from hoarding warrants. So if there are at least two non-pricetaking warrantholders one non-pricetaker buys all warrants from the other non-pricetakers such that an economy remains with one large trader and a competitive fringe.

This investigation can be extended in several directions. For instance, a model to determine a fair price for warrants traded between large traders is needed. Furthermore, there is at least one fact not considered in the model: The pricetakers do not know the distribution of the warrant ownership. Perhaps less interesting is the analysis of a situation where warrantholders own shares of the common stock of the firm in addition: Non-pricetaking warrantholders will exercise less warrants at maturity, because the stock price is decreasing in the number of warrants exercised.

A Proofs

Proof of Lemma 1:

Proof of part (a): Since the first derivative of $\overline{S}_T(V_{T^-} + mK) - (N + m)K$ with respect to the number of warrants exercised is strictly negative,

$$\frac{\partial}{\partial m}\left(\overline{S}_T(V_{T^-} + mK) - (N + m)K\right) = K\Delta(V_{T^-} + mK) - K < 0\,,$$

there is at most one solution $\widehat{m}$ of $\overline{S}_T(V_{T^-} + mK)/(N + m) = K$. For $m < \widehat{m}$, we have obviously $\overline{S}_T(V_{T^-} + mK)/(N + m) > K$. Otherwise (i.e. for $m > \widehat{m}$) we have $\overline{S}_T(V_{T^-} + mK)/(N + m) < K$.

Proof of part (b): The stock price is above the strike price for all exercise policies $m \leq \widehat{m}$. For $m \in [0, \widehat{m})$ the first and second derivative of the stock price with respect to the number of warrants exercised reads:

$$\begin{aligned} \frac{\partial}{\partial m}\left(\frac{\overline{S}_T\left(V_{T^-}+mK\right)}{N+m}\right) &= \frac{K\Delta_T(V_{T^-}+mK)}{N+m} - \frac{\overline{S}_T(V_{T^-}+mK)}{(N+m)^2} \\ &\leq \frac{K\Delta_T(V_{T^-}+mK)-K}{N+m} < 0 \\ \frac{\partial^2}{\partial m^2}\left(\frac{\overline{S}_T\left(V_{T^-}+mK\right)}{N+m}\right) &= \frac{K^2\Gamma_T(V_{T^-}+mK)}{N+m} - \frac{2K\Delta_T(V_{T^-}+mK)}{(N+m)^2} \\ &\quad + \frac{2\overline{S}_T(V_{T^-}+mK)}{(N+m)^3} \\ &> 2\frac{K-K\Delta_T(V_{T^-}+mK)}{(N+m)^2} > 0\,. \end{aligned}$$

Proof of part (c): We prove that there is at most one $m_A^* \in [0, \widehat{m} - m_{-A})$ which fulfills $\partial\pi_A(m_A^*, m_{-A}, V_{T^-})/\partial m_A = 0$. This implies that the payoff function of warrantholder A is quasi-concave. Furthermore we show that m_A^* maximises the payoff function of warrantholder A.

If $[0, \widehat{m} - m_{-A})$ is not empty $m_{-A} < \widehat{m}$ and $\overline{S}_T(V_{T^-})/N > K$ must hold. We use the conventions $S(m_A) \equiv \overline{S}_T(V_{T^-} + m_{-A}K + m_AK)/(N + m_A + m_{-A})$ and $S'(m_A) = \partial S(m_A)/\partial m_A$. Then the payoff function and its first derivative with respect to the number of warrants exercised by warrantholder A reads as

$$\begin{aligned} \pi_A(m_A, m_{-A}, V_{T^-}) &= m_A(S(m_A) - K) \\ \frac{\partial}{\partial m_A}\pi_A(m_A, m_{-A}, V_{T^-}) &= S(m_A) - K + m_AS'(m_A)\,. \end{aligned}$$

The set of exercise policies with positive payoffs for warrantholder A is given by $\mathcal{M}_A = [0, \widehat{m} - m_{-A}]$ (we admit exercise policies $m_A \in \mathcal{M}_A$ with $m_A > n_A$, if $\widehat{m} - m_{-A} > n_A$). Let us now assume that $m_A^1, m_A^2 \in \mathcal{M}_A$ with $m_A^1 < m_A^2$ and $\alpha > 1$, such that the number of warrants exercised $m_A^\alpha = \alpha m_A^2 + (1-\alpha)m_A^1$ belongs to the set $\mathcal{M}_A$. Then m_A^α is above m_A^1 and m_A^2. According to part (b) of this proof $S(\cdot)$ is strictly convex and $S'(\cdot)$ strictly increasing in $\mathcal{M}_A$, so we have

$$\begin{aligned} \frac{\partial}{\partial m_A}\pi_A(m_A^\alpha, m_{-A}, V_{T^-}) &= S(m_A^\alpha) - K + m_A^\alpha S'(m_A^\alpha) \\ &> \alpha S(m_A^2) + (1-\alpha)S(m_A^1) - K + m_A^\alpha S'(m_A^\alpha) \\ &= \alpha(S(m_A^2) - K + m_A^2 S'(m_A^\alpha)) \\ &\quad + (1-\alpha)(S(m_A^1) - K + m_A^1 S'(m_A^\alpha)) \end{aligned}$$

$$
\begin{aligned}
&> \alpha(S(m_A^2) - K + m_A^2 S'(m_A^2)) \\
&\quad +(1-\alpha)(S(m_A^1) - K + m_A^1 S'(m_A^1)) \\
&= \alpha \frac{\partial}{\partial m_A} \pi_A(m_A^2, m_{-A}, V_{T^-}) \\
&\quad +(1-\alpha) \frac{\partial}{\partial m_A} \pi_A(m_A^1, m_{-A}, V_{T^-}) \,. \qquad \text{(A1)}
\end{aligned}
$$

Now we consider two cases:

<u>Case 1:</u> $m_{-A} < \widehat{m} < \infty$. We have $S(\widehat{m} - m_{-A}) = K$ and

$$
\begin{aligned}
\frac{\partial}{\partial m_A} \pi_A(\widehat{m} - m_{-A}, m_{-A}, V_{T^-}) &= S(\widehat{m} - m_{-A}) - K + (\widehat{m} - m_{-A}) S'(\widehat{m} - m_{-A}) \\
&= (\widehat{m} - m_{-A}) S'(\widehat{m} - m_{-A}) < 0 \,.
\end{aligned}
$$

If there are $m_A^1, m_A^2 \in \mathcal{M}_A$ with $m_A^1 < m_A^2$ and

$$
\frac{\partial}{\partial m_A} \pi_A(m_A^1, m_{-A}, V_{T^-}) = \frac{\partial}{\partial m_A} \pi_A(m_A^2, m_{-A}, V_{T^-}) = 0
$$

then there exists an $\alpha > 1$ with $m_A^\alpha = \widehat{m} - m_{-A}$ and $\partial \pi_A(m_A^\alpha, m_{-A}, V_{T^-})/\partial m_A > 0$. This is not possible, so there is only one $m_A^* \in \mathcal{M}_A$ with $\partial \pi_A(m_A^*, m_{-A}, V_{T^-})/\partial m_A = 0$. Therefore we must have

$$
\pi_A(m_A^*, m_{-A}, V_{T^-}) \geq \pi_A(m_A, m_{-A}, V_{T^-}) \quad \text{for all } m_A \in \mathcal{M}_A \,.
$$

<u>Case 2:</u> $\widehat{m} = \infty$. If there are $m_A^1, m_A^2 \in \mathcal{M}_A$ with $m_A^1 < m_A^2$ and

$$
\frac{\partial}{\partial m_A} \pi_A(m_A^1, m_{-A}, V_{T^-}) = \frac{\partial}{\partial m_A} \pi_A(m_A^2, m_{-A}, V_{T^-}) \,,
$$

then there are also a $m_A^2 \in \mathcal{M}_A$ with $m_A^1 < m_A^2$ and

$$
\frac{\partial}{\partial m_A} \pi_A(m_A^1, m_{-A}, V_{T^-}) < \frac{\partial}{\partial m_A} \pi_A(m_A^2, m_{-A}, V_{T^-}) \,.
$$

According to the equation (A1) we can easily find an $\alpha > 1$ with

$$
\begin{aligned}
\frac{\partial}{\partial m_A} \pi_A(m_A^\alpha, m_{-A}, V_{T^-}) &> S(0) - K \\
&\geq \frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_{T^-}) \quad \text{for all } m_A \in \mathcal{M}_A \,.
\end{aligned}
$$

Since this is not possible, $\partial \pi_A(\cdot, m_{-A}, V_{T^-})/\partial m_A$ is a strictly decreasing function, i.e. $\pi_A(\cdot, m_{-A}, V_{T^-})$ is strictly concave (and therefore quasi-concave) and $m_A^* \in \mathcal{M}_A$ with $\partial \pi_A(m_A^*, m_{-A}, V_{T^-})/\partial m_A = 0$ is a Maximum of $\pi_A(\cdot, m_{-A}, V_{T^-})$. □

Proof of Proposition 2:

Proof of part (a): If $\overline{S}_T(V_{T^-}+m_{-A}K)/(N+m_{-A}) \leq K$ the non-pricetakers A optimal exercise policy is to exercise no warrant, because if he exercises a positive number of warrants $m_A > 0$ the stock price drops below the strike price — according to Lemma 1 — and warrantholder A loses money. From Proposition 1 we know that if warrants finish out of the money ($V_{T^-} < \underline{V}$) the pricetakers let expire their warrants. If warrants finish at the money, i.e. $V_{T^-} \in [\underline{V}, \underline{V}_A)$, the pricetakers exercise as many warrants as necessarly to equalize stock price and strike price. (For $V_{T^-} = \underline{V}_A$ the pricetakers exercise all their warrants, $m^*_{-A} = n_{-A}$, and the non-pricetaker still none, $m^*_A = 0$.)

If warrants finish in the money all pricetakers exercise all their warrants, i.e. $m^*_{-A} = n_{-A}$. In this situation we have $V_{T^-} > \underline{V}_A$. According to Lemma 1 the optimal exercise policy of the non-pricetaker A is a null of the first derivative of the payoff function with respect to the number of warrants exercised. If $V_{T^-} \in [\underline{V}_A, \overline{V}_A)$ the optimal exercise policy of A is a partial exercise, $m^*_A < n_A$, and if $V_{T^-} \geq \overline{V}_A$ the optimal exercise policy is to exercise all warrants.

Proof of part (b): We denote by (m^*_A, m^*_{-A}) the optimal exercise policy in the presence of one non-pricetaker and by m^* the optimal exercise policy in a competitive economy. Let $\overline{V}$ be defined like in Proposition 1. For $V_{T^-} \in (\underline{V}_A, \overline{V}_A)$ we have $m^*_A \in (0, n_A)$.

So for all $V_{T^-} \in [\overline{V}, \overline{V}_A) \subset (\underline{V}_A, \overline{V}_A)$ we have $m^*_{-A} = n_{-A}$ and therefore

$$m^*_A + m^*_{-A} < n_A + n_{-A} = n = m^* \,.$$

Let $V_{T^-} \in (\underline{V}_A, \overline{V}) \subset (\underline{V}_A, \overline{V}_A)$. Then the stock price is above the strike price in the presence of one non-pricetaker:

$$\frac{\overline{S}_T(V_{T^-} + m^*_A K + m^*_{-A} K)}{N + m^*_A + m^*_{-A}} > K = \frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*} \,.$$

According to Lemma 1 it follows that $m^*_A + m^*_{-A} < m^*$. If $V_{T^-} \notin (\underline{V}_A, \overline{V}_A)$ the statement $m^*_A + m^*_{-A} = m^*$ follows immediately from Proposition 1 and part (a). □

Proof of Proposition 3:

Proof of part (a): We consider all four cases of statement (a) of Proposition 3. If $V_{T^-} \in [0, \underline{V})$ the stock price is below the strike price for all exercise policies (the warrants finish out of the money). Rational warrantholders let all warrants expire, i.e. $(m_b^*, m_B^*) = (0,0)$ is an optimal exercise policy.

Let $V_{T^-} \in [\underline{V}, \overline{V}_b)$. According to Lemma 1 the exercise policies (m_b^*, m_B^*) are optimal, if we have

$$\begin{aligned} 0 = \frac{\partial}{\partial m_b}\pi_b(m_b^*, m_B^*, V_{T^-}) &= \frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*} - K + m_b^* \left(\frac{\partial}{\partial m}\frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*}\right) \\ 0 = \frac{\partial}{\partial m_B}\pi_B(m_B^*, m_b^*, V_{T^-}) &= \frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*} - K + m_B^* \left(\frac{\partial}{\partial m}\frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*}\right) \end{aligned}$$

with $m^* = m_b^* + m_B^*$. This implies

$$m_b^* = \frac{\frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*} - K}{-\left(\frac{\partial}{\partial m}\frac{\overline{S}_T(V_{T^-} + m^* K)}{N + m^*}\right)} = m_B^* \,.$$

Accordingly, if it is optimal for one non-pricetaker to exercise only a fraction of his holdings, the same is true for the other non-pricetaker, and both exercise the same number of warrants. This is a Nash equilibrium for all firm values in the given range.

Finally consider cases 3 and 4, where $V_{T^-} \geq \overline{V}_b$. Then warrantholder b exercises all his warrants (he would do that even for $V_{T^-} = \overline{V}_b$), i.e. $m_b^* = n_b$, so the situation of the warrantholder B is that of warrantholder A in the presence of only one non-pricetaker and a competitive fringe, if we set $n_{-A} = n_b$ $(V_{T^-} \geq \underline{V}_A)$.

Proof of part (b): We denote by (m_b^*, m_B^*) the optimal exercise policy in the presence of two non-pricetakers and by (m_A^*, m_{-A}^*) the optimal exercise policy in the presence of one non-pricetaker and a competitive fringe. Let $\underline{V}_A$ be defined as in Proposition 2.

Let $V_{T^-} \in (\underline{V}, \underline{V}_A]$. Then we have $m_b^* \in (0, n_b)$ and $m_b^* = m_B^*$ in the presence of two non-pricetaker and $m_A^* = 0$ and $\overline{S}_T(V_{T^-} + m_{-A}^* K) = (N + m_{-A}^*)K$ in the presence of one non-pricetaker. The optimization of the payoff functions leads to a positive payoff of the two non-pricetakers. This is only possible if

$$\frac{\overline{S}_T(V_{T^-} + m_b^* K + m_B^* K)}{N + m_b^* + m_B^*} > K = \frac{\overline{S}_T(V_{T^-} + m_A^* K + m_{-A}^* K)}{N + m_A^* + m_{-A}^*} \,.$$

According to Lemma 1 this implies $m_b^* + m_B^* < m_A^* + m_{-A}^*$. Since $m_A^* = 0$ we have also $m_B^* > m_A^*$ and $m_b^* < m_{-A}^*$.

Let $V_{T^-} \in (\underline{V}_A, \overline{V}_b)$. Then we have $m_b^* \in (0, n_b)$ and $m_b^* = m_B^*$ in the presence of two non-pricetakers and $m_A^* \in (0, n_A)$ and $m_{-A}^* = n_{-A}$ in the presence of one non-pricetaker and therefore $m_b^* < m_{-A}^*$.

We assume $m_A^* + m_{-A}^* = m_b^* + m_B^* = m^*$. Since $\partial\pi_B(m_B^*, m_b^*, V_{T^-})/\partial m_B = 0$ and $\partial\pi_A(m_A^*, m_{-A}^*, V_{T^-})/\partial m_A = 0$ we obtain

$$m_B^* = \frac{\frac{\overline{S}_T(V_{T-}+m^*K)}{N+m^*} - K}{-\left(\frac{\partial}{\partial m}\frac{\overline{S}_T(V_{T-}+m^*K)}{N+m^*}\right)} = m_A^* \,.$$

From the assumption it follows that $m_b^* = m_{-A}^*$, so the assumption cannot be correct. Since m_A^* and m_B^* are continuous in V_{T^-} and the inequality $m_b^* + m_B^* < m_A^* + m_{-A}^*$ is correct for $V_{T^-} = \underline{V}_A$, the inequality must be correct for all firm values in the given range.

Using the same argumentation as in the proof of statement (c) of Lemma 1 we can show that there exists at most one m_{-A}^* which solves $\partial\pi_A(m_A^*, \cdot, V_{T^-})/\partial m_A = 0$. Therefore we cannot have $m_A^* = m_B^*$. Since m_A^* and m_B^* are continuous in V_{T^-} and inequality $m_B^* > m_A^*$ is correct for $V_{T^-} = \underline{V}_A$, the inequality must be correct for all firm values in the given range.

Proof of part (c): We denote by (m_b^*, m_B^*) the optimal exercise policy in the presence of two non-pricetakers and by m_A^* the optimal exercise policy in a monopoly. Let $\overline{V}_A$ be defined like in Proposition 2.

Let $V_{T^-} \in [\overline{V}_B, \overline{V}_A)$. Then we have $m_b^* = n_b$ and $m_B^* = n_B$ in the presence of two non-pricetakers and $m_A^* \in (0, n)$ in the monopoly, i.e. $m_b^* + m_B^* = n > m_A^*$.

Let $V_{T^-} \in (\underline{V}, \overline{V}_B)$. Then we get $m_b^* \in (0, n_b]$ and $m_B^* \in (0, n_B)$ in the presence of two non-pricetakers and $m_A^* \in (0, n)$ in the monopoly. We assume $m_A^* = m_b^* + m_B^* = m^*$. Since $\partial\pi_B(m_B^*, m_b^*, V_{T^-})/\partial m_B = 0$ and $\partial\pi_A(m_A^*, 0, V_{T^-})/\partial m_A = 0$ we obtain

$$m_B^* = \frac{\frac{\overline{S}_T(V_{T-}+m^*K)}{N+m^*} - K}{-\left(\frac{\partial}{\partial m}\frac{\overline{S}_T(V_{T-}+m^*K)}{N+m^*}\right)} = m_A^* \,.$$

From the assumption it follows that $m_b^* = 0$, so the assumption cannot be correct. Since m_A^*, m_b^* and m_B^* are continuous in V_{T^-} and inequality $m_b^* + m_B^* < m_A^*$ is correct for $V_{T^-} = \overline{V}_B$, the inequality must be correct for all firm values in the given range.

□

Proof of Proposition 6:

We denote the value of the firm's initial assets by A_t which follows the same Geometric Brownian Motion as the firm value. So before the warrantholders exercise m

warrants at time $t = 0$ the firm value equals the asset value $V_{0^-} = A_0$ and thereafter we have $V_0 = A_0 + mK$, because exercise proceeds are used to rescale the firm's investment. At time T^- we get $V_{T^-} = (A_0 + mK)A_T/A_0$. We use the following notation:

$$\begin{aligned} \underline{\overline{S}}_T(A_T, m) &\equiv \overline{S}_T\left(\frac{A_0 + mK}{A_0}A_T\right) \qquad \text{and} \\ \overline{\overline{S}}_T(A_T, m) &\equiv \overline{S}_T\left(\frac{A_0 + mK}{A_0}A_T + (n-m)K\right) \end{aligned}$$

denote the total value of common stock when *no* warrant and *all* warrants are exercised at time T, respectively. We write $\underline{\Delta}_T$ and $\overline{\Delta}_T$ for the partial derivative of $\underline{\overline{S}}_T(A_T, m)$ and $\overline{\overline{S}}_T(A_T, m)$ with respect to the asset value A_T, respectively. Assuming that warrants that are not exercised at $t = 0$ are sold to pricetakers, we get according to Proposition 1 two critical firm values $\underline{A}(m)$ and $\overline{A}(m)$ at time $t = T$ with

$$\underline{\overline{S}}_T(\underline{A}(m), m) = (N + m)K \quad \text{and} \quad \overline{\overline{S}}_T(\overline{A}(m), m) = (N + n)K\,.$$

If the asset value A_T is less than $\underline{A}(m)$, no warrant is exercised and the stock price is less than the strike price, whereas if $A_T \geq \overline{A}(m)$ all warrants are exercised in a competitive market. So the stock price and its first derivative with respect to $m = m_A + m_{-A}$ can be written as

$$\begin{aligned} S_T(A_T, m) &= \begin{cases} \frac{1}{N+m}\underline{\overline{S}}_T(A_T, m) & \text{for } A_T \in (0, \underline{A}(m)) \\ K & \text{for } A_T \in [\underline{A}(m), \overline{A}(m)) \\ \frac{1}{N+n}\overline{\overline{S}}_T(A_T, m) & \text{for } A_T \in [\overline{A}(m), \infty) \end{cases} \\ \frac{\partial}{\partial m_A}S_T(A_T, m) &= \begin{cases} \frac{1}{N+m}K\frac{A_T}{A_0}\underline{\Delta}_T - \frac{\underline{\overline{S}}_T(A_T,m)}{(N+m)^2} & \text{for } A_T \in (0, \underline{A}(m)) \\ 0 & \text{for } A_T \in [\underline{A}(m), \overline{A}(m)) \\ \frac{1}{N+n}K\left(\frac{A_T}{A_0} - 1\right)\overline{\Delta}_T & \text{for } A_T \in [\overline{A}(m), \infty) \end{cases} \end{aligned} \tag{A2}$$

The warrant price and its first derivative with respect to the number of warrants exercised at time $t = 0$ (m) reads as

$$\begin{aligned} W_T(A_T, m) &= \begin{cases} 0 & \text{for } A_T < \overline{A}(m) \\ \frac{1}{N+n}\overline{\overline{S}}_T(A_T, m) - K & \text{for } A_T \geq \overline{A}(m) \end{cases} \\ \frac{\partial}{\partial m_A}W_T(A_T, m) &= \begin{cases} 0 & \text{for } A_T < \overline{A}(m) \\ \frac{1}{N+n}K\left(\frac{A_T}{A_0} - 1\right)\overline{\Delta}_T & \text{for } A_T \geq \overline{A}(m) \end{cases} \end{aligned} \tag{A3}$$

This implies

$$\begin{aligned}\frac{\partial}{\partial m_A} W_0(A_0+mK) &= e^{-rT}\int_{\overline{A}(m)}^{\infty} \frac{\partial}{\partial m} W_T(A_T,m)dQ \\ &\leq e^{-rT} \int\limits_{\max\{A_0,\overline{A}(m)\}}^{\infty} \frac{1}{N+n_A} K\left(\frac{A_T}{A_0}-1\right) dQ \qquad \text{(A4)}\end{aligned}$$

since $\overline{\Delta}_T \leq 1$ and $n_A \leq n$. According to the equations (A2) and (A3) we have

$$\begin{aligned}&\frac{\partial}{\partial m_A} \quad [m_A\left(S_T(A_T,m)-K-W_T(A_T,m)\right)] \\ &= \begin{cases} \frac{N+m_{-A}}{(N+m)^2}\underline{S}_T(A_T,m)+\frac{m_A}{N+m}K\frac{A_T}{A_0}\underline{\Delta}_T - K & \text{for} \quad A_T \in (0,\underline{A}(m)) \\ 0 & \text{for} \quad A_T \in [\underline{A}(m),\overline{A}(m)) \\ 0 & \text{for} \quad A_T \in [\overline{A}(m),\infty) \end{cases} \\ &\leq \begin{cases} \frac{m_A}{N+m}K\left(\frac{A_T}{A_0}\underline{\Delta}_T - 1\right) & \text{for} \quad A_T \in (0,\underline{A}(m)) \\ 0 & \text{for} \quad A_T \in [\underline{A}(m),\infty) \end{cases} \\ &\leq \begin{cases} \frac{n_A}{N+n_A}K\left(\frac{A_T}{A_0} - 1\right) & \text{for} \quad A_T \in [A_0,\max\{A_0,\overline{A}(m)\}) \\ 0 & \text{for} \quad A_T \notin [A_0,\max\{A_0,\overline{A}(m)\}) \end{cases} \qquad \text{(A5)}\end{aligned}$$

Lemma A.2 *If the firm uses the exercise proceeds to rescale the firm's investment the marginal payoff of the non-pricetaking warrantholder A is bounded by*

$$\frac{\partial}{\partial m_A}\pi_A^a(m_A,m_{-A},V_0) \quad < \quad K\left(\frac{n_A}{N+n_A}\frac{W_0^{am}(V_0)}{V_0} - \left(1-e^{-rT}\right)\right) \qquad \text{(A6)}$$

for all (sequential) exercise strategies $(m_i)_{i\in I}$, where W_0^{am} is an at-the-money warrant on the firm value with maturity T. For a pricetaking warrantholder i the marginal payoff is always negative.

Proof of Lemma A.2: The marginal payoff of a *pricetaking* warrantholder is always negative, since $S_0(V_0)-K-W_0(V_0) \leq -K(1-e^{-rT})$ due to the warrant's European lower bound.

The payoff of the non-pricetaker A is defined by equation (3). We rewrite the payoff function in the following way:

$$\begin{aligned}\pi_A^a(m_A,m_{-A},V_0) &= e^{-rT}\int_0^{\infty} m_A\left(S_T(A_T,m)-K-W_T(A_T,m)\right)dQ \\ &\quad +e^{-rT}\int_{\overline{A}(m)}^{\infty} n_A W_T(A_T,m)dQ - m_A K(1-e^{-rT})\,.\end{aligned}$$

Using relations (A4) and (A5) we get an upper bound for the marginal payoff of warrantholder A:

$$
\begin{aligned}
\frac{\partial}{\partial m_A}\pi_A^a(m_A, m_{-A}, V_0) \;\leq\; & e^{-rT}\int\limits_{A_0}^{\max\{A_0,\overline{A}(m)\}} \frac{n_A}{N+n_A}K\left(\frac{A_T}{A_0}-1\right)dQ \\
& +e^{-rT}\int\limits_{\max\{A_0,\overline{A}(m)\}}^{\infty} \frac{n_A}{N+n_A}K\left(\frac{A_T}{A_0}-1\right)dQ \\
& -K(1-e^{-rT}) \\
= \; & \frac{n_A}{N+n_A}K\frac{W_0^{am}(A_0)}{A_0} - K(1-e^{-rT})\,.
\end{aligned}
$$

This completes the proof since $W_0^{am}(A_0)/A_0 = W_0^{am}(V_0)/V_0$. □

If the upper bound for the marginal payoff (A6) is negative a sequential exercise strategy is never optimal. So if a sequential exercise strategy should be optimal it is a necessary condition that the upper bound is positive, i.e. we get

$$
\begin{aligned}
0 \;<\; & K\left(\frac{n_A}{N+n_A}\frac{W_0^{am}(V_0)}{V_0} - \left(1-e^{-rT}\right)\right) \\
<\; & K\left(\frac{n_A}{N+n_A} - \left(1-e^{-rT}\right)\right)\,. \qquad \text{(A7)}
\end{aligned}
$$

Relation (A7) is equivalent to the lower bound (5). The upper bound (A6) implies an other lower bound for the interest rate. □

B Examples

Computations of Examples 2 and 3

The firm value in Examples 2 and 3 follows a binomial process as illustrated in the following figure:

$$V_T = 60,000 + 100m \quad \begin{cases} V_{T_D}^u = 75,000 + 125m \\ S_{T_D}^u = \frac{1}{N+m}[V_{T_D}^u - F]^+ \\ \qquad = \frac{1}{100+m}(21,000 + 125m) \\ \\ V_{T_D}^d = 45,000 + 75m \\ S_{T_D}^d = \frac{1}{N+m}[V_{T_D}^d - F]^+ = 0 \end{cases}$$

Please note that for any exercise policy the firm's down-state value is less than the face value of the debt so that in this situation the firm defaults. We define now a subset of the non-pricetaking warrantholders $J \subseteq \{A \in I | P(\{A\}) > 0\}$ and assume that the optimal exercise policy for all warrantholders not in J is to exercise all warrants, i.e. if $(m_i^*)_{i \in I}$ are optimal policies we have $m_i^* = n_i$ for all $i \in I \backslash J$. Furthermore, we assume that according to Proposition 3 all warrantholders in J have the same optimal exercise policy, i.e. $m_A^* = m_B^*$ for all $A, B \in J$. If j denotes the number of warrantholders in J and $n_{-J} = \int_{I \backslash J} m_i^* dP$ the optimal exercise policy of the warrantholders not in J we get $m^* = n_{-J} + j \cdot m_A^*$ for an arbitrary $A \in J$.

Using the conventions $\widehat{S} = q((V_{T^-} + n_{-J}K) \cdot 1.25 - F)/(1+r)$ and $\delta = q \cdot 1.25/(1+r)$ the stock price equals $S_T = (\widehat{S} + jm_A^* K\delta)/(N + n_{-J} + jm_A^*)$. Then the payoff function of warrantholder A and its first derivative with respect to the number of warrants exercised satisfy

$$\begin{aligned} \pi_A(m_A^*, m_{-A}^*, V_{T^-}) &= m_A^* \left(\frac{\widehat{S} + jm_A^* K\delta}{N + n_{-J} + jm_A^*} - K \right), \\ \frac{\partial}{\partial m_A} \pi_A(m_A^*, m_{-A}^*, V_{T^-}) &= \frac{N + n_{-J} + (j-1)m_A^*}{(N + n_{-J} + jm_A^*)^2} (\widehat{S} + jm_A^* K\delta) \\ &\quad + \frac{m_A^*}{N + n_{-J} + jm_A^*} K\delta - K \,. \end{aligned}$$

The equation $\partial \pi_A(m_A^*, m_{-A}^*, V_{T^-})/\partial m_A = 0$ has a unique, positive solution:

$$\begin{aligned} m_A^* &= (N + n_{-J}) \frac{(j-1)\frac{\widehat{S}}{N+n_{-J}} + (j+1)K\delta - 2jK}{2(j^2K - j^2K\delta)} \\ &+ (N + n_{-J}) \sqrt{\left(\frac{(j-1)\frac{\widehat{S}}{N+n_{-J}} + (j+1)K\delta - 2jK}{2(j^2K - j^2K\delta)} \right)^2 + \frac{\frac{\widehat{S}}{N+n_{-J}} - K}{j^2K - j^2K\delta}} \,. \end{aligned} \tag{B1}$$

For any market structure we can compute the optimal exercise policy according to the following algorithm:[12]

1. If $\overline{S}_T(V_{T^-}) \leq NK$ the optimal exercise policy is $m_i^* = 0$ for all pricetaking and non-pricetaking warrantholder $i \in I$ and the algorithm stops.

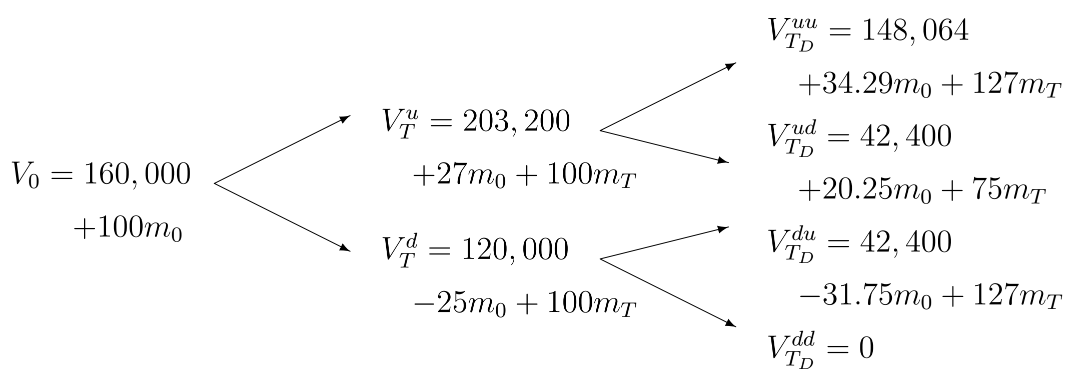
2. Define $J = \{A \in I | P(\{A\}) > 0\}$. If $\widehat{S} \leq (N + n_{-J})K$ the pricetakers exercise as many warrants as necessary to equalize stock and strike price and the algorithm stops.

3. Compute m_A^* according to relationship (B1). If $m_A^* < n_A$ for all $A \in J$ the optimal exercise policies are $m_i^* = n_i$ for all $i \in I \backslash J$ and $m_B^* = m_A^*$ for all $B \in J$ and the algorithm stops.

4. Define $J^{new} = J \backslash \{B \in J | n_B < m_A^*\}$ and continue with step 3.

Computation of Example 4

The firm value in Example 4 follows a two-period binomial process as illustrated in the following figure including the redemption of the additional debt or the default at time T_D. Please note that the warrantholders exercise $m_T - m_0$ warrants at time T implying the up-state firm value $V_T^u = V_0 \cdot 1.27 + (m_T - m_0)K$.



If $V_T = V_T^u$ the stock is for all exercise policies m_T above the strike price, so the *pricetaking* warrantholders exercise $m_T^* = n - m_0$ warrants. Therefore the stock price, the warrant price, and the debt value satisfy

$$S_T(V_T^u) = \frac{1}{1+r}\left(526.66 + \frac{5454}{40,000}m_0\right), \qquad W_T(V_T^u) = S_T(V_T^u) - 100$$

[12]In Examples 2 and 3 the steps 1 and 2 have only to be done once.

$$\text{and} \qquad D_T(V_T^u) = \frac{1}{1+r} 110,000\,,$$

respectively. Even if $V_T = V_T^d$ the stock price is above the strike price for all exercise policies (m_0, m_T). Therefore the pricetaking warrantholders will again rationally exercise $m_T^* = n - m_0$ warrants and we obtain

$$S_T(V_T^d) = \frac{1}{1+r}\left(137.75 - \frac{3175}{40,000} m_0\right), \qquad W_T(V_T^d) = S_T(V_T^d) - 100$$

$$\text{and} \qquad D_T(V_T^d) = \frac{1}{1+r}\left(103,750 - \frac{75}{8} m_0\right).$$

In time $t = 0$ the stock price, warrant price and the debt value satisfy

$$S_0(V_0) = \frac{1}{(1+r)^2}\left(332.205 + \frac{2279}{80,000} m_0\right), \ W_0(V_0) = S_0(V_0) - \frac{1}{1+r} 100 \qquad \text{(B2)}$$

$$\text{and} \quad D_0(V_0) = \frac{1}{(1+r)^2}\left(106,875 - \frac{75}{16} m_0\right),$$

respectively. Since $S_0(V_0) - K - W_0(V_0) < 0$ a pricetaking warrantholder is better off not to exercise warrants, i.e. in a competitive economy we get $m_0^* = 0$. In an economy with one large trader A with $n_A \in (0, n]$ and a competitive fringe the payoff function of warrantholder A and its first derivative with respect to the number of warrants exercised is equal to

$$\begin{aligned}
\pi_A^a(m_A, 0, V_0) &= m_A(S_0(V_0) - K - W_0(V_0)) + n_A W_0(V_0) \\
&= m_A\left(\frac{1}{1+r} 100 - 100\right) \\
&\quad + n_A\left(\frac{1}{(1+r)^2} 332.205 + \frac{1}{(1+r)^2}\frac{2279 m_0}{80,000} - \frac{1}{1+r} 100\right), \\
\frac{\partial}{\partial m_A}\pi_A^a(m_A, 0, V_0) &= \left(\frac{1}{1+r} 100 - 100\right) + n_A \frac{1}{(1+r)^2}\frac{2279}{80,000}.
\end{aligned}$$

The first derivative of the payoff function of warrantholder A is constant in the number of warrants exercised. So warrantholder A will exercise either all warrants or no warrant at all. The equation $\partial \pi_A^a(m_A, 0, V_0)/\partial m_A > 0$ is equivalent to

$$n_A > \frac{80,000}{2279}(1+r)^2\left(100 - \frac{1}{1+r} 100\right) \approx 35.45\,.$$

If warrantholder A owns more than 35.45 warrants he exercises all his warrants, otherwise none. With $m_0 = m_A^*$ the stock and warrant price and the debt value can be calculated with equation (B2).

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