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Robustness of option-like warrant valuation

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Abstract

This paper presents a methodology for arriving at the unobserved asset value and its volatility of a firm with outstanding warrants. This enables us to price warrants correctly and to examine the robustness of "option-like" warrant valuation where the dilution that occurs when warrants are exercised is ignored. Our analysis helps to justify the frequent simplifying option-like warrant valuation. Furthermore, we examine 50,960 daily prices for 37 American-type warrants written on 16 German stocks over the period 1979–1990. The empirical results confirm our theoretical analysis: there is virtually no dilution-related pricing bias of the American constant variance diffusion model with the stock price as the state variable.

Keywords: Warrant pricing; Equity volatility; Robustness; Equity dilution; Unobserved parameters

JEL classification: G13, G32

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1. Introduction

Since 1980, substantial parts of corporate capital have been raised by corporations in the form of warrants attached to bond issues. Nevertheless, there has been little theoretical and empirical analysis of warrant valuation in the spirit of the option valuation work of Black and Scholes (1973). Possibly, the more complicated analysis of warrants compared to the analysis of simple call options has led to this situation. Although both instruments carry the right to buy a share of the underlying stock at a certain price during a given time period, it is well-known that they differ in at least one important way: while the call option is issued by an individual, the warrant is issued by the firm and its proceeds increase the firm's equity. Furthermore, when a warrant is exercised, new shares are issued, and the cash payment that is made increases the assets of the issuing firm. Because of this there is some dilution of equity and dividend. [1]

Black and Scholes (1973, pp. 648–649) already discussed some effects of the potential dilution and proposed to value warrants as options on shares of the *equity* of the firm. Assuming that the ratio of the number of outstanding warrants to the number of outstanding shares of common stock prior to exercise (so called dilution factor) is $\lambda = 1$, Black and Scholes point out (p. 649): "We can look at the warrants as options to buy shares in the equity rather than shares of common stock, at half the stated exercise price rather than at the full exercise price. The value of a share in the equity is defined as the sum of the value of the warrants and the value of the common stock, divided by twice the number of outstanding shares of common stock. If we take this point of view, then we will take v^2 in Eq. (13) (Black–Scholes formula) to be the variance rate of the return on the company's equity, rather than the variance rate of the return on the company's common stock".

Unfortunately, both the actual value of the firm's equity and the variance of the rate of return on these assets are neither observable nor inferable when the outstanding warrants are not traded in an informationally efficient market. The same obstacle to empirical application of a dilution-adjusted valuation model occurs in case of an equivalent warrant valuation model where warrants are reconstructed as call options on the *total value* of the firm. [2] This is why warrants are often valued as otherwise identical call options on the firm's common stock in a Black–Scholes world. In the following such a non-dilution-adjusted valuation of

[1] In the sample period from January 1979 to December 1990 the potential dilution was substantial for German companies. For example, the exercise of all outstanding warrants would have increased the number of outstanding shares of CONTI-GUMMI and BAYER (the German company with the highest nominal capital of all) by 49% and 25%, respectively.

[2] See, for example, Cox and Rubinstein (1985, pp. 392–399) and Ingersoll (1987, pp. 411–418).

warrants is referred to as *option-like* warrant valuation. [3] To account for the dilution effect, sometimes the option-like warrant value is adjusted by a factor equal to $1/(1+\lambda)$ where λ is the dilution factor. [4] This is, however, an *inadequate* application of Galai and Schneller's (1978) result that the value of a warrant must equal $1/(1+\lambda)$ times the value of an otherwise identical call option on the stock of an otherwise identical firm *without* warrants.

The purpose of this paper is twofold. First, we explore how the presence of warrants in the capital structure affects the applicability of the Black–Scholes formula, where a call option or a warrant is valued relative to its underlying stock. Under the assumption that the unobserved value of the firm follows a continuous process with constant volatility we present a method which allows to infer the firm's current value and its volatility from the actual stock price and the observable variance of the rate of return on common stock. This enables us to examine the non-stationarity of stock volatility when warrants are outstanding. Moreover, we show that option-like warrant valuation is very precise, if: (1) the potential dilution of equity is anticipated in the current stock price; (2) the warrant to be valued is in the money; and (3) sequential exercise of American-type warrants is not optimal. [5] Second, we examine empirically the robustness of option-like warrant valuation based on 50,960 daily prices of American-type warrants written on German stocks and listed on the Frankfurt Securities Exchange. Since German companies pay cash dividends commonly once a year, the dividend amount is large enough to induce early exercise of the warrants. Therefore we use the American constant variance (CV) diffusion model instead of the dividend-adjusted Black–Scholes formula.

The paper is organized as follows. Section 2 presents the theoretical analysis on the robustness of option-like warrant valuation. Section 3 reports on the observed empirical robustness. Section 4 provides a brief conclusion.

2. Theoretical robustness

2.1. A simple model for an unlevered firm

To isolate the impact of equity dilution induced by the warrants *issuance* on the warrant value, we now make the following basic assumptions:

[3] For convenience, we use the term "option-like" although it might be misleading in the following sense: the value of an otherwise identical call option on the stock of a firm with outstanding European-type warrants should equal the warrant value and should therefore also be calculated in a dilution-adjusted fashion.

[4] See, for example, Noreen and Wolfson (1981).

[5] In a recent paper which closely parallels our work, Bensoussan et al. (1992) confirm this important result.

(A1) The value of the firm, V, follows a constant variance diffusion process during the lifetime of the outstanding warrants. That is, the (unobserved) instantaneous standard deviation of the rate of return on the value of the firm's assets, σ_V, is constant.

(A2) Dividends are not paid during the lifetime of the outstanding warrants, and sequential exercise of the warrants is not optimal for warrantholders.

Assumption (A1) ensures that the distribution of the rates of return on total assets is not affected if the proceeds from selling the warrants are invested in the firm. Assumption (A2) avoids the issues of early and sequential exercise of American-type warrants. [6] Sequential exercise can be optimal for a warrantholder even without regular dividends if the firm uses the proceeds from early exercise to expand the scale of the firm or to repurchase shares of common stock. Both firm policies increase the variance of the stock price and therefore the value of the unexercised warrants. According to Spatt and Sterbenz (1988), there exist for any number of warrantholders parameter values such that this increase in value exceeds the forfeiture of the premium above parity on those exercised, leading to sequential exercise. [7] Fortunately, Spatt and Sterbenz (1988) provide in addition simple policies for the firm to eliminate both the gain from sequential exercise and the advantage of monopolization. [8] Their analysis of the obstacles to sequential exercise is our main justification for ignoring the sequential exercise aspect of American-type warrants.

We use the following notation:

N ≡ number of outstanding shares of common stock,
n ≡ number of outstanding warrants,
λ ≡ dilution factor ($\lambda = n/N$),
S_t ≡ price per share of common stock at time t,
W_t ≡ price per warrant at time t,
$\underline{W}_t^{E}$ ≡ European lower bound of the warrant price at time t,
$\overline{S}_t$ ≡ market value of total common stock at time t ($= NS_t$),
$\overline{W}_t$ ≡ market value of all outstanding warrants at time t ($= nW_t$),
V_t ≡ unobserved value of the firm at time t,

[6] For a detailed discussion of these issues, see Emanuel (1983), Constantinides (1984), Constantinides and Rosenthal (1984), Cox and Rubinstein (1985, pp. 392–399), Ingersoll (1987, pp. 435–445), and Spatt and Sterbenz (1988).

[7] A large number of simulations with the model where the proceeds are used to expand the scale of the firm led Veld (1992, p. 83), however, to the conclusion that "the increase in value of the unexercised warrants generally does *not* exceed the forfeiture in the premium above parity for the exercised warrants." Therefore we emphasize that use of the funds to expand the firm's investment policy does not necessarily lead to sequential exercise instead of block exercise.

[8] One of these policies is described as follows (Proposition 2, p. 496): "If the firm invests the proceeds from any warrant exercise in a riskless zero-coupon bond that matures at the warrant's expiration, then holding all warrants until maturity is the unique Nash equilibrium and maximizes the total value of the warrants."

σ_V ≡ instantaneous standard deviation of the rate of return on the value of the firm's assets (asset volatility),
σ_V^P ≡ instantaneous standard deviation of the rate of return on the value of the firm's assets after the warrants have been exercised ("post-exercise" asset volatility),
σ_S ≡ instantaneous standard deviation of the rate of return on common stock (stock volatility),
σ_S^P ≡ instantaneous standard deviation of the rate of return on common stock after the warrants have been exercised ("post-exercise" stock volatility),
σ_W ≡ instantaneous standard deviation of the rate of return on the outstanding warrants (warrant volatility),
$\varepsilon_{S,V}$ ≡ elasticity of the stock price with respect to the value of the firm,
T ≡ maturity date of the outstanding warrants,
τ ≡ time until maturity of the outstanding warrants ($\tau = T - t$),
K ≡ exercise price of the warrants,
r ≡ instantaneous riskless rate of interest (riskless interest rate),
$N(\cdot)$ ≡ standardized cumulative normal distribution function.

For the sake of simplicity, the subscript t is omitted for $t = 0$ (current time).

Now, let us assume that a company has N shares of common stock and n warrants outstanding. Each warrant entitles the owner to receive one share of stock upon the payment of K dollars. Stocks and warrants are the only financing that the company is using. Hence, the company has a current value of

$$V = NS + nW \tag{1}$$

If it is known that the warrants will be exercised, then at the maturity date T each share of stock will be worth $S_T = (V_T + nK)/(N + n)$. At the point where the warrant owners are indifferent about exercising ($V_T = NK$), each share is worth K, so the warrants are exercised only if $S_T \geqslant K$, just as with call options. If the warrant owners strictly prefer exercising ($V_T > NK$), then the common stock is worth less than the value of the firm's assets. The same is generally true prior to the maturity of the warrants. The value of the common stock is less than the value of the assets. If we think of the assets as being primitive, it is natural to price the warrants as contingent claims not on the stock but on the *firm* as a whole. Then the maturity value of a warrant is given by

$$W_T = \begin{cases} \dfrac{V_T + nK}{N + n} - K, & \text{if} \quad V_T > NK \\ 0, & \text{if} \quad V_T \leqslant NK \end{cases} \tag{2}$$

or equivalently by $W_T = \max[0, V_T/N - K]/(1 + \lambda)$. Now, if we make the standard assumptions of the Black–Scholes option pricing model (see, e.g., Merton, 1973), we can immediately write the value of a warrant as

$$W(V,\sigma_V) = \frac{1}{1 + \lambda} C^{\mathrm{BS}}(V/N,\tau,K,\sigma_V,r), \tag{3}$$

where

$$C^{BS}(V/N,\tau,K,\sigma_V,r) = \frac{V}{N}N(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_1 = \frac{\ln(V/NK) + (r + \sigma_V^2/2)\tau}{\sigma_V\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma_V\sqrt{\tau}. \tag{4}$$

The chief obstacle to empirical application of this valuation formula lies in the fact that neither the true value of the firm, V, nor its instantaneous volatility, σ_V, can be observed. In the context of our model, however, the following representation for the observable current stock price $S \equiv S(V,\sigma_V)$ and for the current stock volatility $\sigma_S \equiv \sigma_S(S(V,\sigma_V))$, respectively, must hold:

$$S = \frac{V}{N} - \frac{n}{N}W(V,\sigma_V) \tag{5}$$

$$\sigma_S = \sigma_V \cdot \varepsilon_{S,V} \tag{6}$$

where

$$W(V,\sigma_V) \equiv \frac{1}{1+\lambda} \cdot C^{BS}(V/N,\tau,K,\sigma_V,r)$$

and

$$\varepsilon_{S,V} \equiv \frac{\partial S}{\partial V}\frac{V}{S} = \left(\frac{1}{N} - \frac{n}{N}\frac{\partial W(V,\sigma_V)}{\partial V}\right)\frac{V}{S}.$$

Representation (5) follows directly from relation (1). Relationship (6) is a standard result in option pricing theory where the stock's elasticity, $\varepsilon_{S,V}$, gives the percentage change in the stock's value for a percentage change in the firm's value (see, e.g., Jarrow and Rudd, 1983, p. 110). According to (6), the stock's elasticity scales the firm's asset volatility to stock volatility. Furthermore, because the stock's elasticity is a function of the firm value and time, it follows that the stock volatility, σ_S, changes over the life of the outstanding warrants even if the asset volatility, σ_V, is constant.

Now, the nonlinear Eqs. (5) and (6) can be solved simultaneously for the two unknowns, V and σ_V, by a numerical routine [9] for each observed S and estimated σ_S. Then, given the solution (V, σ_V), an estimate of the warrant value can be

[9] We used subroutine ZSCNT of the International Mathematical and Statistical Library (IMSL). The initial estimate that we used for the firm value, V, was the market value of common stock, $NS = \overline{S}$, while that for σ_V was σ_S.

computed using Eq. (3).[10] Since σ_S does change over the life of the outstanding warrants, this parameter should be estimated from a *short* historical return time series of the stock, or, if the stock has listed options, estimated from the implied volatility of *short*-lived options.[11]

2.2. Behaviour of stock volatility

Since we assume that the firm value follows a stochastic process with constant volatility it is interesting to check how the stock volatility changes with respect to the money ratio or the time to maturity of the outstanding warrants. Thus we present some simulation results which are all based on an extremely high dilution factor $\lambda = 1$. First, we analyze the sensitivity of the stock volatility $\sigma_S(S(V),\tau)$ to changes of the firm value per share of common stock just before maturity, i.e., in the limiting case $\tau \to 0$, $\tau > 0$. Table 1 shows this behaviour when $K = 100$, $\lambda = 1$ and $\sigma_V = 0.30$. It may be noted that, while the stock volatility remains constant with increasing money ratio for out-of-the-money warrants ($V/N < K$), it changes dramatically when V/N reaches K: the stock volatility falls from 0.30 to 0.225. Furthermore, when V/N exceeds K, the stock volatility falls to just above $\sigma_V/(1 + \lambda) = 0.15$, and then rises again with decreasing increments for increasing money ratio. The stock's elasticity $\varepsilon_{S,V}$ converges to 1 as V goes to infinity, and therefore by (6) the stock volatility σ_S converges to the asset volatility $\sigma_V = 0.30$ as V goes to infinity. Since we have for $\lambda = 1$

$$\lim_{\tau \to 0} \sigma_S = \left(\frac{1}{2} \frac{V/N}{S} \right) \sigma_V = \left(\frac{V}{V + nK} \right) \sigma_V \tag{7}$$

when the warrants are in-the-money ($V/N > 100$),[12] stock volatility behaves *as if* the exercise proceeds are invested in a riskless asset. For example, if the firm value per share of common stock (before the warrants are exercised) is $V/N = 200$, then the influx of $K = 100$ per outstanding warrant, invested in a riskless asset, would reduce the asset volatility according to

$$\sigma_V^{\mathrm{P}} = \left(\frac{200N}{200N + 100n} \right) \sigma_V = 0.20 \tag{8}$$

[10] Alternatively, we can solve Eq. (6) for σ_V, and insert the resulting analytical expression for σ_V in formula (3). Then, we simply find the value of W that satisfies Eq. (3) through the same sort of numerical search procedure.

[11] As long as V and σ_V are known, solving the Black–Scholes partial differential equation with the stock price as the underlying state variable subject to the boundary condition $W_T = \max\{S_T - K, 0\}$ and using the non-constant stock volatility $\sigma_S(S(V),\tau)$ also yields the correct warrant value. The relationship between both approaches is discussed in Geske (1979) in the context of compound options, and more recently in Jarrow and Trautmann (1991) and Bensoussan et al. (1992) in the context of equity warrants.

[12] See the footnotes of Table 1 for elaboration about $\lim_{\tau \to 0} \sigma_S$.

Table 1
Behaviour of stock volatility $\sigma_S \equiv \sigma_S(S(V),\tau)$ in the limiting case $\tau \to 0$, $\tau > 0$ for different firm values per share of common stock V/N

V/N	$\lim_{\tau \to 0} S$	$\lim_{\tau \to 0} W$	$\lim_{\tau \to 0} \frac{\partial \bar{S}(V,\tau)}{\partial V}$ [a]	$\lim_{\tau \to 0} \varepsilon_{S,V}$ [b]	$\lim_{\tau \to 0} \sigma_S(S,\tau)$
0^+ [c]	0^+	0	1.00	1.000	0.300
50	50	0	1.00	1.000	0.300
80	80	0	1.00	1.000	0.300
90	90	0	1.00	1.000	0.300
100^-	100^-	0	1.00	1.000	0.300
100	100	0	0.75	0.750	0.225
100^+	100^+	0^+	0.50	0.500^+	0.150^+
110	105	5	0.50	0.524	0.157
120	110	10	0.50	0.545	0.164
150	125	25	0.50	0.600	0.180
200	150	50	0.50	0.667	0.200
400	250	150	0.50	0.800	0.240
1000	550	450	0.50	0.909	0.273
5000	2550	2450	0.50	0.980	0.294

The calculations are based on the parameter values $K = 100$ (exercise price), $\lambda = 1$ (dilution factor) and $\sigma_V = 0.30$ (asset volatility).

[a] $$\lim_{\tau \to 0} \frac{\partial \bar{S}(V,\tau)}{\partial V} = 1 - \frac{n}{1+\lambda}\left(\lim_{\tau \to 0} \frac{\partial C^{BS}(V/N,\tau,K)}{\partial V}\right)$$

where

$$\frac{\partial C^{BS}(\cdot)}{\partial V} = \frac{\partial C^{BS}(\cdot)}{\partial (V/N)} \frac{\partial (V/N)}{\partial V}$$

and

$$\lim_{\tau \to 0} \frac{\partial C^{BS}(V/N,\tau,K)}{\partial (V/N)} = \begin{Bmatrix} 1 \\ 0.5 \\ 0 \end{Bmatrix} \text{ for } \begin{Bmatrix} V/N & > & K \\ V/N & = & K \\ V/N & < & K \end{Bmatrix}.$$

[b] $$\varepsilon_{S,V} \equiv \frac{\partial S}{\partial V}\frac{V}{S} = \frac{\partial \bar{S}}{\partial V}\frac{V/N}{S}$$

where $S \equiv S(V,\tau)$ and $\bar{S} \equiv \bar{S}(V,\tau)$.

[c] The notation a^+ and a^- means $\lim_{\epsilon \to 0} a + \epsilon$ and $\lim_{\epsilon \to 0} a - \epsilon$, respectively, where $\varepsilon > 0$.

where σ_V^P denotes the "post-exercise" asset volatility. Since σ_V^P must equal the "post-exercise" stock volatility σ_S^P, we have

$$\lim_{\tau \to 0} \sigma_S = \sigma_S^P. \tag{9}$$

But this relation holds only as long as the exercise proceeds are invested in a riskless asset. In general, there will be a jump in stock volatility at the warrant's expiration date, whose size depends on the assumed reinvestment risk of the exercise proceeds.

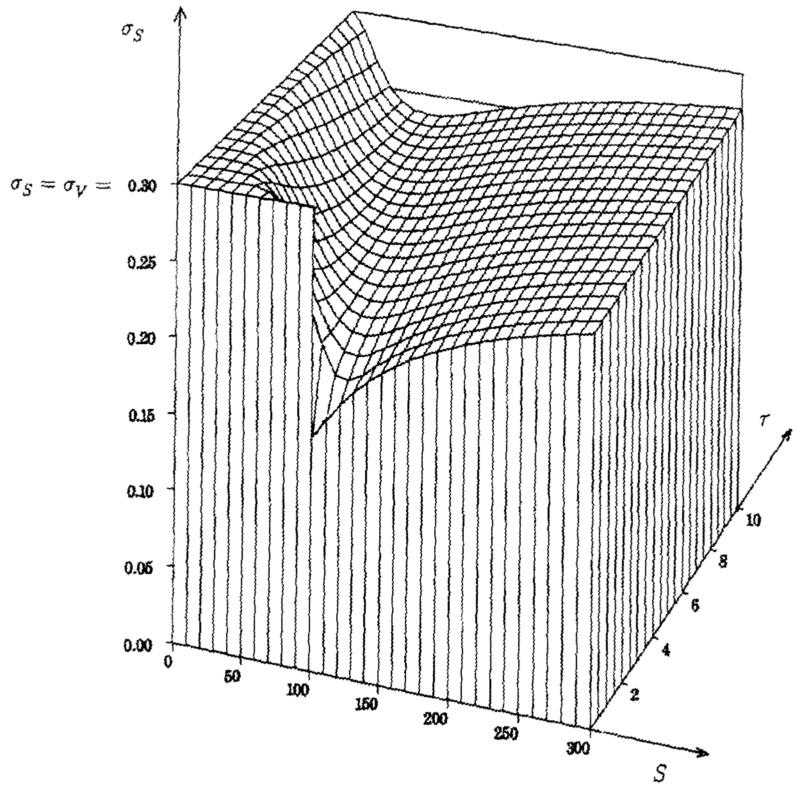

Fig. 1. Non-stationarity of stock volatility. Parameters: exercise price, $K = 100$; dilution factor, $\lambda = 1$; asset volatility, $\sigma_V = 0.30$; riskless interest rate, $r = 0.07$.

Fig. 1 visualizes the non-stationarity of stock volatility as a function of the stock price level and the time to maturity of the outstanding warrants. [13] For $\tau \to 0$ the figure displays the relationship between $S(V,\tau)$ and $\sigma_S(S(V),\tau)$ which is already contained in Table 1. It can readily be seen that the function $\sigma_S(S,\tau)$ is continuous in S for $\tau > 0$ but the limit $\lim_{\tau \to 0} \sigma_S(S,\tau)$ is discontinuous in S. Furthermore, Fig. 1 clearly demonstrates two important properties of stock volatility behaviour:

(1) Stock volatility is *below* asset volatility as long as $0 < \tau < \infty$ and $0 < S(V,\tau) < \infty$.

(2) Stock volatility is *most sensitive* to changes in stock price when the outstanding warrants are *near maturity* and *at the money*.

[13] Given the parameters S, τ, K, σ_V, r and λ, the stock volatility $\sigma_S \equiv \sigma_S(S,\tau)$ has been calculated by solving the nonlinear Eqs. (5) and (6) simultaneously for the two unknowns, V/N and σ_S, by a numerical routine (see footnote 9).

Since the returns of the stock and the warrant are perfectly correlated, the asset volatility can be split up as follows

$$\sigma_V = \sigma_S\left(\frac{NS}{V}\right) + \sigma_W\left(\frac{nW}{V}\right). \tag{10}$$

Property (1) holds because the warrant's volatility σ_W is strictly positive, while property (2) holds because option volatility is most sensitive with respect to stock price changes for near-maturity, at-the-money options.

2.3. Valuation error of option-like warrant valuation

Several studies on warrants, theoretical as well as empirical, have ignored the dilution effect and valued the warrant as an otherwise identical call option on common stock of a firm without warrants. [14] We now investigate the bias resulting from this option-like warrant valuation. Assuming again (A1) and (A2), we compare the warrant values resulting from our valuation model to those obtained by simply using the Black–Scholes formula. More precisely, for given parameters S, τ, K, λ, σ_V and r we solve the Eqs. (5) and (6) simultaneously for the model input parameters σ_S and V/N and compare the option-like Black–Scholes warrant value

$$C^{BS} = C^{BS}(S,\tau,K,\sigma_S,r) \tag{11}$$

to the *true* warrant value according to Eq. (3)

$$W = C^{BS}(V/N,\tau,K,\sigma_V,r)/(1+\lambda).$$

Figs. 2 and 3 show the difference between C^{BS} and W as a function of S and τ. The other model input parameters are fixed at $K=100$, $\lambda=1$, $\sigma_V=0.30$ and $r=0.07$, like in the stock volatility simulation of the previous section. Since option and warrant pricing models primarily explain and predict the *time value* of options and warrants, respectively, we first analyse the difference $C^{BS}-W=(C^{BS}-\underline{W}^E)-(W-\underline{W}^E)$ relative to the true time value of the warrant, $W-\underline{W}^E$, where

$$\begin{aligned}\underline{W}^E &= \mathrm{Max}\{0,\ S-K\cdot e^{-r\tau}\}\\ &= \mathrm{Max}\{0,\ V/N-K\cdot e^{-r\tau}\}/(1+\lambda)\end{aligned} \tag{12}$$

denotes the (European) lower bound or intrinsic value of the warrant.

The u-shaped graph of Fig. 2 tells us that this relative difference in time value *is negligible* for at-the-money ($S\approx Ke^{-r\tau}$) warrants. This was, at least for us,

[14] See, for example, Schwartz (1977) and Trautmann (1986).

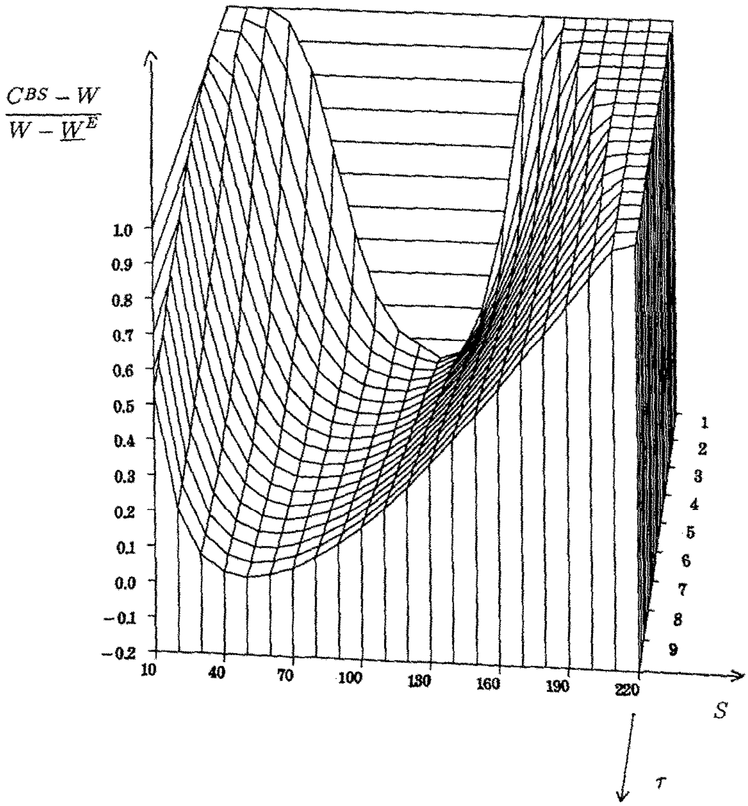

Fig. 2. Valuation error (in percentage points) of option-like warrant valuation relative to the true time value. Parameters: exercise price, $K = 100$; dilution factor, $\lambda = 1$; asset volatility, $\sigma_V = 0.30$; riskless interest rate, $r = 0.07$.

very *surprising* since the time value to be explained by a warrant valuation model is highest in absolute terms just for these at-the-money warrants.

Only for deep-out-of-the-money ($S \ll Ke^{-r\tau}$) warrants and deep-in-the-money ($S \gg Ke^{-r\tau}$) warrants option-like warrant valuation produces *significantly higher* time values than the "true" warrant model. Now, it is well-known that the time value of deep-in-the-money options is small relative to their intrinsic value. Therefore it is reasonable to expect that for *in-the-money* warrants option-like warrant valuation produces only a *small* percentage valuation error defined as $(C^{BS} - W)/W$. This conjecture is confirmed by the graph of Fig. 3. Even for *out-of-the-money* warrants with *more* than, say, two years until maturity a relative valuation error different from the zero level is hard to recognize. Only for deep-out-of-the-money warrants and out-of-the-money, near-maturity warrants the

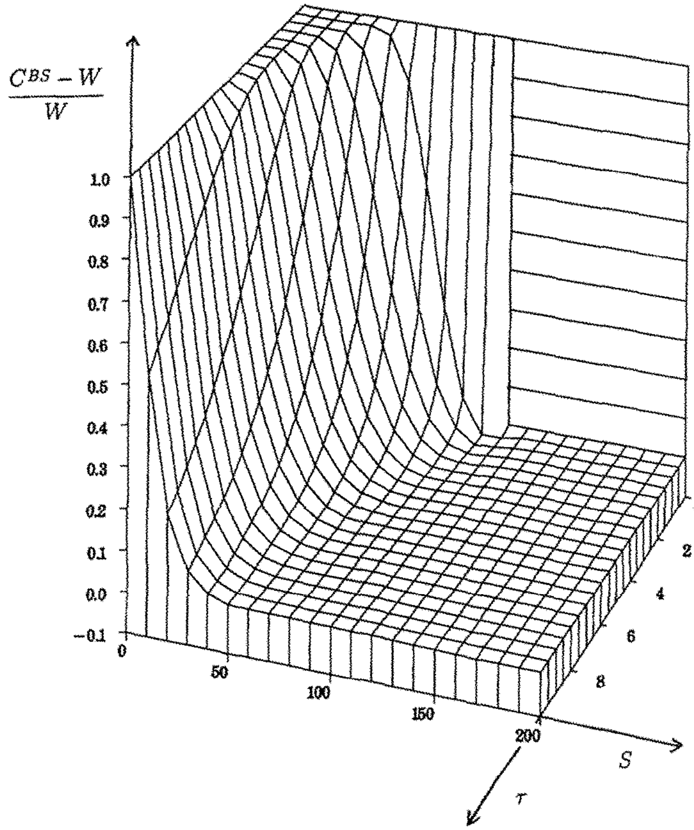

Fig. 3. Relative error (in percentage points) of option-like warrant valuation. Parameters: exercise price, $K = 100$; dilution factor, $\lambda = 1$; asset volatility, $\sigma_V = 0.30$; riskless interest rate, $r = 0.07$.

option-like warrant value is substantially higher than the true warrant value. [15] Since this result is based on an extremely high dilution factor, namely $\lambda = 1$, it was again *unexpected* to us.

Moreover, we found the seemingly paradox result that option-like warrant valuation leads for in-the-money, near-maturity warrants to a slightly lower warrant value compared to the true value. Obviously, this situation occurs because omitting the coefficient $1/(1 + \lambda)$ in valuation formula (3) is overcompensated by the *lower* model input parameters $S < V/N$ and $\sigma_S < \sigma_V$. In sum, the bias from option-like warrant valuation is small even for an extreme potential dilution since according to our warrant valuation model potential equity dilution is already

[15] For a time to maturity of less than ten years and otherwise identical parameters the absolute valuation error, $C^{BS} - W$, is *less* than 1.0 in absolute terms.

anticipated in the current stock price. To obtain warrant values with acceptable accuracy, adjustments to the Black–Scholes formula are *not* needed except perhaps for deep-out-of-the-money warrants. This holds especially for a more realistic, less extreme potential dilution.

3. Empirical robustness

The theoretical analysis in Section 2 proves the robustness of option-like warrant valuation in an all equity firm where the asset volatility is constant, dividends are not paid, and sequential exercise is not optimal for the warrantholders. [16] As these conditions do not hold simultaneously in real valuation problems, it is interesting to know whether there is a dilution-related pricing bias when valuing listed warrants in an option-like fashion. In doing this, we choose a constant variance (CV) diffusion model, which admits the possibility of early exercise of the American-type warrants. Our finite difference approach to approximate solutions to a partial differential equation under boundary conditions, as pioneered by Schwartz (1977), checks just before each ex-dividend date whether premature exercise is optimal. While this numerical approach is very flexible, it has the disadvantage that the estimation of implied standard deviations is very time-consuming. Therefore the following analysis relies only on historical volatility estimates.

3.1. Data

The data employed in this study consisted of 50,960 daily market prices for 37 warrants written on 16 German stocks (''blue chips'') and listed on the *Frankfurter Wertpapierbörse* (Frankfurt Securities Exchange) during the period from January 1, 1979 through December 30, 1990. [17] The potential increase of nominal capital due to a single warrant issue (dilution factor) ranges between 1 and 23 percent, while on a cumulative basis, taking all simultaneously outstanding issues into account, the potential increase ranges between 2 and 49 percent. This potentially high equity dilution, even for companies with a large capital stock, underscores the practical importance of our analysis. Warrant prices as well as the exercise conditions, stock prices (odd lot prices), daily stock return data and dividend data were taken from the *Karlsruher Kapitalmarktdatenbank* (KKMDB,

[16] The robustness result would hold also for a levered firm as long as its *equity* volatility is constant.

[17] At the end of our sample period there were almost twice as many (68) warrants written on German stocks outstanding. Many warrants have been excluded from the sample because of one of the following reasons: (1) virtual illiquidity of the corresponding market; (2) no access to price data; (3) the exercise price is denominated in US-dollar, complicating the valuation procedure. Descriptive statistics for the warrants included in the sample are available from the authors upon request.

Karlsruhe Capital Market Data Base). Wherever necessary, adjustments of the observed stock prices were made for the following reason. New shares issued because of warrant exercise are for the first time entitled to dividend for the business year in which the warrant exercise takes place. Assuming that the stock price falls at the ex-dividend date for the old shares, t_j, by the known amount D_j, the warrant valuation should be based on the stock price net of the present value of the escrowed dividend payment

$$S_t = S_t^{\text{obs}} - D_j \mathrm{e}^{-r(t_j - t)}$$

if date t lies between the end of a business year and the next ex-dividend date t_j, and $S_t = S_t^{\text{obs}}$ otherwise. S_t^{obs} is the observed odd lot stock price (Kassakurs) available from the KKMDB tapes. The continuously compounded risk-free rate of interest was obtained from the average yield of treasury bonds maturing nearest in time to the warrant expiration date. The term structure of these bonds (assumed to be default free) was taken from various issues of the *Statistische Beihefte zu den Monatsberichten der Deutschen Bundesbank.*

Of the determinants in the call option pricing formulas, all but the future stock volatility are known or can be estimated with little difficulty. In our context we have at least one additional estimation problem: the prediction of the date and amount of dividend payments during the lifetime of outstanding warrants. Inspection of the stock's historical dividend series tells us that dividend policy can change dramatically within the lifetime of long-lived warrants. Future dividend prediction is therefore a difficult task. For the sake of simplicity, we assume a constant dividend policy during a warrant's lifetime. As a predictor of the yearly dividend amount per share we use the average dividend amount per share of the 4-year period preceding the price observation date. As a proxy for the future volatility a historical estimate is taken, obtained from the most recent 200 observations of the daily return of the underlying stock.

3.2. Results

A representative summary of the empirical results is presented in Tables 2 and 3. Table 2 presents the mean, standard deviation (SD), minimum and maximum of the percentage prediction error as well as the absolute percentage prediction error of the total sample and each warrant in the sample. Percentage prediction error is defined as the model value minus the market price (W) divided by the market price, $(C^{\text{CV}} - W)/W$. The mean percentage error demonstrates that there are substantial differences between market prices and CV model values if we look at a certain warrant issue, ranging from −38% (DRESDNER BANK 86–91) to 90% (METALLGES. 87–97). Especially for those warrants of CONTI-GUMMI, KAUFHOF, METALLGES., and VW which have been issued and listed in 1986 and later on model prices are significantly higher than the corresponding market prices. With four exceptions (BAYER 87–97, BAY. HYP.-BANK 86–96,

Table 2
Magnitude of deviation of American constant variance (CV) diffusion model values from observed prices for listed warrants over the period January 1, 1979 through December 30, 1990

Warrant		Obs.	$(C^{CV} - W)/W$				$\|(C^{CV}\text{-}W)/W\|$			
			mean	SD	min.	max.	mean	SD	min.	max.
ALLIANZ	89–96	354	0.06	0.08	−0.15	0.26	0.08	0.05	0.00	0.26
BASF	74–86	1733	−0.19	0.17	−0.64	0.16	0.21	0.14	0.00	0.64
BASF	85–94	1456	−0.02	0.12	−0.33	0.45	0.08	0.08	0.00	0.45
BAYER	79–89	2429	0.04	0.25	−0.58	0.93	0.17	0.18	0.00	0.93
BAYER	82–87	1146	−0.13	0.17	−0.54	0.17	0.16	0.14	0.00	0.54
BAYER	84–94	1682	0.00	0.10	−0.26	0.42	0.07	0.08	0.00	0.42
BAYER	85–95	1452	0.04	0.15	−0.35	0.61	0.10	0.11	0.00	0.61
BAYER	87–97	806	−0.03	0.28	−0.51	0.68	0.25	0.13	0.00	0.68
BAY. HYP.-BANK	85–94	1217	−0.18	0.23	−0.67	0.34	0.26	0.15	0.00	0.67
BAY. HYP.-BANK	86–96	1091	−0.02	0.27	−0.53	0.84	0.22	0.15	0.00	0.84
COMMERZBANK	78–88	2304	−0.26	0.21	−0.73	0.37	0.29	0.17	0.00	0.73
COMMERZBANK	84–89	1185	−0.05	0.09	−0.30	0.31	0.08	0.06	0.00	0.31
CONTI-GUMMI	84–94	1701	0.03	0.10	−0.27	0.38	0.08	0.07	0.00	0.38
CONTI-GUMMI	86–96	1051	0.37	0.33	−0.17	1.14	0.39	0.30	0.00	1.14
CONTI-GUMMI	87–97	797	0.66	0.51	−0.10	1.96	0.67	0.51	0.00	1.96
DEGUSSA	83–93	1767	−0.15	0.16	−0.69	0.28	0.18	0.12	0.00	0.69
DEUTSCHE BANK	86–96	1184	0.22	0.34	−0.35	1.29	0.30	0.28	0.00	1.29
DEUTSCHE BANK	87–92	757	0.03	0.34	−0.46	0.90	0.28	0.19	0.00	0.90
DRESDNER BANK	83–90	1351	−0.08	0.16	−0.54	0.43	0.14	0.11	0.00	0.54
DRESDNER BANK	83–93	1766	0.00	0.16	−0.25	0.71	0.12	0.11	0.00	0.71
DRESDNER BANK	84–92	1534	−0.02	0.11	−0.26	0.42	0.08	0.07	0.00	0.42
DRESDNER BANK	86–96	1171	0.16	0.25	−0.45	0.83	0.24	0.18	0.00	0.83
DRESDNER BANK	86–91	966	−0.38	0.21	−0.82	0.28	0.39	0.20	0.00	0.82
HOECHST	75–90	2806	0.01	0.18	−0.53	0.83	0.12	0.14	0.00	0.83
HOECHST	79–89	2366	0.03	0.21	−0.44	0.85	0.14	0.16	0.00	0.85
HOECHST	83–93	1786	−0.05	0.08	−0.33	0.19	0.07	0.07	0.00	0.33
KAUFHOF	84–94	1495	0.21	0.20	−0.18	0.87	0.21	0.19	0.00	0.87
KAUFHOF	85–95	1229	0.33	0.19	−0.13	0.81	0.33	0.18	0.00	0.81
KAUFHOF	86–98	963	0.75	0.34	0.12	1.76	0.75	0.34	0.12	1.76
METALLGES.	86–96	1099	0.68	0.39	0.06	1.72	0.68	0.39	0.06	1.72
METALLGES.	87–97	725	0.90	0.64	0.23	2.37	0.90	0.64	0.23	2.37
PREUSSAG	84–91	1507	−0.22	0.28	−0.77	0.59	0.31	0.17	0.00	0.77
SCHERING	83–90	1640	−0.05	0.13	−0.42	0.30	0.11	0.09	0.00	0.42
SIEMENS	83–90	1706	−0.05	0.10	−0.39	0.23	0.08	0.08	0.00	0.39
VW	86–95	1217	0.05	0.17	−0.46	0.93	0.13	0.12	0.00	0.93
VW	86–01	998	0.52	0.29	−0.12	1.80	0.53	0.29	0.00	1.80
VW	88–98	523	0.46	0.17	0.02	0.80	0.46	0.17	0.02	0.80
All warrants	50960	0.05	0.34	−0.82	2.37	0.22	0.26	0.00	2.37	

The model values are based on historical volatility estimates.

DRESDNER BANK 86–91) this result holds for all warrants issued in or after 1986. This is mainly due to the crash in October 1987, causing a downward adjustment in price and volatility expectations afterwards. Hence, it is remarkable

Table 3
Parameter estimates of the multiple regression [a] testing for systematic pricing biases of the American constant variance (CV) diffusion model (p-values are shown in parentheses; total sample: daily prices of German warrants in the period from January 1, 1979 through December 30, 1990)

Sample [b]	Obs.	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\alpha}_6$	R^2
79–87	26731	−0.399	0.033	0.055	0.054	−0.007	−0.010	0.597	0.330
		(0.00)	(0.27)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	
79–87/in	23690	−0.405	0.524	−0.013	0.046	−0.009	0.006	0.681	0.322
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	
79–87/at	2069	−2.035	0.244	0.939	0.096	−0.001	0.024	0.910	0.561
		(0.00)	(0.02)	(0.00)	(0.00)	(0.60)	(0.00)	(0.00)	
79–87/out	972	−1.207	1.273	0.574	0.026	−0.003	−0.006	−0.531	0.274
		(0.00)	(0.00)	(0.00)	(0.05)	(0.56)	(0.22)	(0.00)	
87–90	24229	−0.502	1.098	0.077	0.077	−0.019	−0.019	0.181	0.488
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	
87–90/in	19013	0.029	0.700	0.034	0.069	−0.045	−0.012	−0.268	0.479
		(0.15)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	
87–90/at	2559	−1.963	2.609	0.302	0.098	0.004	0.009	1.168	0.824
		(0.00)	(0.00)	(0.00)	(0.00)	(0.09)	(0.03)	(0.00)	
87–90/out	2657	−3.863	4.989	0.874	0.083	0.077	−0.007	1.858	0.755
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.26)	(0.00)	

[a] $$\frac{(C^{CV} - W)}{W} = \alpha_0 + \alpha_1 \hat{\sigma}_S + \alpha_2 \frac{S}{Ke^{-r\tau}} + \alpha_3 \tau + \alpha_4 D + \alpha_5 r + \alpha_6 \lambda^c$$

Test specification notation:
$W \equiv$ market price of the warrant;
$C^{CV} \equiv$ American CV model value of the warrant;
$\hat{\sigma}_S \equiv$ historical volatility of stock return (based on 200 daily returns);
$S \equiv$ stock price;
$K \equiv$ exercise price;
$\tau \equiv$ years to expiration;
$D \equiv$ yearly dividend amount (assumed to be constant);
$r \equiv$ annualized riskless rate of return;
$\lambda^c \equiv$ cumulative dilution factor, $\lambda^c = \Sigma_{i=1}^m \lambda_i$, where m denotes the number of simultaneously outstanding warrant issues of a company and λ_i denotes the dilution factor of the ith warrant issue.
For convenience the subscript j and the disturbance term ε_j are omitted from the test specification.
[b] 79–87 denotes the subperiod before the crash on October 19, 1987, and 87–90 denotes the subperiod thereafter. Subsamples with respect to the money ratio are formed according to the following classification: in $= S/Ke^{-r\tau} \geqslant 1.1$; at $= 0.9 \leqslant S/Ke^{-r\tau} < 1.1$; out $= 0.9 > S/Ke^{-r\tau}$.

that the pooling of all observations results in the close correspondence between market prices and CV model values shown in the last row of Table 2. This confirms earlier results based on different samples reported in Trautmann (1986) and Schulz and Trautmann (1989). Not surprisingly, mean absolute percentage prediction errors, provided in column six of Table 2, are higher. But even Lauterbach and Schultz (1990, table VII), working with implied volatility estimates, show average absolute percentage errors for the dividend and dilution-adjusted Black–Scholes model of more than 13.5% compared to 22% in Table 2. [18]

Table 3 contains the results of a multiple regression testing for systematic relation between relative prediction error and option model determinants and the cumulative dilution factor λ^c, respectively. The cumulative dilution factor is defined as $\lambda^c \equiv \Sigma_{i=1}^{m} \lambda_i$, where m denotes the number of simultaneously outstanding warrant issues of a company and λ_i denotes the dilution factor of the ith warrant issue. This multiple regression was performed on the pooled observations as well as on subsamples. The latter are formed with respect to the money ratio, observation period and capital structure of the underlying firm. [19]

Examination of the second last column of Table 3 indicates that the null hypothesis, that there is no relationship between prediction error and magnitude of potential dilution, is soundly rejected. In the pre-crash period 79–87, the model appears to overprice in- and at-the-money-warrants on stocks with high potential dilution and to underprice warrants on stocks with low potential dilution. We believe, however, that this is a spurious correlation since high dilution factors were especially observed in the overpricing period from July 1983 through October 1987. As a second explanation serves the conjecture that in contradiction to our model assumption, potential dilution was not completely anticipated in observed stock prices. Unexpectedly, the opposite relationship holds for out-of-the-money warrants in this subperiod. The results of the post-crash subperiod 87–90 reflect almost the expected behaviour. Out-of-the-money warrants are significantly overpriced by the CV model, while for in-the-money warrants there holds a negative relationship between prediction error and cumulative dilution factor. For all other subsamples formed with respect to money ratio and capital structure, not explicitly discussed here, the parameter estimate $\hat{\alpha}_6$ is also not stable over time. Therefore, we conclude that there is no constant dilution-related pricing bias of the American CV model with the stock price as the state variable. This supports the empirical robustness of option-like warrant valuation.

4. Concluding remarks

In sum, we were able to demonstrate feasibility of option-like warrant valuation for at- and in-the-money warrants. We presented a methodology for arriving at the

[18] In addition, Lauterbach and Schultz examine the pricing performance of a dividend and dilution-adjusted constant elasticity of variance (CEV) diffusion model whose average absolute percentage error is slightly lower (11.3%). Schulz and Trautmann (1989), however, get the opposite result based on historical volatilities when comparing the CEV model with the CV model, although the latter is a special case of the former. So obviously when using *historical* volatilities, the CEV model does not provide sufficient improvement in price forecasts to overcome the noise associated with estimating an additional parameter.

[19] The results for different capital structures are not reported in Table 3.

correct warrant value working only with the market evaluated data on common stock. This enabled us to study the relationship between true and option-like warrant values. As a by-product, our approach allowed a detailed analysis of the non-stationarity of stock volatility for a firm with outstanding warrants. The present approach assumes a very simple capital structure of the firm. An extension to include different kinds of debt is straightforward. However, the corresponding simulation results (not reported here) cloud the issue with respect to the non-stationarity of stock volatility while option-like warrant valuation is still robust when warrants are at- and in-the-money. Out-of-the-money warrants should be valued either according to our methodology or according to the approach laid out in a very general setting (including stochastic interest rates and stochastic volatility) in Jarrow and Trautmann (1991).

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