# Robust Recovery Risk Hedging: Only the First Moment Matters*

by

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Credit derivatives are subject to at least two sources of risk: the default time and the recovery payment. This paper examines the impact of modeling the recovery payment on hedging strategies in reduced-form models. We show that all hedging approaches based on a quadratic criterion do only depend on the *expected* recovery payment at default and not the whole shape of the recovery payment distribution if the underlying hedging instrument (say, a defaultable zero coupon bond with total loss in case of default, or common stock) jumps to/or reaches a pre-specified value when the credit event occurs. This justifies assuming a *certain* recovery rate conditional on default time and interest rate level. Hence, this result allows a simplified modeling of credit risk. Moreover, in contrast to the existing literature, our model yields explicit solutions for the hedge ratio even when all relevant quantities are stochastic.

**JEL: C10, G13, G24**

**Key words:** Credit Risk, Recovery Risk, Hedging

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Credit derivatives are subject to at least two sources of risk: the default time and the recovery payment. This paper examines the impact of modeling the recovery payment on hedging strategies in reduced-form models. We show that all hedging approaches based on a quadratic criterion do only depend on the *expected* recovery payment at default and not the whole shape of the recovery payment distribution if the underlying hedging instrument (say, a defaultable zero coupon bond with total loss in case of default, or common stock) jumps to/or reaches a pre-specified value when the credit event occurs. This justifies assuming a *certain* recovery rate conditional on default time and interest rate level. Hence, this result allows a simplified modeling of credit risk. Moreover, in contrast to the existing literature, our model yields explicit solutions for the hedge ratio even when all relevant quantities are stochastic.

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# Contents

# 1 Introduction

In contrast to a large amount of theoretical and empirical work available on the valuation of credit derivatives (see Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Lando (2004) for reviews), hedging of credit derivatives remains a largely unexplored avenue of research. When valuing and hedging credit derivatives, two quantities are crucial. The first is the *probability of default* (or default intensity, if it exists), and the second is the *default recovery* (or recovery rate) in the event of default. While in traditional models the recovery rate is given exogenously as a known constant at the default time[1], this rate is stochastic in reality, even conditional on the default time. This uncertainty in the *default recoveries* of both the underlying instrument (e.g., equity) and particularly the credit derivative (e.g., a convertible bond) is perhaps the most important reason why hedges in practice are not self-financing.

The main purpose of this paper is therefore not valuation but hedging credit derivatives in the presence of recovery risk in a reduced-form framework. Since in general, the common objective of arbitrageurs in credit derivatives markets is to minimize the variance of the hedging costs, we focus on the *locally risk-minimizing* hedging strategy. Föllmer and Sondermann (1986) pioneered this approach in the special case where the underlying instrument follows a martingale. At each point in time they require that the risk, defined as the expected quadratic hedging costs, is minimized. However, in semimartingale models a risk-minimizing strategy does not always exist. Therefore, Schweizer (1991) introduced a locally risk-minimizing (LRM) hedging strategy and showed that – under certain assumptions – a strategy is *locally* risk-minimizing if the cost process is a martingale which is orthogonal to the martingale part of the underlying instrument process. The LRM-strategy is *mean-self-financing*, that is at each point in time the expected sum of discounted cash infusions or withdrawals until maturity is zero. The value of the hedge portfolio is then the discounted expected terminal payoff of the option under the so-called *minimal equivalent martingale measure*.

Hedging strategies for credit derivatives within the reduced-form framework have been studied in the literature. On the one hand, there exist quite tractable models where the hedge ratio is explicitly given. For instance, Bielecki, Jeanblanc and Rutkowski (2007) derived a hedging strategy for credit derivatives using credit default swaps (CDS) and a position in the riskless money market account. The model

[1] One exception is Guo, Jarrow and Zeng (2009). They model the recovery rate process itself.

is easily implemented due to the fact that the interest rate level is assumed to be flat at level null and both the default intensity and the recovery payment are deterministic, i.e. the default time is the only random quantity. On the other hand, there exist models that allow all of the relevant quantities to be stochastic, but only yield hedge ratios that contain the predictable process appearing in the above mentioned martingale representation of the claim to hedge. Therefore, using these strategies, one has to calculate this process. If all relevant quantities are stochastic and possibly dependent, the situation quickly becomes hopeless. Models of this type can be found, for instance, in Bielecki, Jeanblanc and Rutkowski (2008, 2011).[2] Biagini and Cretarola (2007, 2009, 2012) apply the local risk-minimization approach to credit derivatives. However, they assume the recovery payment to be *constant conditional on default* and provide explicit hedge ratios under the additional assumption of either the interest rate or the default rate being stochastic. In this paper, we fill the gap between those two classes of models and, based on a result from Heath (1995), provide explicit representations of the LRM-hegde ratio in case the recovery payment is *stochastic conditional on default* and both stochastic but independent interest and default rates. However, due to a result from Brigo and Mercurio (2006), this independence assumption is no major restriction.

We derive LRM-hedging strategies for reduced-form models when there are two hedging instruments: a locally riskless money market account and a risky underlying instrument. We denote the recovery rate as *single-stochastic* if the recovery amount depends only on the default event and the interest rate. We call the recovery rate *doubly-stochastic* if the recovery amount also depends on the realization of another random variable. Corresponding model variants are examined for the reduced-form model framework. In this framework we assume the existence of a tradable zero coupon bond with total loss at default of the firm under consideration. However, we emphasize that the defaultable zero coupon bond can be replaced by stocks, if the stock is assumed to fall to a prespecified level at the time of default.

It turns out that the corresponding LRM-strategy is not only mean-self-financing but also self-financing if the default recovery is single-stochastic. That is, as long as the recovery amount is known in the event of default, there exists a self-financing replication strategy for credit derivatives. Moreover, we find that in the more realistic case of doubly-stochastic default recoveries, the LRM-hedging strategy does

[2] In fact, there exists a large amount of sometimes overlapping published and unpublished papers by the same and related authors. For a complete list, we refer to Bielecki and Rutkowski (2002) and Chesney, Jeanblanc and Yor (2009).

only depend on the *expected* recovery amount, not on other characteristics of its distribution. This key result of the paper helps to justify the frequently made simplifying assumption that the default recovery is a constant, conditional on the default event, when pricing and hedging credit derivatives.

At first glance this result seems to contradict the result of Grünewald and Trautmann (1996) when deriving LRM-strategies for stock options in the presence of jump risk. In that setting the LRM-strategy depends additionally on the variance of the stock's jump amplitude. This key difference is due to the fact that in our model default of the firm implies that the underlying instrument's price jumps always to *zero* while in Merton's (1976) jump diffusion setting assumed by Grünewald and Trautmann (1996), the option's underlying stock price jumps to an arbitrary price level.

We also run a simulation to test the impact of the different model assumptions on the cumulative hedging costs. It will turn out that the latter are nearly unaffected by the whether the interest rate is deterministic or stochastic. However, they are affected by the assumptions imposed on the default rate. Therefore, our simulation results suggest that both the recovery and the default rate should be modelled as stochastic processes when hedging credit derivatives. We also test the LRM-strategy against alternative strategies (and alternative hedging instruments). First, we consider an extension of the duplication strategy using CDS contracts by Bielecki, Jeanblanc and Rutkowski (2007) to the case of doubly-stochastic recovery payment. Finally, we also consider two cross-hedging strategies. The first of them involves a junior bond of a comparable firm, i.e. an instrument with the same default intensity but a distinct default time. The second cross-hedging strategy involves a position in a credit index of the type investigated in Brigo and Morini (2011), i.e. a pool of credit names with the same credit quality (the same default rate) as the instrument we wish to hedge.

The paper is organized as follows: Section 2 describes hedging as a sequential regression and illustrates the paper's basic insight. Section 3 looks at locally risk-minimizing hedging policies in a reduced-form model when recovery is single-stochastic and doubly-stochastic, respectively. In Section 4, we also consider model extensions by assuming that either the interest rate or the default intensity or both are stochastic. In Section 5, we use simulated data to test the impact of the different model assumptions on the cumulative hedging costs. Section 6 concludes the paper. All technical proofs are given in Appendix A.

# 2 Hedging by Sequential Regression

In incomplete financial markets not every contingent claim is replicable. For this reason a lot of different hedging strategies have been evolved in the literature. On the one hand there exist hedging approaches searching self-financing strategies which reproduce the derivative at the best. On the other hand there are hedging strategies replicating the derivative exactly at maturity by taking into account additional costs during the trading period. While the first class of hedging strategies optimizes the *hedging error*, to be more precisely the difference between the pay-off of the derivative $F_T$ and the liquidation value of the hedging strategy, the other class minimizes the *hedging costs*. In a discrete time set-up Föllmer and Schweizer (1989) developed a hedging approach of the latter type, the so-called *locally risk-minimizing hedging*.

Table 1: *Hedging Concepts: An Overview.*

<table>
<tr><td></td><td>Complete Financial Market</td><td>Incomplete Financial Market</td><td></td></tr>
<tr><td>No Shortfall</td><td>Delta-Hedging<br>Black, Merton, Scholes (1973)</td><td>Superhedging<br>Naik and Uppal (1992)</td><td rowspan="3">No Restriction on Initial Costs</td></tr>
<tr><td rowspan="3">Shortfall Risk</td><td></td><td>Risk- & Variance-Minimizing Hedging<br>Föllmer and Sondermann (1986)<br><br>Locally Risk-Minimizing Hedging<br>Föllmer and Schweizer (1989)</td></tr>
<tr><td colspan="2">Globally Risk- and Variance-Minimizing Hedging<br>Schweizer (1995)</td></tr>
<tr><td colspan="2">Shortfall-Hedging<br>Föllmer and Leukert (1999)<br>(Global) Expected Shortfall-Hedging<br>Föllmer and Leukert (2000)<br>Local Expected Shortfall-Hedging<br>Schulmerich (2001), Schulmerich and Trautmann (2003)</td><td>Restriction on Initial Costs</td></tr>
</table>

When using two hedging instruments, the underlying asset with price process $S$ and the money market account with price process $B$, $\mathbf{H} = (h^S, h^B)$ describes the hedging strategy composed of $h^S$ shares in the underlying and $h^B$ shares in the money market account. In a discrete-time setting $\mathrm{V}_t(\mathbf{H}) = h^S_{t+1} S_t + h^B_{t+1} B_t$ denotes the *liquidation value* of the strategy, $G_t(\mathbf{H}) = \sum_{i=1}^{t} (h_i^S \Delta S_i + h_i^B \Delta B_i)$ the *cumulated gain* and finally $\mathcal{C}_t(\mathbf{H}) = \mathrm{V}_t(\mathbf{H}) - G_t(\mathbf{H})$ the cumulated *hedging costs* at time $t$. To achieve a locally risk-minimizing hedging strategy, Föllmer and Schweizer (1989) solve the following

**Problem 1 (Locally risk-minimizing hedging in discrete time)**
*Search the trading strategy* $\mathbf{H}$ *which replicates exactly the derivative* $F_T$ *at maturity* $T$ *and in addition minimizes the expected quadratic growth of the hedging cost at every point in time:*

$$E^P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2\,|\mathcal{G}_{t-1}\right] \to \min \;\; \textit{for all } t = 1,\ldots,T \textit{ and } \mathbf{H} \in \mathbb{H} \textit{ with } V_T(\mathbf{H}) = F_T\,.$$

A solution of Problem 1 is called *locally risk-minimizing hedging strategy* or *LRM-hedge*[3]. Föllmer and Schweizer (1989) have pointed out that Problem 1 is a sequential regression task that can be solved by backwards induction: In a first step we determine $h_T^S$ and $h_T^B$ by identifying the solution of the subproblem

$$\mathrm{E}^P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2|\mathcal{G}_{t-1}\right] \to \min \quad \text{for all} \quad h_t^S,\, h_t^B \quad \text{given} \quad \mathrm{V}_t(\mathbf{H}) \qquad (1)$$

for $t = T$ with $\mathrm{V}_T(\mathbf{H}) = F_T$. Since we have $\mathrm{V}_t(\mathbf{H}) = h_{t+1}^S S_t + h_{t+1}^B B_t$ for all dates $t = 0,\ldots,T-1$ we know $\mathrm{V}_{T-1}(\mathbf{H})$ and then we can solve the subproblem (1) for $t = T-1$ and thus obtain $h_{T-1}^S$ (as slope of the regression line) and $h_{T-1}^B$ (as intercept), and so on. Since $\Delta\mathcal{C}_t(\mathbf{H}) = \mathrm{V}_t(\mathbf{H}) - (h_t^S S_t + h_t^B B_t)$ holds, (1) is a linear regression problem which can be solved by the least square approach. Figure 2 illustrates this idea.

In the following, we show that this relation shows directly that two different ways of modeling recovery payments lead to the same locally risk-minimizing strategy when hedging a short position in credit derivatives. The first kind of recovery model assumes that the recovery rate is *single-stochastic* since it only depends on the default-time and perhaps the interest rate level as illustrated in part (a) of Figure 1 for a two period set-up. Thus, the recovery amount depends only on the time of default (and the interest rate level).

In the second kind of recovery model the recovery rate is called *doubly-stochastic* allowing in addition (to the default time and the term structure) other risk factors to influence the recovery payment (see part (b) of Figure 1). For example these additional factors can characterize the uncertain costs of financial distress or the uncertain time delay of the promised recovery payment. Thus, in this model the default time and the interest rate level do not uniquely determine the recovery payment.

[3]An LRM-hedge also solves the problem

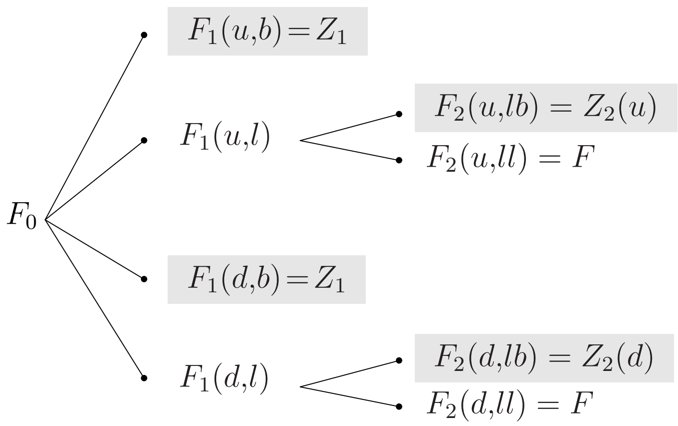
$$\mathrm{E}^P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2\,|\mathcal{G}_{t-1}\right] \to \min \text{ for all } t = 1,\ldots,T \text{ and } \mathbf{H} \in \mathbb{H} \text{ with } \mathrm{V}_T(\mathbf{H}) = F_T\,,$$

where $\Delta C_t(\mathbf{H}) = \Delta\mathcal{C}_t(\mathbf{H})/B_t$ denotes the discounted growth of the hedging costs and $B_t$ is the value of the money market account at time $t$.

Figure 1: *Single-stochastic versus doubly-stochastic recovery.*

Part (a) of this figure depicts the price process of a credit derivative with a recovery payment depending only on the default time (”$l$” denotes liquidity, ”$b$” bankruptcy) and the term structure (”$u$” denotes an up-tick and ”$d$” a down-tick of the interest rate). Conditional on default (and the given term structure) the recovery payment is known. The latter is not the case if the recovery payment is doubly-stochastic. Part (b) of the figure shows that conditional on default (and the given term structure) the recovery payment can take on $m$ different values $Z^1$, …, $Z^m$.

(a) Price process when recovery is single-stochastic.




(b) Price process when recovery is doubly-stochastic.

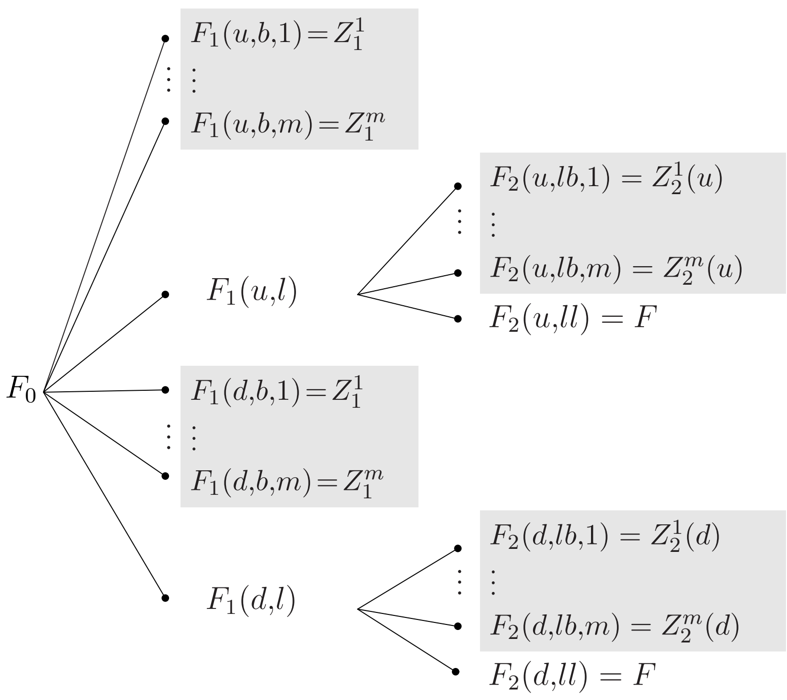
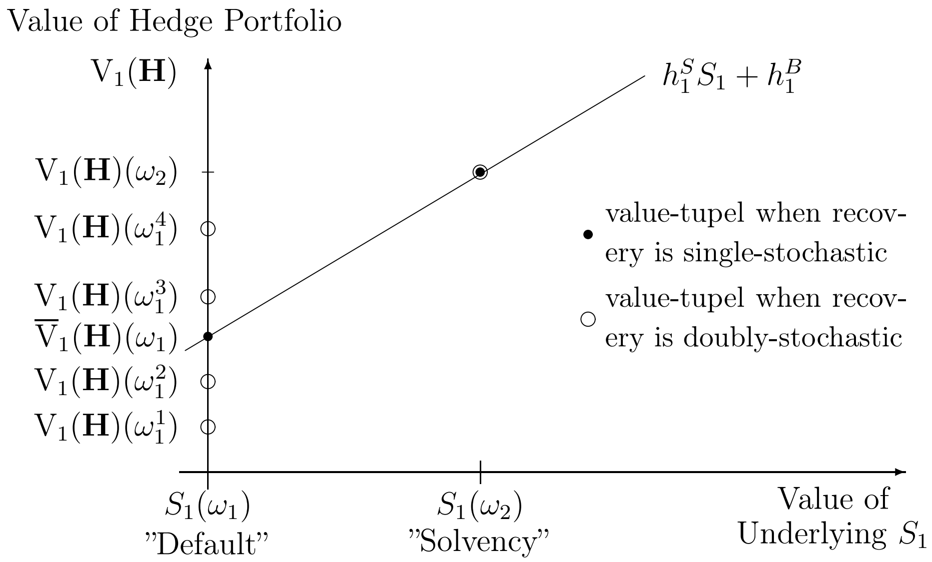


Figure 2 already illustrates the key result of this paper: the locally risk-minimizing hedging strategy for the credit derivative is the same for single- and doubly-stochastic recovery modeling, provided that the *expected* doubly-stochastic recovery payment conditional on the default time (and the term structure) coincides with the single-stochastic recovery payment conditional on the default time (and the interest rate level).

Figure 2: *LRM-strategy when recovery is doubly-stochastic.*

When recovery is doubly-stochastic the payment at default does not only depend on the default time and the interest rate level but also on another risk factor. Different realizations of this risk factor are denoted by the superscript $j$, so the state of the world at date $t = 1$ is $\omega_1^j$. Since the underlying (say, shares of common stock of the firm, or a corporate zero-bond with total loss at default written on the underlying firm) does not depend on the additional factor, its price is always zero at default, $S_1(\omega_1^1) = S_1(\omega_1^2) = \ldots = 0$. The symbol " $\circ$ " describes a possible realization of the discounted value of the hedge portfolio. To determine the LRM-hedge we have to run a regression for the five value tuples represented by the $\circ$-symbol. Alternatively, we can calculate in a first step the average value of the hedge portfolio $\overline{\mathrm{V}}_1(\mathbf{H})(\omega_1)$, conditional on the default event occurring. The latter pairs of values are denoted with the " $\bullet$" symbol. In a second step, we identify the slope for the regression line for the points " $\bullet$" (only two tuples, as you can see) which equals the slope of the first regression.



The insight provided by Figure 2 can be proven in a more formal way. We show that the single-stage regression approach (delivers the LRM-hedge of a defaultable claim assuming doubly-stochastic recovery) and a two-stage procedure (delivers the LRM-hedge of a defaultable claim assuming single-stochastic recovery which coincides at any default time with the expectation of the doubly-stochastic recovery conditional on the default time) provide the same result. With the conventions $p_i = \sum_j p(\omega_i^j)$, $\overline{S}_t(\omega_i^j) = \sum_k S_t(\omega_i^k)p(\omega_i^k)/p_i$, and $\overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j) = \sum_k \mathrm{V}_t(\mathbf{H})(\omega_i^k)p(\omega_i^k)/p_i$ for all $j$, we obtain

$$\mathrm{E}^P[\mathrm{V}_t(\mathbf{H})|\mathcal{G}_{t-1}] = \sum_{i,k} p(\omega_i^k)\mathrm{V}_t(\mathbf{H})(\omega_i^k) = \sum_i p_i \overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j) = \mathrm{E}^P[\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{G}_{t-1}] \ ,$$

and in an analogous manner, we have $\mathrm{E}^P[(S_t)^2|\mathcal{G}_{t-1}] = \mathrm{E}^P\left[(\overline{S}_t)^2|\mathcal{G}_{t-1}\right]$, $\mathrm{E}^P[S_t|\mathcal{G}_{t-1}] = \mathrm{E}^P[\overline{S}_t|\mathcal{G}_{t-1}]$, $\mathrm{E}^P[\mathrm{V}_t(\mathbf{H})S_t|\mathcal{G}_{t-1}] = \mathrm{E}^P[\overline{\mathrm{V}}_t(\mathbf{H})\overline{S}_t|\mathcal{G}_{t-1}]$. From this, it follows that the hedge ratio (slope of the regression line) and the shares in the money market account (intercept of the regression line) of the one-stage regression approach,

$$h_t^S = \frac{\mathrm{Cov}^P[\mathrm{V}_t(\mathbf{H}),S_t|\mathcal{G}_{t-1}]}{\mathrm{Var}^P[S_t|\mathcal{G}_{t-1}]} \quad \text{and} \quad h_t^B = \frac{\mathrm{E}^P[V_t(\mathbf{H})|\mathcal{G}_{t-1}]}{B_t} - h_t^S \mathrm{E}^P[S_t/B_t|\mathcal{G}_{t-1}] \ ,$$

coincide with these of the two-stage procedure:

$$\overline{h_t^S} = \frac{\mathrm{Cov}^P[\overline{\mathrm{V}}_t(\mathbf{H}),\overline{S}_t|\mathcal{G}_{t-1}]}{\mathrm{Var}^P[\overline{S}_t|\mathcal{G}_{t-1}]} \quad \text{and} \quad \overline{h_t^B} = \frac{\mathrm{E}^P[\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{G}_{t-1}]}{B_t} - \overline{h_t^S}\mathrm{E}^P\overline{S}_t/B_t|\mathcal{G}_{t-1}] \ .$$

# 3 Hedging in Reduced-Form Models

Below we will determine hedging strategies for credit derivatives, e.g. defaultable bonds and credit default swaps. We envision a situation where a hedger owns a portfolio of such credit derivatives and tries to hedge this portfolio against all kinds of risk, namely default risk, interest rate risk and recovery rate risk. Suitable hedging instruments are then money market accounts, CDSs, junior bonds and so on.

In the following we assume that the hedger tries to hedge a *short position* in a coupon-paying defaultable bond. This defaultable bond delivers time-continuous cash flows $C_t$ in $0 \leq t \leq T$ as long as no default has occurred. If the firm is still solvent at the time of maturity a payment $F$ will also be paid. Otherwise the owner of the credit derivative receives (in addition to the cash flow stream $C$ during the period $[0,\tau)$) the *uncertain* recovery payment $Z(\tau)$ depending on default time $t = \tau$ and paid out at default. We denote the defaultable coupon bond by $(Z, C, F)$.[4] We assume that the recovery amount does not exceed the remaining value of the credit derivative's cash flow when no default occurs:

$$0 \leq \frac{B_T}{B_\tau} Z(\tau) \leq B_T \int_\tau^T \frac{1}{B_t}\, dC_t + F \quad P\text{-a.s.}, \tag{2}$$

where $B_t = \exp\{\int_0^t r_s\, ds\}$ denotes the value of the money market account at time $t$ and $P$ denotes the statistical probability measure. At any time $t < \tau$, the recovery payment for a credit event occurring at time $\tau = u$ has an expected value of $\mu^Z(u)$ and a standard deviation of $\sigma^Z(u)$ under $P$. Because of (2) we have also

$$0 \leq \frac{B_T}{B_u} \mu^Z(u) \leq B_T \int_u^T \frac{1}{B_t}\, dC_t + F \quad \text{for} \quad 0 < u \leq T.$$

For technical reasons we assume $\sup_{u \in [0,T]} \sigma^Z(u) < \infty$. Assumption (2) guarantees that the value of the defaultable claim $(Z, C, F)$ is lower than the value of a default-

[4] When hedging a CDS, we have a different hedging situation. In this case, one hedges a claim of the form $(F - Z,\, -C, 0)$, since a CDS pays the difference between the recovery payment and the promised face value, $F - Z$, and the buyer of the CDS does not receive but has to pay a time-continuous premium.

free but otherwise identical derivative $(C, F)$.
The cumulative value of the credit derivative at maturity amounts to

$$F_T = \begin{cases} B_T \int_0^T 1/B_t \, dC_t + F \;, & \text{if } \tau > T \\ B_T \int_0^\tau 1/B_t \, dC_t + B_T/B_\tau \cdot Z(\tau) \;, & \text{if } \tau \leq T \end{cases} .$$

The stochastic *recovery rate* of the credit derivative $(Z, C, F)$ is defined as follows:

$$\delta(\tau) = \frac{B_T \int_0^\tau 1/B_t \, dC_t + B_T/B_\tau \cdot Z(\tau)}{B_T \int_0^T 1/B_t \, dC_t + F} = \frac{B_T/B_\tau \widetilde{C}_\tau + B_T/B_\tau \cdot Z(\tau)}{\widetilde{C}_T + F} \quad \in [0{,}1], \tag{3}$$

where $\widetilde{C}_t = \int_0^t B_t/B_s \, dC_s$ denotes the time-$t$ value of the cash flow stream $C$ during $[0,t]$ when default has not occurred until $t$. Relation (3) relates the final value of the defaultable claim's cash flows $(Z, C, F)$ to the final value of the default-free, but otherwise identical derivative's cash flows $(C, F)$.[5] Because of assumption (2) the recovery rate is lower than one. If the recovery only depends on the uncertain *default time* and the uncertain *interest rate*, we will call it *single-stochastic*. If it is subject to another source of risk, we will denote the recovery *doubly-stochastic*.

We assume that the seller of this defaultable claim $(Z, C, F)$ can hedge his short position with strategy $\mathbf{H} = (h^S,\, h^B)$ consisting of $h^S$ defaultable zero bonds with total loss in case of default and $h^B$ money market accounts. To simplify the following presentation we start with a *deterministic* term structure, i.e. the short rate $(r_t)_{t \in [0,T]}$ is a deterministic function of time.

## 3.1 A Simple Intensity Model

This section presents a simple intensity model in continuous time which describes the possible default of a firm at time $\tau > 0$ during the time horizon $[0,T]$, where interest rates are deterministic. The credit event is specified in terms of an exogenous jump process, the so-called *default process* $H_t = 1\!\!1_{\{\tau \leq t\}}$. In the following we assume that $H$ is an inhomogeneous Poisson process stopped at the first jump – the default time:

$$P(\tau \leq t) = P(H_t = 1) = 1 - \exp\left\{ - \int_0^t \lambda(s) \, ds \right\} \quad \text{for every} \quad t \geq 0 \;,$$

[5]Bakshi, Madan and Zhang (2006, p. 22) define the recovery rate by means of the out-standing payments. But the definition above simplifies the following formulae for the hedging strategies.

where $\lambda$ is a deterministic, non-negative function of time with $\int_0^T \lambda(t)\,dt < \infty$ representing the *default intensity* under the statistical probability measure $P$. The model is based on a probability space $(\Omega,\mathcal{G},P)$, where $\Omega$ denotes the state space in the economy. The information available to the market participants at time $t$ is given by the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ generated by the marked inhomogeneous Poisson process $H^Z = (H, Z)$ stopped at the first jump: $\mathcal{G}_t = \sigma(H_t^Z)$ for $t \in [0,T]$. $S = (S_t)_{t\in[0,T]}$ denotes the price process of the traded defaultable zero coupon bond with maturity date $T$ *and total loss in case of default* given by

$$S_t = \frac{B_t}{B_T} \exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}(1 - H_t) \tag{4}$$

if financial markets are frictionless and arbitrage-free. Since the hedging instrument faces a total loss in case of default while the credit derivative only suffers a partial loss, we call the hedging instrument the *junior bond* while we refer to the claim to hedge as the *senior bond*.

The deterministic non-negative function $\widehat{\lambda}$ with $\int_0^T \widehat{\lambda}(t)\,dt < \infty$ can be estimated via market values of defaultable financial instruments[6] and specifies the default intensity under the martingale measure $Q \in \mathbb{Q}$. In particular,

$$\begin{aligned} \mathrm{E}^Q\left[\frac{S_t}{B_t}\middle|\mathcal{G}_s\right] &= \mathbb{1}_{\{\tau>s\}}\left(\frac{S_t}{B_t}\cdot Q(\tau > t|\tau > s) + 0\cdot Q(\tau \le t|\tau > s)\right) \\ &= (1-H_s)\frac{B_t}{B_T}\exp\left\{-\int_t^T \widehat{\lambda}(u)\,du\right\}\exp\left\{-\int_s^t \widehat{\lambda}(u)\,du\right\} = \frac{S_s}{B_s}\,. \end{aligned}$$

The discounted price process $S/B$ admits the decomposition $S/B = S_0/B_0+A+M$, since

$$\begin{aligned} d\left(\frac{S_t}{B_t}\right) = \widehat{\lambda}(t)\frac{S_{t-}}{B_t}dt - \frac{S_{t-}}{B_t}dH_t &= \frac{S_{t-}}{B_t}(\widehat{\lambda}(t) - \lambda(t))dt - \frac{S_{t-}}{B_t}d\widetilde{H}_t \\ &= dA_t + dM_t\,. \end{aligned}$$

Here, $\widetilde{H}_t = H_t - \int_0^{t\wedge\tau} \lambda(s)\,ds$ denotes the *compensated default process*, $A$ describes the continuous drift component with $A_0 = 0$, $M$ denotes a square integrable $P$-martingale[7] with $M_0 = 0$, and finally $S_0 = \exp\left\{-\int_0^T \widehat{\lambda}(s)\,ds\right\}/B_T$ denotes the bond

[6]See, e.g., Jarrow and Turnbull (1995) and Jarrow, Lando and Turnbull (1997).

[7]Since the process $\widetilde{H}$ is a square integrable martingale with $[\widetilde{H},\widetilde{H}] = H$ and since the process $S_-/B$ is predictable with $\mathrm{E}^P[\int_0^T (S_{t-}/B_t)^2\,d[\widetilde{H},\widetilde{H}]_t] = \mathrm{E}^P[\int_0^T (S_{t-}/B_t)^2\,dH_t] < \infty$, $M$ is also a square integrable martingale (see Protter, 1990, p. 142).

price at $t = 0$. Due to properties of the conditional quadratic variation (see, e.g., Protter, 1990) it follows that

$$d\langle M\rangle_t = \left(\frac{S_{t-}}{B_t}\right)^2 d\langle \widetilde{H}\rangle_t = \left(\frac{S_{t-}}{B_t}\right)^2 \lambda(t)d(t\wedge\tau) = \left(\frac{S_{t\wedge\tau-}}{B_{t\wedge\tau}}\right)^2 \lambda(t)d(t\wedge\tau)\ .$$

Since $dA_t = S_{t-}/B_t \cdot (\widehat{\lambda}(t) - \lambda(t))dt = S_{t\wedge\tau-}/B_{t\wedge\tau} \cdot (\widehat{\lambda}(t) - \lambda(t))d(t\wedge\tau)$ we obtain

$$A_t = \int_0^t \widetilde{\alpha}_s\, d\langle M\rangle_s \quad \text{with} \quad \widetilde{\alpha}_t = \frac{B_{t\wedge\tau}}{S_{t\wedge\tau-}}\left(\frac{\widehat{\lambda}(t)}{\lambda(t)} - 1\right)\ ,$$

and therefore $S/B = S_0/B_0 + \int \widetilde{\alpha}\, d\langle M\rangle + M$.

Now we determine hedging strategies for defaultable claims which minimize the risk locally. More precisely, we solve Problem 2 as stated in the appendix. This rather technical formulation is due to Schweizer (1991) and can be seen as continuous-time analogue of Problem 1. To identify the LRM-hedge for credit derivatives we use the minimal martingale measure[8] $\widehat{P}$ defined by the density [9]

$$\begin{aligned}
\widehat{Z}_t &= \mathcal{E}\left\{-\int \widetilde{\alpha}\, dM\right\}_t \\
&= \mathcal{E}\left\{\int_0^{t\wedge\tau}\left(\lambda(s) - \widehat{\lambda}(s)\right)\, ds + \left(\frac{\widehat{\lambda}(\tau)}{\lambda(\tau)} - 1\right) H_t\right\} \\
&= \begin{cases} \exp\{\int_0^t (\lambda(s) - \widehat{\lambda}(s))\, ds\}\ , & \text{if } t < \tau, \\ \frac{\widehat{\lambda}(\tau)}{\lambda(\tau)} \exp\{\int_0^\tau (\lambda(s) - \widehat{\lambda}(s))\, ds\}\ , & \text{if } t \geq \tau. \end{cases}
\end{aligned} \tag{5}$$

Thus, the distribution of the recovery payment remains unaffected by the measure change and the default intensity under $\widehat{P}$ is given by $\widehat{\lambda}$. More precisely, from Theorem $T2$ in Brémaud (1981, p. 165f.) it follows that (5) coincides with the density corresponding to the measure change from $P$ to $Q$. Hence, we have $\widehat{P} \equiv Q$. Moreover, since the orthogonality structure is preserved by the measure change from $P$ to $\widehat{P}$, the distribution of the recovery payment remains unaffected as well. In particular, at any time prior to default, the expected recovery payment in case default occurs at a later date $t \in (0,T]$ is given by $\mu^Z(t)$ also under the minimal martinagle measure $\widehat{P}$. Note that the recovery payment is not priced under $Q$,

[8]The notion "minimal martingale measure" is motivated by the fact that apart from turning $S/B$ into a martingale this measure disturbs the overall martingale and orthogonality structures as little as possible.

[9]For evaluating the stochastic exponential see, e.g., Protter (1990, p. 77).

since $Z(t)$ is not a tradable asset. The result that $\widehat{P}$ coincides with the pricing measure $Q$ was already obtained in Biagini and Cretarola (2007). However, in Biagini and Cretarola (2007, 2009, 2012), it is the default intensity rather than the recovery payment distribution that is unaffected by the measure change from $P$ to $\widehat{P}$, since the process $\widetilde{H}$ and, hence, the default intensity $\lambda$ does not constitute a tradable asset. In our model, the junior bond with default risk premium $\widehat{\lambda}$ is traded and, therefore, priced under the pricing measure $Q$. We emphasize once more that, in this work, we focus on the robust hedging of the recovery risk and the incompleteness of our market model arises from the doubly-stochastic recovery payment rather than from the default risk alone. In the remainder of this work, we will derive explicit hedge ratios under the standing assumption of a doubly-stochastic recovery payment, whereas, in Biagini and Cretarola (2007, 2009, 2012), such explicit representations are provided only under the additional assumption that the recovery payment is deterministic or even constant conditional on default.

The present value of the recovery payment conditional on the event that default takes place in $(t,T]$ is given by the deterministic function

$$\begin{aligned} g_t^Z &= \widehat{E}\left[\frac{B_t}{B_\tau} Z(\tau)\mathbb{1}_{\{\tau\leq T\}}\middle|\tau > t\right] \\ &= \widehat{E}\left[\frac{B_t}{B_\tau} Z(\tau)\mathbb{1}_{\{\tau\leq T\}}\middle| t<\tau\leq T\right]\cdot \widehat{P}(\tau\leq T|\tau>t) \\ &= \int_t^T \frac{B_t}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\widehat{\lambda}(u)\mu^Z(u)\,du. \end{aligned} \tag{6}$$

Likewise, the present value $g^F$ of the payment $F$ being paid out in case of no default up to time $T$ and the present value $g^C$ of the future coupon payments of the senior bond being paid out until the time of default are, respectively, given by

$$g_t^F = \frac{B_t}{B_T}\exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}\cdot F, \tag{7}$$

$$g_t^C = \int_t^T \frac{B_t}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\,dC_u. \tag{8}$$

Due to the results of Schweizer (1991) and with the convention

$$V_t^F = \widehat{E}\left[\frac{B_t}{B_T}F_T\middle|\mathcal{G}_t\right],$$

Lemma 1 provides the LRM-hedge ratio via the Föllmer-Schweizer-decomposition, see Föllmer and Schweizer (1991).

**Lemma 1 (FS-Decomposition of a Senior Bond)**
*The discounted cumulative value $F_T/B_T$ of the senior bond $(Z, C, F)$ at maturity has the following strong Föllmer-Schweizer-decomposition:*

$$\frac{F_T}{B_T} = F_0 + \int_0^T h_t^S \, d\left(\frac{S_t}{B_t}\right) + \frac{L_T^F}{B_T} \, ,$$

*where*

$$h_t^S \;=\; \frac{d\langle V^F, \, S\rangle_t^{\widehat{P}}}{d\langle S, \, S\rangle_t^{\widehat{P}}} \;=\; \begin{cases} \frac{g_t^C + g_t^F + g_t^Z - \mu^Z(t)}{S_{t-}} & : \; t \le \tau, \\ 0 & : \; t > \tau, \end{cases}$$

*is the locally risk-minimizing hedge ratio, $F_0 = g_0^C + g_0^F + g_0^Z$ is a constant, and $L^F/B$ is a martingale which is orthogonal to $M$, given by $L_t^F = \int_0^t \frac{B_t}{B_s}(Z(s) - \mu^Z(s)) \, d\widetilde{H}_s$.*

## 3.2 Single-Stochastic Recovery Payment

We first consider the case of a single-stochastic recovery payment, i.e. $Z(t)$ is a deterministic function of time. The time-$t$ value of the recovery payment under the martingale measure $Q$, given the credit event takes place in $(t,T]$, is represented by the deterministic function

$$g_t^Z = \mathrm{E}^Q \left[ \frac{B_t}{B_\tau} \mu^Z(\tau) \mathbb{1}_{\{\tau \le T\}} \middle| \tau > t \right] = \int_t^T \frac{B_t}{B_u} \exp\left\{ - \int_t^u \widehat{\lambda}(s) \, ds \right\} \widehat{\lambda}(u) Z(u) \, du. \quad (9)$$

Replacing $\mu^Z(t)$ by $Z(t)$ in Lemma 1 results in

**Proposition 1 (Replication for Single-Stochastic Recovery)**
*The senior bond $(Z, C, F)$ with single-stochastic recovery is duplicated by the hedging strategy $\mathbf{H} = (h^S, h^B)$ with*

$$\begin{aligned} h_t^S &= \frac{g_t^C + g_t^F + g_t^Z - Z(t)}{S_{t-}} \, , \\ h_t^B &= \frac{V_{t-}^F}{B_t} - h_t^S \cdot \frac{S_{t-}}{B_t} = \frac{\widetilde{C}_t}{B_t} + \frac{Z(t)}{B_t}, \end{aligned}$$

*for $t \le \tau$, and $h_t^S = 0$, $h_t^B = h_\tau^B$ for $t > \tau$.*

According to this duplication strategy at every point in time $t$ the value of the money market accounts equals the cumulative value of the senior bond in the case of default at time $\tau = t$. The value of the position in the defaultable zeros at time $t < \tau$ equals the expected future payments of the senior bond less the discounted recovery payment in the case of default at time $t$, i.e.

$$h_t^S S_t = g_t^C + g_t^F + g_t^Z - Z(t).$$

The expected recovery rate given that default occurs in $(t,T]$, denoted by $\widetilde{\mu}^\delta(t)$, is given by

$$\widetilde{\mu}^\delta(t) = \int_t^T \delta(u)\widehat{\lambda}(u) \exp\left\{-\int_t^u \widehat{\lambda}(s)ds\right\} du.$$

It is worth mentioning that, see Müller (2008), in the special case when the recovery rate is *constant*, $\delta(u) = \delta$ for all default times $\tau$, we have

$$\begin{aligned}\widetilde{\mu}^\delta(t) &= \delta \left[-\exp\left\{-\int_t^u \widehat{\lambda}(s)ds\right\}\right]_t^T \\ &= \delta \left(1 - \frac{B_T}{B_t} S_t\right), \end{aligned} \tag{10}$$

and it will then be possible to replicate the senior bond $(Z, C, F)$ with single-stochastic recovery by a *static* hedge: Buy

$$h^S = (1 - \delta)(\widetilde{C}_T + F)/B_T$$

junior bonds (with total loss in case of default) and buy

$$h^B = \delta(\widetilde{C}_T + F)/B_T$$

money market accounts.

### 3.3 Doubly-Stochastic Recovery Payment

We now consider the case of a doubly-stochastic recovery payment, i.e. $Z$ is now a stochastic process. Every probability measure $Q \in \mathbb{Q}$ with corresponding default intensity $\widehat{\lambda}$ and arbitrary distribution of the recovery rate with values in [0,1] represents an equivalent martingale measure if the null sets of the distribution of the recovery rate under $Q$ and $P$ are the same. The financial market will be arbitrage-free. But it will be incomplete if the recovery rate is not known $P$-a.s. given that default occurs in $\tau = t$. For this reason defaultable claims with a doubly-stochastic recovery can *not* be duplicated. The incompleteness of the financial market model can also be recognized as follows: There are two sources of risk – the default time and the amount of the recovery are uncertain. But there exists only one financial instrument (besides the money market account) for hedging the default risk.

**Proposition 2 (LRM-Hedge)**
*The locally risk-minimizing hedge of the senior bond* $(Z, C, F)$ *amounts to*

$$h_t^S = \frac{g_t^C + g_t^F + g_t^Z - \mu^Z(t)}{S_{t-}}, \tag{11}$$

$$h_t^B = \frac{V_t^F}{B_t} - h_t^S \cdot \frac{S_{t-}}{B_t} = \frac{\widetilde{C}_t}{B_t} + \frac{\mu^Z(t)}{B_t}. \tag{12}$$

*After default, i.e. for* $t > \tau$, *we have*

$$h_t^S = 0, \quad h_t^B = \widetilde{C}_\tau + Z(\tau)/B_\tau .$$

In case of a defaultable claim with single-stochastic recovery the LRM-hedge collapses to the duplication strategy given in Proposition 1. According to this duplication strategy at every point in time $t$ the value of the money market accounts equals the cumulative value of the senior bond in the case of default at time $\tau = t$. At default the share in the money market account makes a jump in the amount of $(Z(\tau) - \mu^Z(\tau))/B_\tau$ such that the value of the hedging strategy at maturity coincides with the cumulative value of the senior bond. The value of the position in the junior bonds at time $t < \tau$ equals the expected future payments of the senior bond less the expected recovery payment in the case of default at time $t$, i.e.

$$h_t^S S_t = g_t^C + g_t^F + g_t^Z - \mu^Z(t).$$

Because of the relation $C_t(\mathbf{H}) = F_0 + L_t^F$ for all $t \in [0,T]$, the LRM-hedge is self-financing at every point in time before and after default. But at default money accrues or flows out, depending on the difference between realized recovery payment, $Z(\tau)$, and the expected payment at default, $\mu^Z(\tau)$. On average, the locally risk-minimizing hedging strategy is self-financing, that is, the strategy is *mean*-self-financing.

If the recovery is single-stochastic the LRM-hedge will even be self-financing and therefore will collapse to a replication strategy. For the special case, see Müller

(2008), that the expected recovery rate does not depend on the default time, i.e. $\mu^\delta(t) = \mu^\delta$ at $0 < t \leq T$, and hence $\widetilde{\mu}^\delta(t) = \mu^\delta(1 - B_T/B_t\, S_{t-})$ for $t \leq \tau$, the locally risk-minimizing hedge simplifies to a static hedge:

$$\mathbf{H} = (h^S, h^B) = (\,(\widetilde{C}_T + F)/B_T(1 - \mu^\delta), (\widetilde{C}_T + F)/B_T\mu^\delta\,)\ .$$

Proposition 2 shows that the locally risk-minimizing hedge depends only on the expected payment at default under the statistical probability measure, but not on other details of the probability distribution of the recovery. Hence we achieve the following result:

**Proposition 3 (Impact of Recovery Modeling on LRM-Hedge)**
*The LRM-hedge for a senior bond* $(Z^d,C,F)$ *with a doubly-stochastic recovery equals the LRM-hedge for a senior bond* $(Z^s, C, F)$ *with single-stochastic recovery for all points in time until default, provided that the expected recovery payments coincide under the statistical probability measure, i.e.* $\mu^{Z^d}(t) = \mu^{Z^s}(t) = Z^s(t)$ *for every* $0 < t \leq T$ .

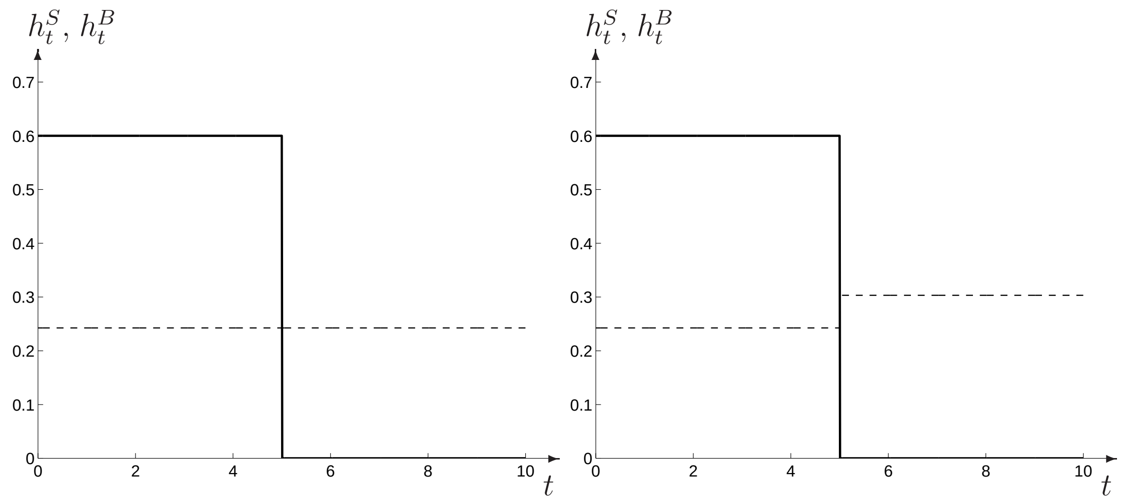
**Example 1** We consider a financial market where a defaultable zero bond of a firm with total loss at default and maturity 10 years is traded. Furthermore, we assume a flat term structure with $r = 5$ %. Default time is exponentially distributed with intensity $\lambda = 0{,}05$ and $\widehat{\lambda} = 0{,}20$ under the statistical probability measure and the martingale measure, respectively. We now calculate hedging strategies of a defaultable zero bond with recovery payment at default. We assume a single-stochastic, even *constant* recovery rate of $\delta^s = 40$ %, and a doubly-stochastic recovery rate with an expected value of $\mu^{\delta^d} = 40$ %.

Figure 3 shows the locally risk-minimizing hedging strategy of a zero with single- and doubly-stochastic recovery. We assume, that the firm defaults after 5 years and that the realized recovery rate amounts to 50 % in the case of doubly-stochastic recovery modeling. According to Proposition 3 the LRM-hedges are equal until default for both the single- and the doubly-stochastic recovery case. After the credit event the shares in the money market account of the locally risk-minimizing strategies differ since the realised payments at default are different.

If an investor prefers a self-financing hedging strategy, the so-called *super-hedging strategy* which assures a liquidation value at maturity at least as high as the pay-off of the derivative, i.e. $\mathrm{V}_T(\mathbf{H}) \geq F_T$ $P$-a.s., then the recovery modeling has the

Figure 3: *LRM-hedges when default occurs at $\tau = 5$*

The left figure illustrates the LRM-hedge for a defaultable zero bond with *constant* recovery. This hedge corresponds to the duplication. The right figure depicts the LRM-hedge for a defaultable zero bond with an *uncertain* recovery payment when default occurs after five years with a realized recovery rate of 50 %. The solid line describes the hedge ratio $h^S$ and the dashed line the number of money market accounts $h^B$ during time.




following impact on the hedging strategy. Assuming a constant recovery payment of 0,40 the super-hedge corresponds to the duplication strategy $\mathbf{H} = (h^S,\, h^B) = (0{,}60\ ;\ 0{,}40/B_T)$ as well as the LRM-hedge. If the payment at default is uncertain, the super-hedge depends on the distribution of the recovery, more precisely, on the domain of the recovery payment. Assuming that the recovery payment can reach values on $[0, 1]$ and $[0, 0{,}95]$, respectively, the resulting super-hedges are $\mathbf{H} = (h^S,\, h^B) = (0;\ 1/B_T)$ and $\mathbf{H} = (h^S,\, h^B) = (0{,}05\ ;\ 0{,}95/B_T)$, respectively. □

## 3.4 Why Does Only the First Moment Matter?

At first glance Proposition 3 seems to contradict the result obtained by Grünewald and Trautmann (1996) when deriving LRM-strategies for stock options in the presence of jump risk. In their setting, the LRM-strategy depends additionally on the variance of the stock's jump amplitude. This key difference is due to the fact that in our model default of the firm implies that the underlying instrument's price jumps always to *zero* while in the Merton (1976) jump diffusion setting assumed by Grünewald and Trautmann (1996), the price of the stock underlying the option jumps to an arbitrary price level.

We now illustrate the effect on the LRM-strategy if the junior bond does not become worthless at default. So suppose now that the junior bond price jumps to a random fraction of its price just prior to default, i.e. we now impose the so-called fractional recovery of market value assumption. Thus

$$\frac{S_\tau}{B_\tau} = \delta^S \frac{S_{\tau-}}{B_\tau}$$

for some random variable $\delta^S$ with $0 \leq \delta^S \leq 1$. In this case, we get

$$d\,\langle S,S\rangle_t^{\widehat{P}} = (\widehat{\lambda}(t) + \sigma_{jump}^2)\left(\frac{S_{t-}}{B_t}\right)^2 dt,$$

where $\sigma_{jump}^2$ denotes the variance of the junior bond jump amplitude. We then have

$$h_t^S = \frac{d\langle V^F, S\rangle_t^{\widehat{P}}}{d\langle S,S\rangle_t^{\widehat{P}}} = \frac{\widehat{\lambda}(t)\widehat{E}[\delta^S]}{\widehat{\lambda}(t) + \sigma_{jump}^2} \cdot \frac{g_t^C + g_t^F + g_t^Z - \mu^Z(t)}{S_{t-}}\,,$$

i.e. the LRM-strategy then also depends on the second moment of the jump amplitude. Moreover, as shown in Grünewald and Trautmann (1996, p. 12 f.), the minimal martingale measure will then be a signed measure unless the market price of risk lies in the interval $[-1,0]$, i.e. $\widehat{P}$ will only be an equivalent martingale measure if the expected excess return over the riskless rate is negative. Otherwise $\widehat{P}$ will attach negative probabilities to certain events. However, Grünewald and Trautmann (1996) show in their simulation study that these events only play a minor role in case realistic paramters are used. Moreover, the minimal measure can only be signed if it is the hedging instrument that is exposed to a random jump amplitude. In our applications, $\widehat{P}$ will never be signed since the corresponding martingale density is not affected by the senior bond, but only by the junior bond.

# 4 Extensions

Explicit solutions of hedging strategies for credit derivatives are rare in the literature. For instance, Bielecki et al. (2008) prove the existence of a hedging strategy for a credit derivative $(Z, C, F)$ in a general setup (including both stochastic interest and stochastic default rates), but do not provide the hedge ratio in closed-form. Biagini and Cretarola (2009) derive locally risk-minimizing strategies, but give closed-form solutions only for the special case of null interest rates, no coupon payments and a single-stochastic recovery payment.

So far, the recovery payment and the time of default were the only random quantities in our model as well. In Section 4.1, we derive the LRM-strategy in case the interest rate is also stochastic while in Section 4.2 we consider the case of a stochastic default intensity instead. In Section 4.3, $r$ and $\widehat{\lambda}$ are then assumed to be both stochastic but independent. This independence assumption, however, will turn out to be no major restriction (see Section 4.3).

## 4.1 Stochastic Interest Rates

We now extend our basic model to the case of a non-trivial reference filtration to investigate to what extent the hedging strategy will be affected. Due to this additional source of risk, we now have $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{F}_t$ describes the time-$t$ information about the evolution of the interest rate and the default rate and $\mathcal{H}_t$ describes the time-$t$ market information about whether default has occured and the recovery risk. In particular, we assume $\mathcal{F}_t = \sigma(\widehat{W}_t)$ for some Brownian motion $\widehat{W}$ under $\widehat{P}$. Denote by $G$ the conditional survival probability with respect to the reference filtration under the minimal martinagle measure, i.e. $G_t = \widehat{P}(\tau > t|\mathcal{F}_t)$.

The discounted price of the junior bond at time $t \in [0,T]$ is now given by

$$\frac{S_t}{B_t} = (1 - H_t)\, G_t^{-1}\, \widehat{E}\left[\frac{G_T}{B_T}\middle|\mathcal{F}_t\right] \tag{13}$$

and its $\widehat{P}$-dynamics are now given by

$$d\left(\frac{S_t}{B_t}\right) = (1 - H_t)G_t^{-1} dm_t^S - \frac{S_{t-}}{B_{t-}} d\widetilde{H}_t,$$

where the martingale $m^S$ is given by

$$m_t^S = \widehat{E}\left[\frac{G_T}{B_T}\middle|\mathcal{F}_t\right]$$

for all $t \in [0,T]$, cf. Proposition 2 in Blanchet-Scalliet and Jeanblanc (2004).

Consider the $(\mathcal{F}_t)$-martingale $m$ given by

$$m_t \;=\; \widehat{E}\left[\int_0^T \frac{1}{B_u} G_u \widehat{\lambda}(u) \mu^Z(u)\,du + G_T \frac{1}{B_T} F + \int_0^T \frac{1}{B_u} G_u dC_u \middle| \mathcal{F}_t\right].$$

Denote by $\xi^m$ and $\xi^S$ the predictable processes appearing in the martingale representations of the processes $m$ and $m^S$, i.e.

$$\begin{aligned} m_t &= m_0 + \int_0^t \xi^m_s \, d\widehat{W}_s \,, \qquad (14) \\ m^S_t &= m_0 + \int_0^t \xi^S_s \, d\widehat{W}_s \,. \end{aligned}$$

Lemma 2 provides the LRM-hedge ratio via the Föllmer-Schweizer-decomposition in case the reference filtration $(\mathcal{F}_t)$ is non-trivial.

**Lemma 2 (FS-Decomposition in case of a Brownian Reference Filtration)** *The discounted cumulative value $F_T/B_T$ of the senior bond $(Z, C, F)$ at maturity has the following strong Föllmer-Schweizer-decomposition:*

$$\frac{F_T}{B_T} = F_0 + \int_0^T h^S_t \, d\left(\frac{S_t}{B_t}\right) + \frac{L^F_T}{B_T}\,,$$

*where*

$$\begin{aligned} h^S_t &= \frac{d\langle V^F, S\rangle^{\widehat{P}}_t}{d\langle S, S\rangle^{\widehat{P}}_t} \\ &= (1 - H_t)\left(\frac{\xi^m_t}{\xi^S_t} + \frac{g^C_t + g^F_t + g^Z_t - \mu^Z(t)}{S_{t-}}\right)\,, \end{aligned}$$

*is the locally risk-minimizing hedge ratio, $F_0 = g^C_0 + g^F_0 + g^Z_0$ is a constant, and $L^F/B$ is a martingale which is orthogonal to $M$, given by $L^F_t = \int_0^t \frac{B_t}{B_s}(Z(s) - \mu^Z(s))\, d\widetilde{H}_s$.*

**Proposition 4 (LRM-Hedge in case of a Brownian Reference Filtration)** *In case of a non-trivial reference filtration, the locally risk-minimizing hedging strategy of the senior bond (Z,C,F) is given by*

$$\begin{aligned} h^S_t &= \frac{\xi^m_t}{\xi^S_t} + \frac{g^C_t + g^F_t + g^Z_t - \mu^Z(t)}{S_{t-}}, \qquad (15) \\ h^B_t &= \frac{V^F_{t-}}{B_{t-}} - h^S_t \cdot \frac{S_{t-}}{B_{t-}}, \end{aligned}$$

*for* $t \leq \tau$, *and*

$$\begin{aligned} h_t^S &= 0, \\ h_t^B &= \int_0^\tau \frac{1}{B_s}\, dC_s + \frac{1}{Z_\tau}, \end{aligned}$$

*for* $t > \tau$.

From Proposition 4 we see that in order to obtain the hedge ratio, we have to calculate the processes $\xi^m$ and $\xi^S$.

Suppose now that the interest rate follows a stochastic process while the default rate is a deterministic function. For $g^Z$, $g^F$ and $g^C$, we then have

$$g_t^Z = \int_t^T \widehat{E}\left[\frac{B_t}{B_u}\Big|\mathcal{F}_t\right] \exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\widehat{\lambda}(u)\mu^Z(u)\,du, \tag{16}$$

$$g_t^F = \widehat{E}\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right] \exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}\cdot F, \tag{17}$$

$$g_t^C = \int_t^T \widehat{E}\left[\frac{B_t}{B_u}\Big|\mathcal{F}_t\right] \exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\,dC_u, \tag{18}$$

respectively, where we have used Fubini's Theorem.

**Example 2** Suppose the short rate follows the CIR model under the minimal martingale measure, i.e.

$$dr_t = \kappa^r(\theta^r - r_t)dt + \sigma^r\sqrt{r_t}d\widehat{W}_t,$$

where $\kappa^r$, $\theta^r$, $\sigma^r$, $r_0 > 0$.

Thus

$$\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] = \exp\left\{-\int_0^t r_s\,ds - r_tC(t,T) - D(t,T)\right\}, \tag{19}$$

where,

$$C(t,T) = \frac{\sinh(\gamma^r(T-t))}{\gamma^r\cosh(\gamma^r(T-t)) + \frac{1}{2}\kappa^r\sinh(\gamma^r(T-t))}, \tag{20}$$

$$D(t,T) = -\frac{2\kappa^r}{(\sigma^r)^2}\ln\left(\frac{\gamma^r e^{\frac{1}{2}\kappa^r(T-t)}}{\gamma^r\cosh(\gamma^r(T-t)) + \frac{1}{2}\kappa^r\sinh(\gamma^r(T-t))}\right), \tag{21}$$

$\gamma^r = \frac{1}{2}\sqrt{(\kappa^r)^2 + 2(\sigma^r)^2}$, $\sinh u = \frac{e^u - e^{-u}}{2}$, and $\cosh u = \frac{e^u + e^{-u}}{2}$.

The discounted junior bond price then follows the dynamics

$$\frac{S_t}{B_t} = (1 - H_t)\,\sigma_t \frac{S_t}{B_t}\, d\widehat{W}_t - \frac{S_{t-}}{B_{t-}} d\widetilde{H}_t,$$

where the junior bond price volatility is given by

$$\sigma_t = -C^r(t,T)\,\sigma^r \sqrt{r_t},$$

for all $t \in [0,T]$, cf. Cox, Ingersoll and Ross (1985, p. 394). In particular,

$$\xi_t^S = G_t\,\sigma_t\,S_t.$$

Note that $G_t = \exp\{-\int_0^t \widehat{\lambda}(s)\,ds\}$ for all $t \in [0,T]$ in the present case, and $m$ writes

$$\begin{aligned} m_t &= \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] G_T \cdot F + \int_0^t \widehat{E}\left[\frac{1}{B_s}\Big|\mathcal{F}_t\right] G_s\,\widehat{\lambda}(s)\,\mu^Z(s)\,ds \\ &\quad + \int_0^t \frac{G_s}{B_s}\,dC_s + \int_t^T \widehat{E}\left[\frac{1}{B_s}|\mathcal{F}_t\right] G_s\,dC_s \\ &=: u(t,r_t). \end{aligned}$$

From Proposition A.1, it follows that the process $\xi^m$ is given by

$$\begin{aligned} \xi_t^m &= B_t \sigma^r \sqrt{r_t}\,\frac{\partial}{\partial r} u(t,r_t) \\ &= \sigma^r \sqrt{r_t} \cdot \Bigg[ -C(t,T)\widehat{E}\left[\frac{1}{B_T}|\mathcal{F}_t\right] G_T \cdot F - \int_0^T C(t,s)\widehat{E}\left[\frac{1}{B_s}|\mathcal{F}_t\right] G_s \\ &\qquad \cdot \widehat{\lambda}(s)\,\mu^Z(s)\,ds - \int_t^T C(t,s)\widehat{E}\left[\frac{1}{B_s}|\mathcal{F}_t\right] G_s\,dC_s \Bigg]. \end{aligned}$$

□

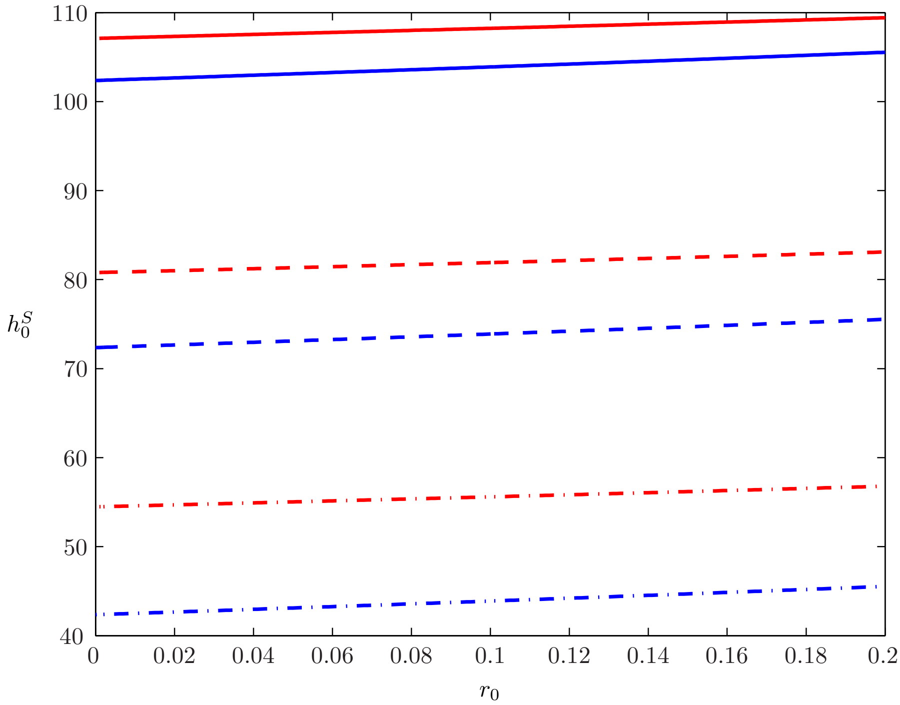
We now discuss the sensitivity of the hedge ratio, the position in the money market and the optimal portfolio value w.r.t. the interest rate level. In particular, we will compare the deterministic and the stochastic interest rates case. The key difference between the LRM-hedges in these two cases is the first term on the right-hand side of (15). In case of a trivial reference filtraton, the LRM-hedge portfolio is only adjusted with respect to the jump-to-default risk and the recovery risk, cf. (11). The additional term in (15) accounts for the diffusion risk arising from the non-trivial reference filtration.

Figure 4 (a) shows that hedge ratio of the LRM-strategy is an increasing function of the interest rate level while the position in the money market account is decreasing in the interest rate level, see Figure 4 (b). This result seems to be counterintuitive at first glance, since one would expect the position in the money market account to be higher for large $r$ due to the higher interest on an investment in the (locally) riskless position. However, we can see from (12) that through the position in the

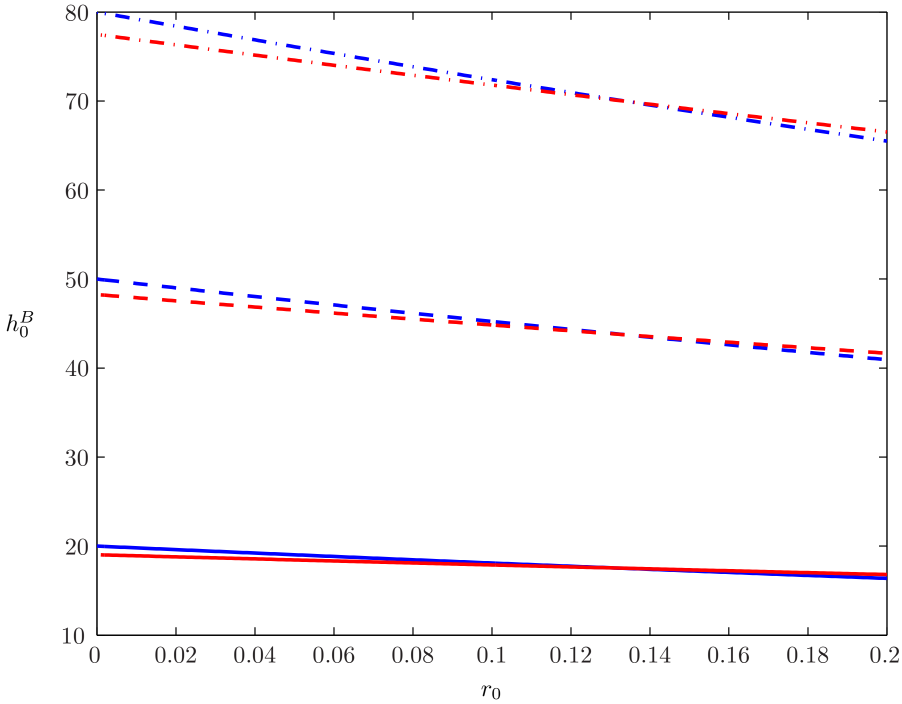
Figure 4: *Sensitivity of the LRM-hedge using the junior bond w.r.t. the interest rate.*

The figure shows the initial hedge ratio, the initial position in the money market account and the initial portfolio value as a function of the initial interest rate level for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic interest rates for parameters $t = 0$, $T = 1$, $\widehat{\lambda} = 2$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic interest rates with CIR dynamics for parameters $\kappa^r = 0.5$, $\theta^r = 0.05$ and $\sigma^r = 0.2$.

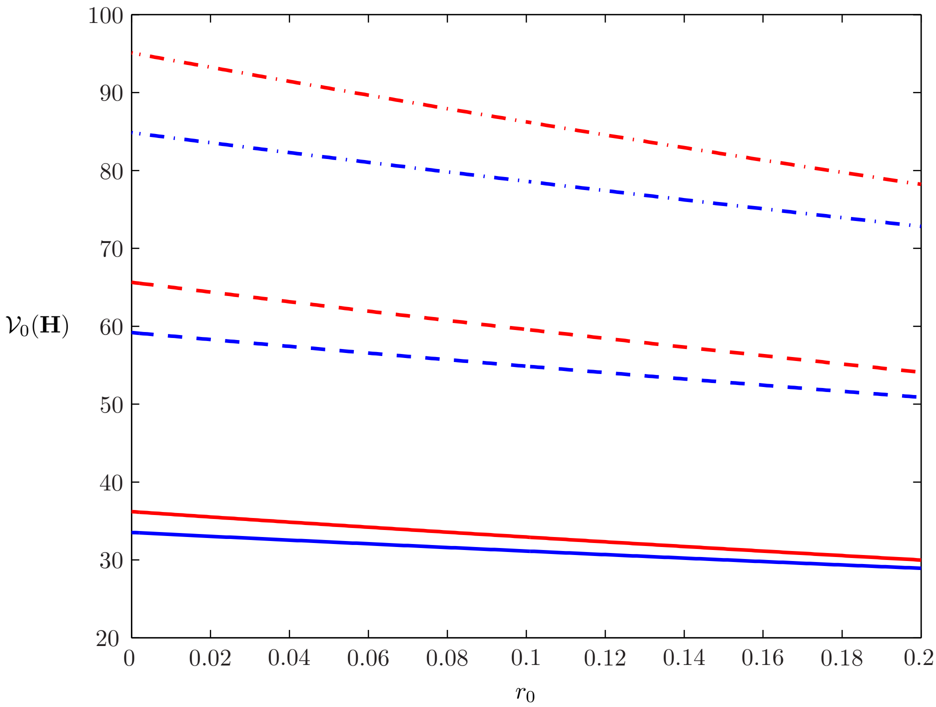
(a) *Initial hedge ratio.*



(b) *Initial position in the money market account.*




(c) *Initial portfolio value.*



money market account, the LRM-hedge only covers the coupon payments up to current time plus the expected recovery payment if default was to occur within the next instant. This position is decreasing in the interest rate level. In case the interest rate is stochastic, we see from Figure 4 (b) that the additional term on the right-hand side of (15) is positive, i.e. this additional source of risk (the volatility of the uncertain interest rate) causes a rebalancement of the LRM-hedge such that more money is invested in the junior bond subject to total loss in case of default and less money is invested in the (locally) riskless money market account. Note that, for a very high interest rate level, the position in the money market account is higher in the stochastic interest rates case. This is due to the mean-reversion property of the CIR dynamics, i.e. if $r_0$ is much larger than the interest rates mean $\theta^r$, the interest rate soonly will fall back in direction of its long-term average. Hence, the investor receives an unusally high return on his investment in the money market account and therefore the LRM-hedge puts more weight on it in this case.

Comparing the blue and the red graphs in Figure 4 (a), one can also see that treating the interest rate (which is stochastic in reality) as a constant will decrease the number of zeros (with total loss in case of default) held in the hedging strategy below the optimal level, hence leading to a position less risky than necessary. Note that from Figure 4 (c), we see that the optimal discounted portfolio value is also decreasing in the interest rate level. So put another way, since the riskless rate is stochastic in reality, the senior bond will be underestimated in the deterministic interest rates model. Therefore, modeling the interest rate as a stochastic process will reduce the hedging costs and thus improve the hedging quality.

## 4.2 Stochastic Intensities

Suppose now that the default rate follows a stochastic process while the interest rate is a deterministic function. The martingale $m^S$ can then be written

$$m_t^S = \frac{1}{B_T}\widehat{E}\left[G_T \middle| \mathcal{F}_t\right],$$

since $r$ and hence $B$ is deterministic now.

For $g^Z$, $g^F$ and $g^C$, we now have

$$g_t^Z = \int_t^T \frac{B_t}{B_u}\widehat{E}\left[\exp\left\{-\int_t^u \widehat{\lambda}_s ds\right\}\widehat{\lambda}_u|\mathcal{F}_t\right]\ \mu^Z(u)\ du, \tag{22}$$

$$g_t^F = \frac{B_t}{B_T}\widehat{E}\left[\exp\left\{-\int_t^T \widehat{\lambda}_s\ ds\right\}|\mathcal{F}_t\right]\cdot F, \tag{23}$$

$$g_t^C = \int_t^T \frac{B_t}{B_u}\widehat{E}\left[\exp\left\{-\int_t^u \widehat{\lambda}_s\ ds\right\}|\mathcal{F}_t\right]\ dC_u, \tag{24}$$

respectively.

**Example 3** Suppose now that it is the intensity that follows the CIR model under the minimal martingale measure, i.e.

$$d\widehat{\lambda}_t = \kappa^{\widehat{\lambda}}(\theta^{\widehat{\lambda}} - \widehat{\lambda}_t)dt + \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}d\widehat{W}_t,$$

where $\kappa^{\widehat{\lambda}}$, $\theta^{\widehat{\lambda}}$, $\sigma^{\widehat{\lambda}}$, $\widehat{\lambda}_0 > 0$. Thus

$$\widehat{E}\left[\exp\left\{-\int_0^T \widehat{\lambda}_s\ ds\right\}|\mathcal{F}_t\right] = \exp\left\{-\int_0^t \widehat{\lambda}_s\ ds - \widehat{\lambda}_t C(t,T) - D(t,T)\right\}, \tag{25}$$

where $C(t,T)$ and $D(t,T)$ are given by (20) and (21) with $\sigma^r$ replaced by $\sigma^{\widehat{\lambda}}$ and $\gamma^{\widehat{\lambda}} = \frac{1}{2}\sqrt{(\kappa^{\widehat{\lambda}})^2 + 2(\sigma^{\widehat{\lambda}})^2}$.

The discounted junior bond price now follows the dynamics

$$\frac{S_t}{B_t} = (1 - H_t)\ \sigma_t \frac{S_t}{B_t}\ d\widehat{W}_t - \frac{S_{t-}}{B_{t-}}d\widetilde{H}_t,$$

where the junior bond price volatility is given by

$$\sigma_t = -C^{\widehat{\lambda}}(t,T)\ \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t},$$

for all $t \in [0,T]$, see Cox, Ingersoll and Ross (1985, p. 394). In particular,

$$\xi_t^S = G_t\ \sigma_t\ S_t.$$

Moreover, we now have

$$\begin{aligned} m_t &= \frac{F}{B_T}\widehat{E}[G_s|\mathcal{F}_t] + \int_0^T \frac{1}{B_s}\widehat{E}[G_s\widehat{\lambda}_s|\mathcal{F}_t]\mu^Z(s)\ ds + \int_0^T \frac{1}{B_s}\widehat{E}[G_s|\mathcal{F}_t]\ dC_s \\ &=: u(t,r_t). \end{aligned}$$

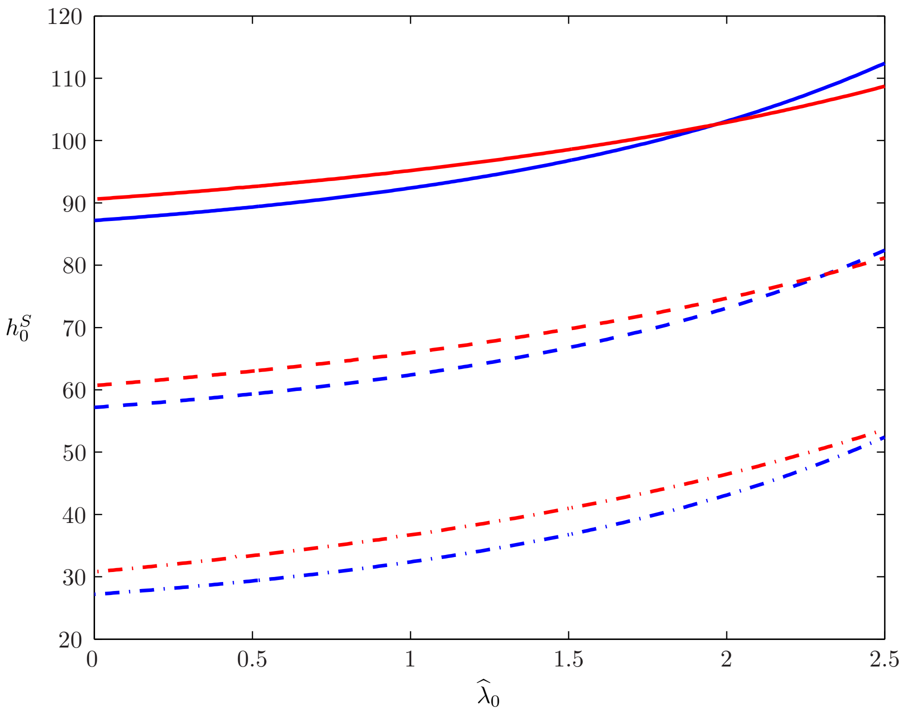
From Brigo and Mercurio (2006, p. 822), we get

$$
\begin{aligned}
& \widehat{E}[G_s \widehat{\lambda}_s | \mathcal{F}_t] \\
= \; & -\frac{\partial}{\partial s} \widehat{E}[G_s | \mathcal{F}_t] \\
= \; & \widehat{E}[G_s | \mathcal{F}_t] \cdot \left[ \left( 1 - \kappa^{\widehat{\lambda}} C(t,s) + \frac{(\sigma^{\widehat{\lambda}})^2}{2} C^2(t,s) \right) \widehat{\lambda}_t + \kappa^{\widehat{\lambda}} \theta^{\widehat{\lambda}} C(t,s) \right] \qquad (26)
\end{aligned}
$$

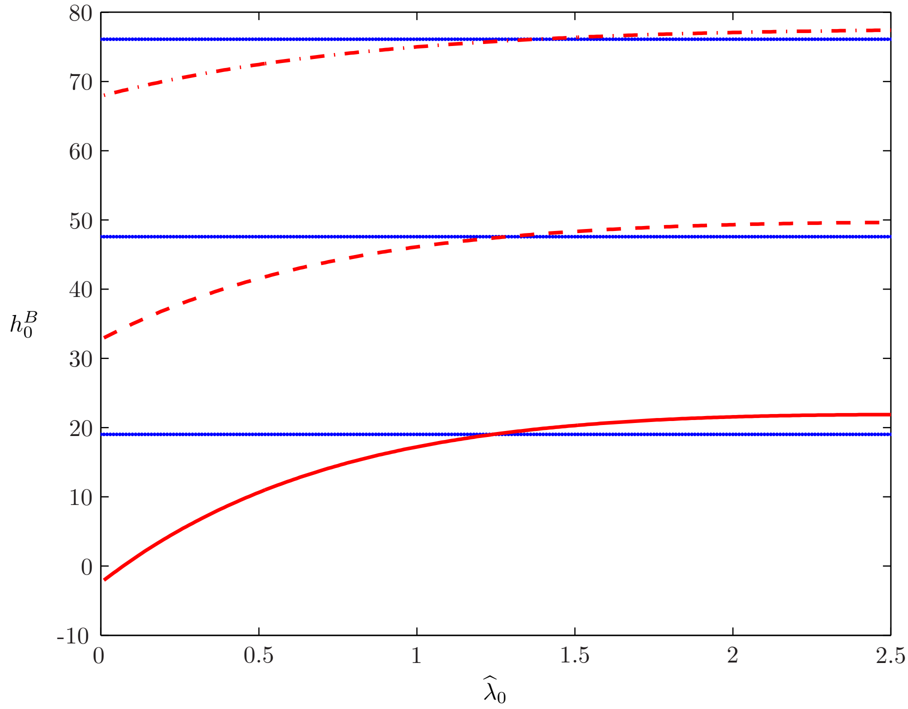
Figure 5: *Sensitivity of the LRM-hedge using the junior bond w.r.t. the intensity.*

The figure shows the initial hedge ratio, the initial position in the money market account and the initial portfolio value as a function of the initial default rate for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic default rates for parameters $t = 0$, $T = 1$, $r = 0.05$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic default rates with CIR dynamics for parameters $\kappa^{\widehat{\lambda}} = 0.5$, $\theta^{\widehat{\lambda}} = 1.25$ and $\sigma^{\widehat{\lambda}} = 0.4$.

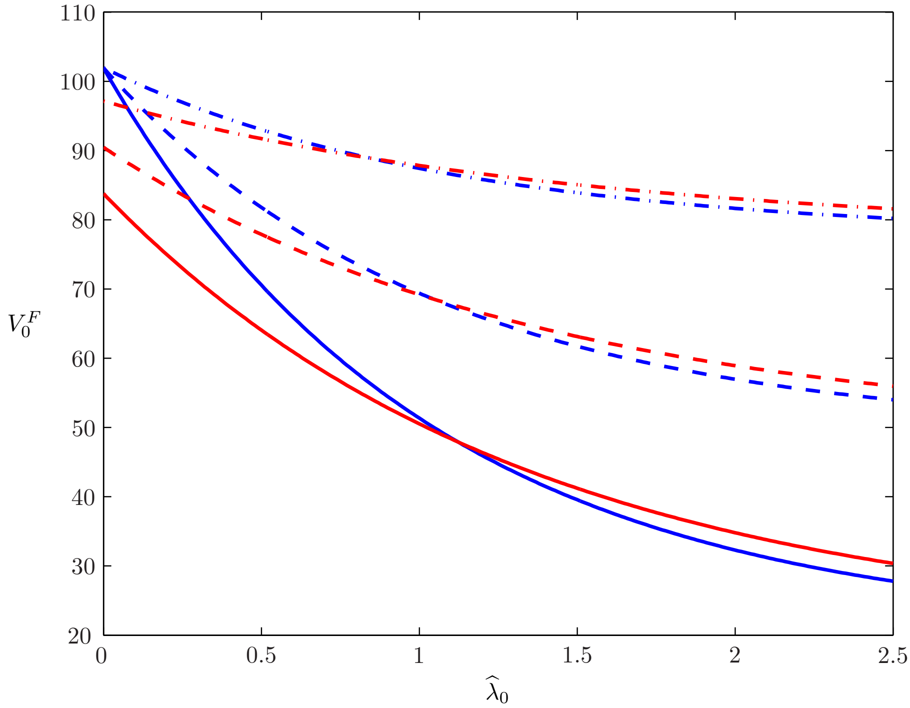
(a) *Initial hedge ratio.*



(b) *Initial position in the money market account.*




(c) *Initial portfolio value.*



From Proposition A.1, it follows that the process $\xi^m$ is given by

$$\begin{aligned}
\xi_t^m &= B_t \sigma^r \sqrt{\widehat{\lambda}_t}\, \frac{\partial}{\partial \widehat{\lambda}} u(t,\widehat{\lambda}_t) \\
&= \sigma^r \sqrt{\widehat{\lambda}_t} \cdot \Bigg[ -C(t,T) \frac{1}{B_T} \widehat{E}[G_T|\mathcal{F}_t] \cdot F + \int_t^T \frac{1}{B_u} \frac{\partial}{\partial \widehat{\lambda}} \widehat{E}[G_s \widehat{\lambda}_s|\mathcal{F}_t] \mu^Z(s)\, ds \\
&\qquad - \int_t^T C(t,s) \frac{1}{B_s} \widehat{E}[G_s|\mathcal{F}_t]\, dC_s \Bigg]
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\frac{\partial}{\partial \widehat{\lambda}} \widehat{E}[G_s \widehat{\lambda}_s|\mathcal{F}_t] \\
= \quad & -C(t,s) \cdot \widehat{E}[G_s \widehat{\lambda}_s|\mathcal{F}_t] + \widehat{E}[G_s|\mathcal{F}_t] \cdot \left( 1 - \kappa^{\widehat{\lambda}} C(t,s) + \frac{(\sigma^{\widehat{\lambda}})^2}{2} C^2(t,s) \right). \quad (27)
\end{aligned}$$

□

Figure 5 (a) shows the hedge ratio of the LRM-strategy as a function of the intensity. One can see that the hedge ratio is an increasing function of the default rate and that it is typically higher in the stochastic intensity case. Thus, the effect on the hedge ratio of an additional source of (diffusion) risk is the same no matter whether the latter comes from the interest or the default rate, cf. Figure 4 (a). An additional source of risk will always increase the optimal number of junior bonds held in the LRM-hedging portfolio. If the current default rate $\widehat{\lambda}_0$ is much higher than its long-term mean $\theta^{\widehat{\lambda}}$, the LRM-strategy anticipates that in will move back towards its mean, whereas this cannot happen if the default rate is deterministic. That is why the blue graphs lie above the red graphs for high $\widehat{\lambda}_0$. However, the effect on the position in the money market account is different when the additional source of risk comes from the default rate rather than the interest rate, cf. 5 (b). In the deterministic default rates case, we see from (12) that the position in the money market account does not depend on the default rate level, since through this position the LRM-hedge only covers the coupon payments up to current time plus the expected recovery payment if default was to occur within the next instant. However, in the stochastic default rates case, the position in the money market account is not decreasing but increasing in the intensity, since the additional term on the right-hand side of (15) is decreasing in $\widehat{\lambda}$. From Figure 5 (b), we can see most clearly how the LRM-hedge accounts for the mean-reversion property of the stochastic intensity. If $\widehat{\lambda} < \theta^{\widehat{\lambda}}$, i.e. it is currently unlikely that default occurs, the position in the (locally) riskless money market account is lower than in the deterministic default rates case. On the contrary, if $\widehat{\lambda} > \theta^{\widehat{\lambda}}$, i.e. default is likely to occur, the LRM-strategy puts more weight on the position $h^B$ compared to the deterministic default rates case. Analogously, the optimal portfolio value for the LRM-strategy is higher in the stochastic default rates case if the current default intensity lies above its long-term average, $\widehat{\lambda} > \theta^{\widehat{\lambda}}$, and it is higher in the deterministic default rates case if the current default intensity lies below its long-term average, $\widehat{\lambda} < \theta^{\widehat{\lambda}}$.

## 4.3 Stochastic Interest Rates and Stochastic Intensities

We now assume that both the interest and the default rate follow a stochastic process and, in particular, that $\mathcal{F}_t = \sigma(W_t^r, W_t^{\widehat{\lambda}})$ for two independent[10] Brownian motions.

The discounted junior bond price follows the dynamics

$$d\left(\frac{S_t}{B_t}\right) = (1 - H_t)G_t^{-1} dm_t^S - \frac{S_{t-}}{B_{t-}} d\widetilde{H}_t,$$

where the martingale $m^S$ is now given by

$$m_t^S = \widehat{E}\left[\frac{G_T}{B_T} | \mathcal{F}_t\right]$$

with both $B$ and $G$ inside the conditional expectation since both $r$ and $\widehat{\lambda}$ are stochastic now. The martingale representations of the processes $m^S$ and $m$ now take the form

$$\begin{aligned} m_t &= m_0 + \int_0^t \xi_s^{m,r}\, dW^r + \int_0^t \xi_s^{m,\widehat{\lambda}}\, dW^{\widehat{\lambda}}, \\ m_t^S &= m_0 + \int_0^t \xi_s^{S,r}\, dW^r + \int_0^t \xi_s^{S,\widehat{\lambda}}\, dW^{\widehat{\lambda}}, \end{aligned}$$

for $(\mathcal{F}_t)$-predictable processes $\xi^{\cdot,\cdot}$.

[10] Brigo and Mercurio (2006, p. 817) consider the case of two correlated Brownian motions with correlation coefficient $\rho$, i.e. $dW^r dW^{\widehat{\lambda}} = \rho dt$, and show that there exists no explicit representation of the zero bond price in case $\rho \neq 0$, but that the impact of $\rho$ is negligible. Thus this independence assumption is no major restriction.

**Lemma 3 (FS-Decomposition, Case of a Two-Dimensional BM)** *The discounted cumulative value $F_T/B_T$ of the senior bond $(Z, C, F)$ at maturity has the following strong Föllmer-Schweizer-decomposition:*

$$\frac{F_T}{B_T} = F_0 + \int_0^T h_t^S \, d\left(\frac{S_t}{B_t}\right) + \frac{L_T^F}{B_T} \, ,$$

*where*

$$\begin{aligned} h_t^S &= \frac{d\langle V^F, S\rangle_t^{\widehat{P}}}{d\langle S, S\rangle_t^{\widehat{P}}} \\ &= (1 - H_t)\left(\frac{\xi_t^m}{\xi_t^S} + \frac{g_t^C + g_t^F + g_t^Z - \mu^Z(t)}{S_{t-}}\right) , \end{aligned}$$

*is the locally risk-minimizing hedge ratio, $F_0 = g_0^C + g_0^F + g_0^Z$ is a constant, $L^F/B$ is a martingale which is orthogonal to $M$, given by $L_t^F = \int_0^t \frac{B_t}{B_s}(Z(s) - \mu^Z(s)) \, d\widetilde{H}_s$, and the processes $\xi^m$ and $\xi^S$ are given by*

$$\begin{aligned} \xi^m &= \xi_t^{m,r} + \xi_t^{m,\widehat{\lambda}}, \\ \xi^S &= \xi_t^{S,r} + \xi_t^{S,\widehat{\lambda}}. \end{aligned}$$

This yields the LRM-strategy in case both the interest rate and the intensity are stochastic.

**Proposition 5 (LRM-Hedge, Case of a Two-Dimensional BM)** *In case of both stochastic interest and default rates, the locally risk-minimizing hedging strategy of the senior bond $(Z,C,F)$ is given by*

$$\begin{aligned} h_t^S &= \frac{\xi_t^m}{\xi_t^S} + \frac{g_t^C + g_t^F + g_t^Z - \mu^Z(t)}{S_{t-}}, \\ h_t^B &= \frac{V_{t-}^F}{B_t} - h_t^S \cdot \frac{S_{t-}}{B_t}, \end{aligned}$$

*for $t \leq \tau$, and*

$$\begin{aligned} h_t^S &= 0, \\ h_t^B &= \int_0^\tau \frac{1}{B_s} \, dC_s + \frac{Z_\tau}{B_\tau}, \end{aligned}$$

*for $t > \tau$.*

**Example 3** Suppose both the interest rate and the intensity follow a CIR-process, i.e.

$$\begin{aligned} dr_t &= \kappa^r(\theta^r - r_t)dt + \sigma^r\sqrt{r_t}d\widehat{W}_t^r, \\ d\widehat{\lambda}_t &= \kappa^{\widehat{\lambda}}(\theta^{\widehat{\lambda}} - \widehat{\lambda}_t)dt + \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}d\widehat{W}_t^{\widehat{\lambda}}. \end{aligned}$$

The martingale $m$ can be written

$$\begin{aligned} m_t &= \widehat{E}\left[\frac{G_T}{B_T}F + \int_0^T \frac{G_s}{B_s}Z_s\widehat{\lambda}_s ds + \int_0^T \frac{G_s}{B_s}dC_s \middle| \mathcal{F}_t\right] \\ &= F \cdot \widehat{E}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right]\widehat{E}\left[G_T\middle|\mathcal{F}_t\right] + \widehat{E}\left[\middle|\mathcal{F}_t\right]\int_0^T \frac{1}{B_s}\widehat{E}\left[G_s\widehat{\lambda}_s\middle|\mathcal{F}_t\right]\mu^Z(s)\,ds \\ &\qquad + \int_0^T \widehat{E}\left[\frac{1}{B_s}\middle|\mathcal{F}_t\right]\widehat{E}\left[G_s\middle|\mathcal{F}_t\right]dC_s \\ &=: u(t,r_t,\widehat{\lambda}_t). \end{aligned}$$

From Proposition A.2, it follows that the processes $\xi^{m,r}$ and $\xi^{m,\widehat{\lambda}}$ are given by

$$\begin{aligned} \xi_t^{m,r} &= \sigma^r\sqrt{r_t}\cdot\frac{\partial}{\partial r}u(t,r,\widehat{\lambda}), \\ \xi_t^{m,\widehat{\lambda}} &= \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}\cdot\frac{\partial}{\partial\widehat{\lambda}}u(t,r,\widehat{\lambda}). \end{aligned}$$

Since

$$\begin{aligned} &\frac{\partial}{\partial r}u(t,r,\widehat{\lambda}) \\ =\; &-C^r(t,T)\,\widehat{E}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right]\cdot F\cdot\widehat{E}\left[G_T\middle|\mathcal{F}_t\right] - \int_0^T C^r(t,s)\widehat{E}\left[\frac{1}{B_s}\middle|\mathcal{F}_t\right]\widehat{E}\left[G_s\widehat{\lambda}_s\middle|\mathcal{F}_t\right]\mu^Z(s)\,ds \\ &\qquad - \int_t^T C^r(t,s)\,\widehat{E}\left[\frac{1}{B_s}\middle|\mathcal{F}_t\right]\widehat{E}\left[G_s\middle|\mathcal{F}_t\right]dC_s \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial\widehat{\lambda}}u(t,r,\widehat{\lambda}) &= -C^{\widehat{\lambda}}(t,T)\cdot F\cdot\widehat{E}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right]\widehat{E}\left[G_T\middle|\mathcal{F}_t\right] \\ &\qquad + \int_0^T \widehat{E}\left[\frac{1}{B_s}\middle|\mathcal{F}_t\right]\frac{\partial}{\partial\widehat{\lambda}}\widehat{E}\left[G_s\widehat{\lambda}_s\middle|\mathcal{F}_t\right]\mu^Z(s)\,ds \\ &\qquad - \int_t^T C^{\widehat{\lambda}}(t,s)\,\widehat{E}\left[\frac{1}{B_s}\middle|\mathcal{F}_t\right]\widehat{E}\left[G_s\middle|\mathcal{F}_t\right]dC_s, \end{aligned}$$

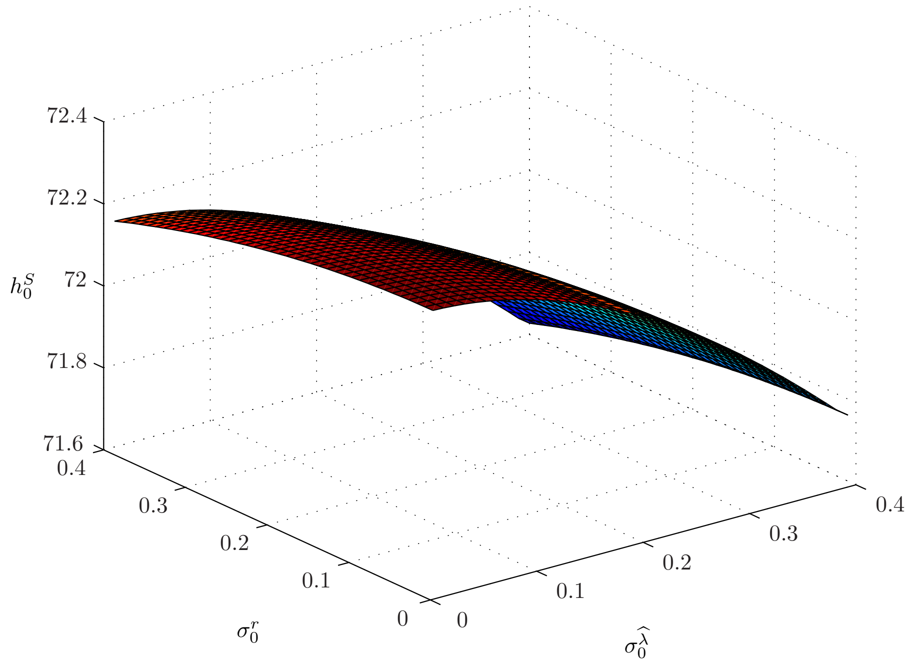
we have

$$\xi_t^{m,r} = \sigma^r \sqrt{r_t} \cdot \Bigg[ -C^r(t,T)\, \widehat{E}\left[\frac{1}{B_T} \Big| \mathcal{F}_t\right] \cdot F \cdot \widehat{E}\left[G_T \big| \mathcal{F}_t\right]$$
$$- \int_0^T C^r(t,s) \widehat{E}\left[\frac{1}{B_s} \Big| \mathcal{F}_t\right] \widehat{E}\left[G_s \widehat{\lambda}_s \big| \mathcal{F}_t\right] \mu^Z(s)\, ds$$
$$- \int_t^T C^r(t,s)\, \widehat{E}\left[\frac{1}{B_s} \Big| \mathcal{F}_t\right] \widehat{E}\left[G_s \big| \mathcal{F}_t\right] dC_s \Bigg], \qquad (28)$$

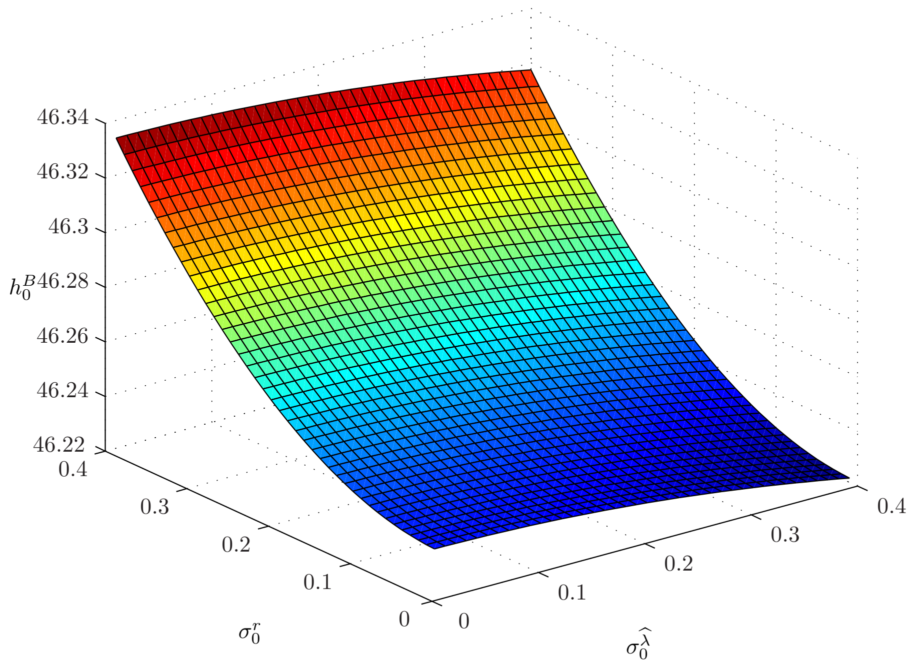
Figure 6: *Sensitivity of the LRM-hedge using the junior bond w.r.t. the interest rate volatility and the default rate volatility.*

The figure shows the initial hedge ratio, the initial position in the money market account and the initial portfolio value as a function of the initial interest rate volatility and the initial default rate volatility for parameters $t = 0$, $T = 2$, $C = 8$, $F = 100$, $r_0 = 0.05$, $\kappa^r = 2.5$, $\theta^r = 0.05$, $\widehat{\lambda}_0 = 0.35$, $\kappa^{\widehat{\lambda}} = 0.5$ and $\theta^{\widehat{\lambda}} = 0.35$.

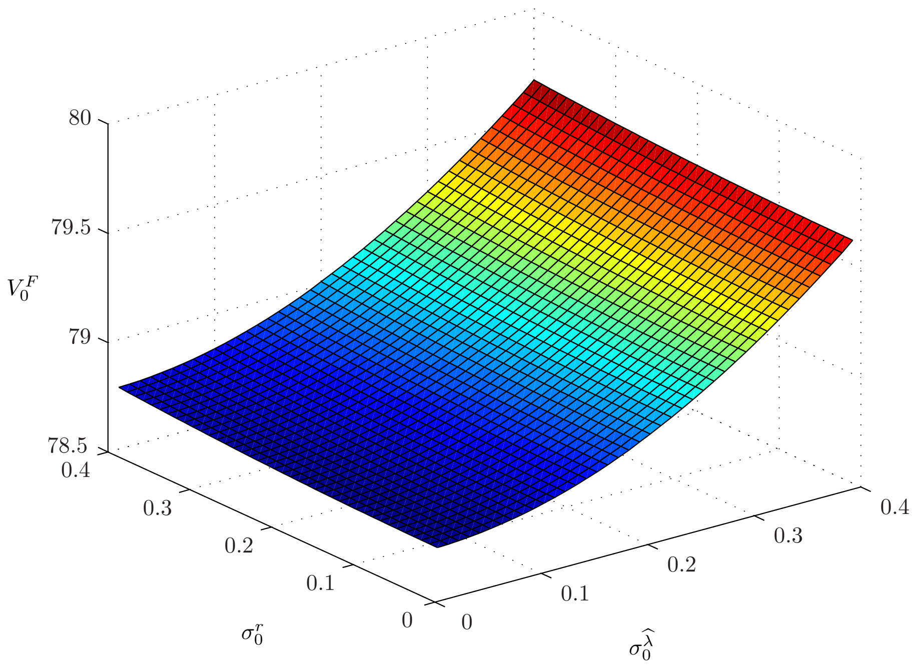
(a) *Initial hedge ratio.*



(b) *Initial position in the money market account.*




(c) *Initial portfolio value.*



$$
\begin{aligned}
\xi_t^{m,\widehat{\lambda}} \;&=\; \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}\cdot\Bigg[-C^{\widehat{\lambda}}(t,T)\cdot F\cdot\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right]\widehat{E}\left[G_T|\mathcal{F}_t\right] \\
&\qquad + \int_0^T \widehat{E}\left[\frac{1}{B_s}\Big|\mathcal{F}_t\right]\frac{\partial}{\partial\widehat{\lambda}}\widehat{E}\left[G_s\widehat{\lambda}_s\Big|\mathcal{F}_t\right]\mu^Z(s)\,ds \\
&\qquad - \int_t^T C^{\widehat{\lambda}}(t,s)\,\widehat{E}\left[\frac{1}{B_s}\Big|\mathcal{F}_t\right]\widehat{E}\left[G_s|\mathcal{F}_t\right]dC_s\Bigg], \qquad (29)
\end{aligned}
$$

Since the martingale $m^S$ can be written

$$
\begin{aligned}
m_t^S \;&=\; \widehat{E}\left[G_T|\mathcal{F}_t\right]\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \\
&=:\; v(t,r_t,\widehat{\lambda}_t),
\end{aligned}
$$

it follows from Theorem 15.4.1 in Bruti-Liberati and Platen (2010) that the processes $\xi^{S,r}$ and $\xi^{S,\widehat{\lambda}}$ are given by

$$
\begin{aligned}
\xi_t^{S,r} \;&=\; \sigma^r\sqrt{r_t}\cdot\frac{\partial}{\partial r}v(t,r,\widehat{\lambda}), \\
\xi_t^{S,\widehat{\lambda}} \;&=\; \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}\cdot\frac{\partial}{\partial\widehat{\lambda}}v(t,r,\widehat{\lambda}).
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{\partial}{\partial r}v(t,r,\widehat{\lambda}) \;&=\; -C^r(t,T)\,\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right]\widehat{E}\left[G_T|\mathcal{F}_t\right], \\
\frac{\partial}{\partial\widehat{\lambda}}v(t,r,\widehat{\lambda}) \;&=\; -C^{\widehat{\lambda}}(t,T)\,\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right]\widehat{E}\left[G_T|\mathcal{F}_t\right],
\end{aligned}
$$

we get

$$
\xi_t^{S,r} \;=\; -\sigma^r\sqrt{r_t}\,C^r(t,T)\,\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right]\widehat{E}\left[G_T|\mathcal{F}_t\right], \qquad (30)
$$

$$
\xi_t^{S,\widehat{\lambda}} \;=\; -\sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}\,C^{\widehat{\lambda}}(t,T)\,\widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right]\widehat{E}\left[G_T|\mathcal{F}_t\right]. \qquad (31)
$$

Plugging (19), (25), (26) and (27) into (28)-(31) yields the hedge ratio. □

Figure 6 depicts the sensitivity of the LRM-hedge w.r.t. the interest rate volatility and the default rate volatility. From Figure 6 (c), we can see that the optimal portfolio value is slightly increasing in the interest rate volatility and considerably increasing in the default rate volatility. This is due to the fact that an increase in either of the two volatilities will also increase the value of the senior bond, and hence the optimal portfolio value must also rise. From Figure 6 (a) and (b) we see how the LRM-hedge accounts for a higher diffusion risk. Consider first the interest rate volatility. If $\sigma^r$ increases, so does the optimal portfolio value and the LRM-hedge will then put more weight on the position in the locally riskless money market account and less weight on the position in the junior bond. If the optimal portfolio value is higher due to a higher default rate volatility, both the position in the junior bond and the position in the money market accoutn will be lower. This is due to the fact that a higher default rate volatiliy will not only increase the value of the senior bond and, hence, the optimal portfolio value but also the value of the junior bond.

# 5 Simulation of Hedging Costs

In this section, we run a simulation with 10,000 iterations to test the impact of the different model assumptions on the cumulative hedging costs. We also test the LRM-strategy against strategies using alternative hedging instruments such as CDS contracts, CoCo-bonds, a junior bond of a comparable firm, and a credit index.

If not specified otherwise, we use the following parameters: The senior bond $(C, F, Z)$ is assumed to pay an annualized coupon at rate $c = 0.08$ and to have a promised payment of $F = 100$. The doubly-stochastic fraction of this payment recoverd in case of default is assumed to have a Beta $(12, 12)$-distribution, i.e. $\mu^Z(t) = 50$ for all $t$. We assume a maturity of $T = 2$ years and that the hedging strategies are adjusted on a weekly basis, i.e. we consider the trading dates $t_0 = 0 < t_1 < ... < t_n = 2$ with $t_i - t_{i-1} = 1/52$ for all $i = 1,...,104$.

For the case of both deterministic interest and default rate, we use constant rates of $r = 0.05$ and $\widehat{\lambda} = 0.35$. To simulate the CIR-model for the stochastic interest respectively default rate, we proceed as described by Glasserman (2003, p. 120ff.). As is mentioned there, a simple Euler discretization of the form

$$\begin{aligned} r(t_{i+1}) &= r(t_i) + \kappa^r(\theta^r - r(t_i)) \cdot (t_{i+1} - t_i) + \sigma^r \sqrt{r(t_i)(t_{i+1} - t_i)} Z^r_{i+1} \\ \widehat{\lambda}(t_{i+1}) &= \widehat{\lambda}(t_i) + \kappa^{\widehat{\lambda}}(\theta^{\widehat{\lambda}} - \widehat{\lambda}(t_i)) \cdot (t_{i+1} - t_i) + \sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}(t_i)(t_{i+1} - t_i)} Z^{\widehat{\lambda}}_{i+1}, \end{aligned}$$

where $Z^r_1,...,Z^r_n$ and $Z^{\widehat{\lambda}}_1,...,Z^{\widehat{\lambda}}_n$ are independent standard normal random variables, will still produce negative values, even if the expressions under the square root are replaced by their positive parts. We therefore use the algorithm from Glasserman (2003, p. 124) that allows to sample from the exact transition law of the processes. The respective parameters are given by $\theta^r = 0.05$, $\kappa^r = 0.01$, $\sigma^r = 0.01$ and $r_0 = 0.05$ for the interest rate and $\theta^{\widehat{\lambda}} = 0.35$, $\kappa^{\widehat{\lambda}} = 0.25$, $\sigma^{\widehat{\lambda}} = 0.4$ and $\widehat{\lambda}_0 = 0.35$ for the default rate. We first examine the basic model with both deterministic interest and default rate. From Table 2 we see that the hedger, on average, faces nearly zero additional costs apart from the initial investment in the amount of the initial value $V_0^F$ of the senior bond $(C, F, Z)$ to set up the strategy, i.e. the strategy is mean-self-financing. Additional costs accrue if default occurs before maturity and the doubly-stochastic recovery payment deviates from its expected value of $\mu^Z(t) = 50$. For instance, the highest cumulative hedging costs of 107.42 in the simulation are due to realized recovery payment of 85.92. In this case, prior to default the position $h_t^B$ in the money market account from Proposition 2 is far too low. Conversely, the lowest cumulative hedging costs of 49.48 in the simulation correspond to a realized recovery payment of only 20.75. In this case, the position in the money market account was far too high. If no default occurs prior to maturity, the hedging strategy reduces to the replication strategy in case of a single-stochastic recovery payment. Hence the strategy is self-financing and no additional costs accrue, i.e. the cumulative costs equal the initial costs of 78.68. In case only the interest rate is stochastic, the hedging costs remain nearly unaffected, see Table 2, since the interest rate risk is perfectly hedgeable, and the small differences in the hedging costs are thus solely due to the discretization error. In contrast, if the default rate is stochastic, the hedging costs are affected, especially in iterations where no default occurs, i.e. when the hedging strategy is adjusted for variation in the intensity at any trading date, but this this turns out to have been unnecessary since there is no default. Moreover, the simulation results show that the hedging costs of the LRM-strategy are even slightly lower in the practically more relevant case of using stocks as the hedging instrument.

Let us now consider the alternative strategies. First, we consider an extension of the duplication strategy using CDS contracts by Bielecki, Jeanblanc and Rutkowski (2007) to the case of doubly-stochastic recovery payment, hence to an incomplete market setting. The corresponding LRM-strategy involves a short position in CDS contracts and a long position in the money market account. It can be seen from Table 2 that the hedging costs due to the discretization are very small. The variance is smaller than for all other strategies considered and both the minimum and the maximum costs are much closer to the expected costs, see also part (a) of Figure 8. However, as mentioned by Bielecki, Jeanblanc and Rutkowski (2008, p. 2512f.),

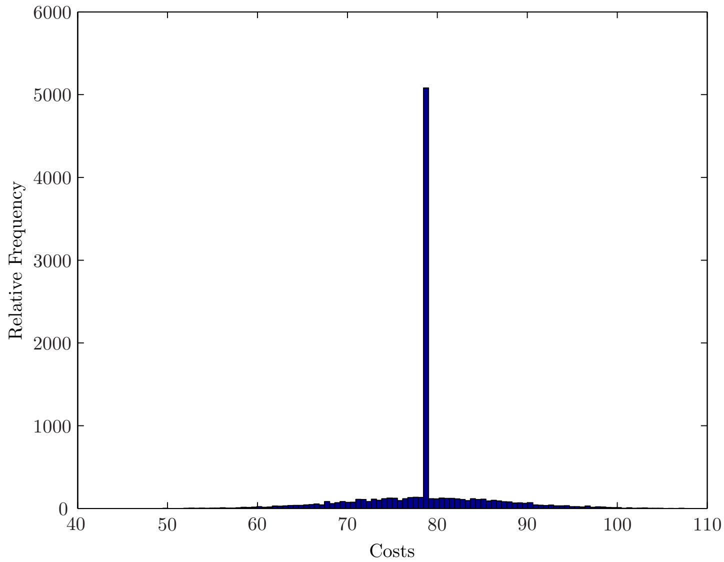
Table 2: Discounted cumulative hedging costs when hedging a senior bond.

| | | | Hedging | Instruments | | | | |
|---|---|---|---|---|---|---|---|---|
| Total Costs | | Junior | Bond | | Stock | CDS | Junior Bond | Credit Index |
| | $r$,$\widehat{\lambda}$ determ. | $r$ stoch. | $\widehat{\lambda}$ stoch. | $r$&$\widehat{\lambda}$ stoch. | | | comp. firm | |
| (Initial Costs) | (78.68) | (78.68) | (79.81) | (79.80) | (78.68) | (78.68) | (78.68) | (78.68) |
| Mean | **78.70** | **78.71** | **79.64** | **79.65** | **78.60** | **79.78** | **78.91** | **79.09** |
| Std Dev | 6.47 | 6.52 | 6.53 | 6.53 | 8.53 | 1.69 | 15.43 | 30.56 |
| Skewness | 0.01 | 0.23 | 0.50 | 0.38 | -0.05 | -1.13 | -0.90 | -0.14 |
| Kurtosis | 5.63 | 5.57 | 5.48 | 5.50 | 3.34 | 9.81 | 4.21 | 1.24 |
| Min | 49.48 | 52.78 | 49.32 | 49.52 | 42.80 | 67.59 | 16.52 | 19.39 |
| Max | 107.42 | 112.73 | 109.76 | 112.04 | 111.21 | 89.68 | 107.42 | 125.82 |
| 99%-quantile | 97.72 | 97.99 | 99.53 | 99.41 | 99.47 | 84.28 | 105.02 | 116.58 |
| 95%-quantile | 90.33 | 90.72 | 92.37 | 92.17 | 92.25 | 81.94 | 102.50 | 114.75 |
| 90%-quantile | 86.47 | 87.00 | 88.38 | 88.20 | 89.19 | 80.90 | 98.70 | 113.48 |
| 75%-quantile | 78.75 | 79.74 | 80.90 | 81.19 | 84.12 | 80.13 | 87.91 | 111.70 |
| 50%-quantile | 78.68 | 78.39 | 78.68 | 79.03 | 78.74 | 80.13 | 78.68 | 77.26 |
| 25%-quantile | 78.55 | 76.96 | 77.50 | 77.39 | 73.03 | 79.74 | 74.87 | 52.36 |
| 10%-quantile | 70.98 | 71.49 | 73.04 | 72.90 | 67.66 | 77.84 | 56.86 | 43.79 |
| 5%-quantile | 66.93 | 67.51 | 69.15 | 68.90 | 64.44 | 76.42 | 45.34 | 39.30 |
| 1%-quantile | 60.02 | 60.53 | 61.98 | 61.70 | 57.45 | 73.47 | 33.74 | 32.16 |

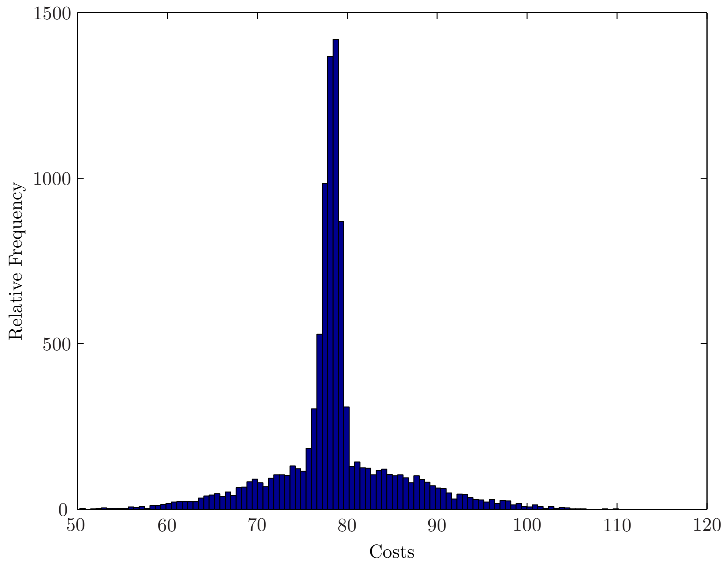
the strategy involves trading a CDS contract issued in the past, i.e. an instrument that is not very liquid in practice. Finally, we also consider two cross-hedging strategies. The first of them involves a junior bond of a comparable firm, i.e. an instrument with the same default intensity but a distinct default time. The second cross- hedging strategy, involves a position in a credit index of the type investigated in Brigo and Morini (2011), i.e. a pool of credit names with the same credit quality (the same default rate) as the instrument we wish to hedge. Taking a long position in such a credit index, the investor makes periodical protection payments and receives a payment at any time one of the credit names in the index defaults. In our simulations, we chose a credit index with $n = 5$ credit names.

Figure 7: *Discounted cumulative hedging costs of LRM-strategy with defaultable zero bond.*

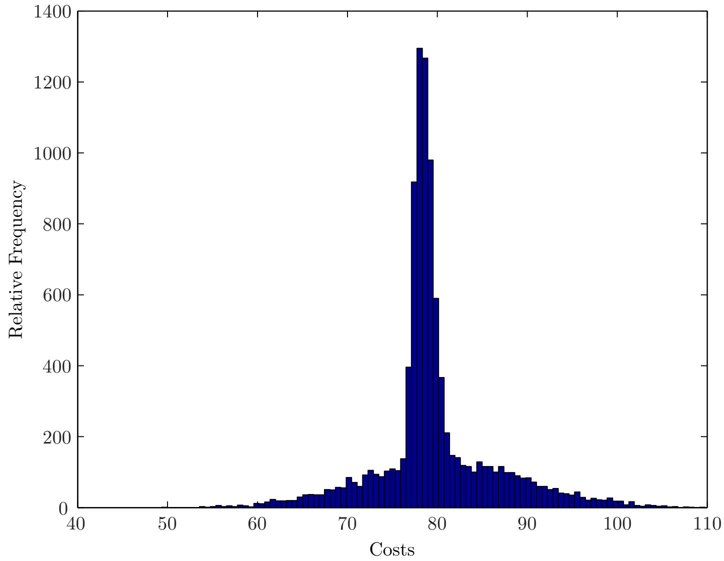
*(a) Deterministic interest rate and deterministic default rate.*




*(b) Stochastic interest rate and deterministic default rate.*



*(c) Deterministic interest rate and stochastic default rate.*




*(d) Stochastic interest rate and stochastic default rate.*

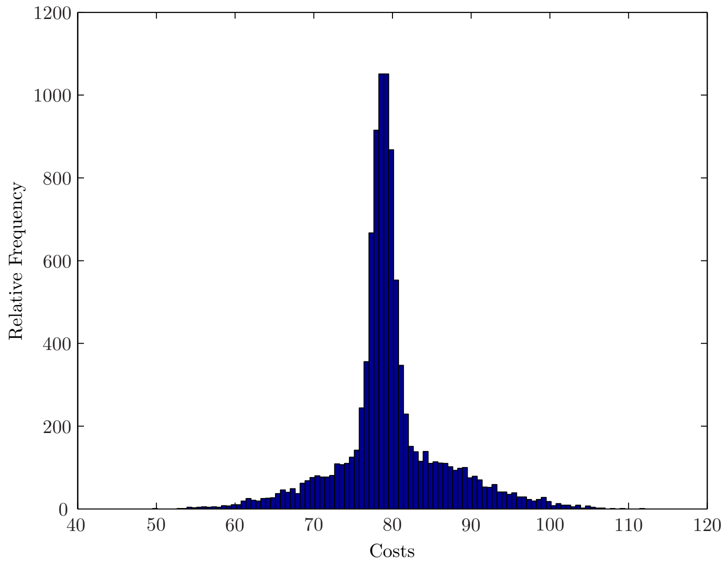
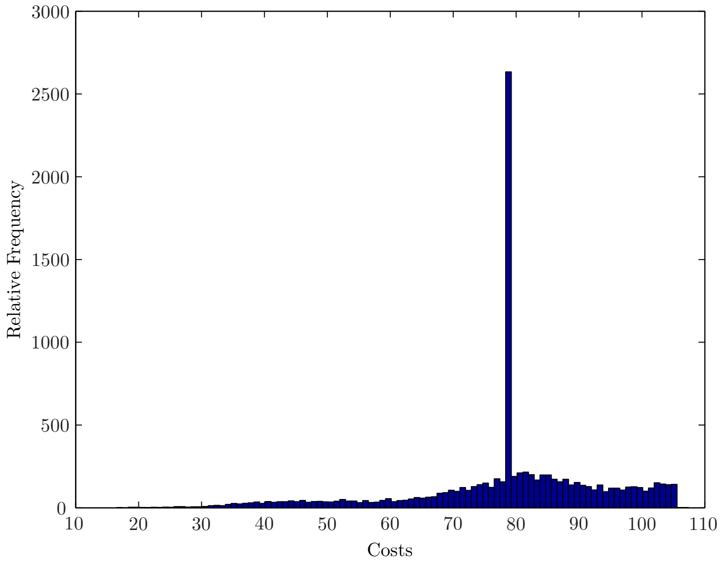
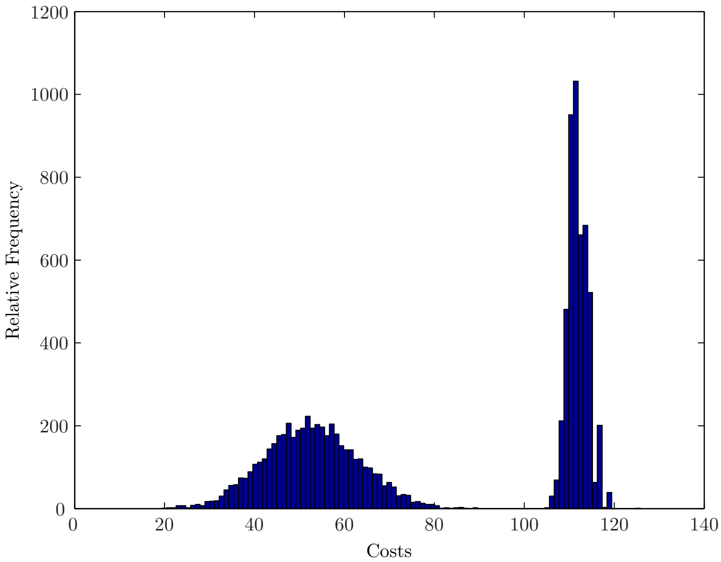


Figure 8: *Discounted cumulative hedging costs for alternative hedging instruments.*

*(a) Junior bond of a comparable firm.*




*(b) Credit Index.*



# 6 Conclusion

There is overwhelming empirical evidence that recovery payments in case of default do not only depend on time of default and the term structure but also on additional sources of risk. Based on the concept of single-stochastic and doubly-stochastic recovery payments introduced in this paper, we derive hedging strategies which are locally risk minimizing (LRM). We denote the recovery rate as *single-stochastic* if the recovery amount depends only on the default event and the interest rate. We denote the recovery rate as *doubly-stochastic* if the recovery amount also depends on the realization of another random variable. Corresponding model variants are examined for the reduced-form model framework.

It turns out that the corresponding LRM-strategy is not only mean-self-financing but also self-financing if the default recovery is single-stochastic. That is, as long as the recovery amount is known in the event of default, there exists a self-financing replication strategy for credit derivatives. Moreover, we find that in the more realistic case of doubly-stochastic default recoveries, the LRM-hedging strategy does only depend on the *expected* recovery amount, not on other characteristics of its distribution. This key result of the paper helps to justify the simplifying assumption frequently made when valuing and hedging credit derivatives, that the default recovery is constant, conditional on the default event.

The key result also holds when replacing the zero coupon with total loss in case of default by another hedging instrument. For instance, under the assumption that the stock price jumps to/or reaches a pre-specified value when the credit event occurs, one may also use common stocks. Moreover, and in contrast to the existing literature, we provide explicit representations for the hedge ratio even when all relevant quantities are stochastic. In our simulations, it turns out that it is crucial to model the default intensity as a stochastic process (in addition to a doubly-stochastic recovery payment).

Our key insight still remains valid when replacing the LRM-concept by another hedging concept which is based on a quadratic criterion. Moreover, it can be shown that the key message also holds when dealing with structural models.

# A Appendix

**Problem 2 (LRM-Hedge in continuous time)**
*A trading strategy* $\mathbf{H}$ *with* $V_T(\mathbf{H}) = F_T$ $P$-a.s. *is called* locally risk-minimizing, *(LRM) for short, if it satisfies*

$$\liminf_{N\to\infty} r^{\mathcal{T}_N}(\mathbf{H},\Delta) \geq 0 \quad P_M\text{-a.s.}^{11}$$

*for every null-convergent sequence of partitions* $\mathcal{T}_N = \{t_0 = 0, t_1, \ldots, t_N = T\}$ *of* $[0,T]$, *i.e.* $\mathcal{T}_N \subset \mathcal{T}_{N+1}$ *and* $\lim_{N\to\infty} \max_{i=1,\ldots,N}(t_i^N - t_{i-1}^N) = 0$, *and every disturbance* $\Delta$. *Here a* disturbance $\Delta = (\delta,\varepsilon)$ *is a trading strategy, such that* $\delta_T = \varepsilon_T = 0$ *and* $\int_0^T |\delta_s|\, d|A|_s$ *is bounded. Furthermore, defining the* remaining risk $R_t(\mathbf{H})$ *measured as the expected quadratic increase of the discounted hedging costs,* $R_t(\mathbf{H}) = E^P\left[(C_T(\mathbf{H}) - C_t(\mathbf{H}))^2 \,|\mathcal{G}_t\right]$, *the expression*

$$r^{\mathcal{T}}(\mathbf{H},\Delta) = \sum_{i=0}^{n-1} \frac{R_{t_i}(\mathbf{H} + \Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\mathbf{H})}{E^P[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}|\mathcal{G}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]}$$

*denotes the* risk quotient *for a trading strategy* $\mathbf{H}$, *a disturbance* $\Delta = (\delta,\varepsilon)$ *and the partition* $\mathcal{T} = \{t_0 = 0, t_1, \ldots, t_n = T\}$.

Hence, a trading strategy is locally risk-minimizing if a disturbance of the strategy will increase the risk measured by the risk quotient.

**Proof of Lemma 1.**
In proving this result,Theorem 2.4 from Schweizer (1991) applies directly, provided that the regularity conditions X(1) - X(5) are satisfied. From these five conditions[12,13] only X(2) is not satisfied since $\langle M\rangle_t = 0$ for $t > \tau$. Nonetheless, we can apply the results from Schweizer (1991) since for $t > \tau$ the financial market is not subject to any risk. In addition, $S_t = 0$ for $t > \tau$.

[11] $P_M = P \times \langle M,M\rangle$ denotes the Doléans Dade measure of $\langle M,M\rangle$ on the product space $\Omega \times [0,T]$ with the predictable $\sigma$-algebra.

[12] X(4) is satisfied if the default intensities fulfill the requirement $\mathrm{E}_M[|\widetilde{\alpha}| \log^+(|\widetilde{\alpha}|)] < \infty$. Here $\mathrm{E}_M[\cdot]$ denotes the expectation under the Doléans-Dade measure $P_M = P \times \langle M,M\rangle$. If the default intensities fulfill for example the condition $\inf_{t\in[0,T]} |\widehat{\lambda}(t) - \lambda(t)|/|\lambda(t)| > 0$, the requirement $\mathrm{E}_M[|\widetilde{\alpha}| \log^+(|\widetilde{\alpha}|)] < \infty$ holds.

[13] X(5) is satisfied since $P(\tau = T) = 0$ and $S_T$ is $P$-a.s. continuous at $T$.

The following proof consists of two steps. In the first step we derive the locally risk-minimizing hedge ratio $h_t^S$, and in the second step we verify that $L^F$ is a square-integrable martingale which is orthogonal to $M$. We have

$$\begin{aligned}
\frac{V_t^F}{B_t} &= \widehat{E}\left[\frac{F_T}{B_T}\middle|\mathcal{G}_t\right] \\
&= \widehat{E}\left[\left(\int_0^T \frac{1}{B_s}\,dC_s + \frac{F}{B_T}\right)\mathbb{1}_{\{\tau>T\}} + \left(\int_0^\tau \frac{1}{B_s}\,dC_s + \frac{Z(\tau)}{B_\tau}\right)\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau\le t\}}\left(\int_0^\tau \frac{1}{B_s}\,dC_s + \frac{Z(\tau)}{B_\tau}\right) + \mathbb{1}_{\{\tau>t\}}\int_0^t \frac{1}{B_s}\,dC_s \\
&\quad + \mathbb{1}_{\{\tau>t\}}\widehat{E}\left[\int_t^{T\wedge\tau} \frac{1}{B_s}\,dC_s + \mathbb{1}_{\{\tau>T\}}\frac{F}{B_T} + \mathbb{1}_{\{\tau\le T\}}\frac{Z(\tau)}{B_\tau}\middle|\mathcal{G}_t\right] \\
&= H_t^Z + \mathbb{1}_{\{\tau>t\}}\int_0^t \frac{1}{B_s}\,dC_s \\
&\quad + \mathbb{1}_{\{\tau>t\}}\int_t^T \frac{1}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\,dC_u \\
&\quad + \mathbb{1}_{\{\tau>t\}}\frac{1}{B_T}\exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}\cdot F \\
&\quad + \mathbb{1}_{\{\tau>t\}}\int_t^T \frac{1}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)ds\right\}\widehat{\lambda}(u)\,\mu^Z(u)\,du \\
&= \frac{H_t^Z}{B_t} + (1-H_t)\left(\int_0^t \frac{1}{B_s}\,dC_s + \frac{g_t^C + g_t^F + g_t^Z}{B_t}\right) \qquad \text{(A1)}
\end{aligned}$$

where

$$H_t^Z = \mathbb{1}_{\{\tau\le t\}}\left(\int_0^\tau \frac{B_t}{B_s}dC_s + \frac{B_t}{B_\tau}Z(\tau)\right). \qquad \text{(A2)}$$

Thus

$$\begin{aligned}
\left\langle\frac{V^F}{B},\frac{S}{B}\right\rangle_t &= \left\langle\frac{H^Z}{B},\frac{S}{B}\right\rangle_t + \left\langle(1-H)\int_0^\cdot \frac{1}{B_s}ds,\frac{S}{B}\right\rangle_t \qquad \text{(A3)} \\
&\quad + \left\langle(1-H)\frac{g^C}{B},\frac{S}{B}\right\rangle_t + \left\langle(1-H)\frac{g^F}{B},\frac{S}{B}\right\rangle_t + \left\langle(1-H)\frac{g^Z}{B},\frac{S}{B}\right\rangle_t.
\end{aligned}$$

For the first term on the right-hand side of (A3),

$$\begin{aligned}
\left[\frac{H^Z}{B},\frac{S}{B}\right]_t &= \frac{H_t^Z}{B_t}\frac{S_{t-}}{B_t} - \int_0^t \frac{H_{s-}^Z}{B_s}d\left(\frac{S_t}{B_t}\right) - \int_0^t \frac{S_{t-}}{B_t}d\left(\frac{H_s^Z}{B_s}\right) \\
&= 0 - 0 - \mathbb{1}_{\{\tau\le t\}}\frac{S_{\tau-}}{B_\tau}\cdot\frac{\Delta H_\tau^Z}{B_\tau} = -\mathbb{1}_{\{\tau\le t\}}\frac{S_{\tau-}}{B_\tau}\cdot\frac{H_\tau^Z}{B_\tau}
\end{aligned}$$

implies

$$
\begin{aligned}
d\left\langle \frac{H^Z}{B}, \frac{S}{B} \right\rangle_t &= \widehat{E}\left[ d\left[ \frac{H^Z}{B}, \frac{S}{B} \right]_t \middle| \mathcal{G}_{t-} \right] \\
&= -\widehat{\lambda}(t) \frac{S_{t-}}{B_{t-}} \left( \int_0^t \frac{1}{B_s} dC_s + \frac{\mu^Z(t)}{B_t} \right) dt \\
&= \frac{-\int_0^t \frac{B_t}{B_s} dC_s - \mu^Z(t)}{S_{t-}} \, d\left\langle \frac{S}{B}, \frac{S}{B} \right\rangle_t .
\end{aligned}
$$

Similarly, for the second term we get

$$
\begin{aligned}
d\left\langle (1-H) \int_0^{\cdot} \frac{1}{B_s} dC_s, \frac{S}{B} \right\rangle_t &= -d\left\langle H \int_0^{\cdot} \frac{1}{B_s} dC_s, \frac{S}{B} \right\rangle_t \\
&= \frac{\int_0^t \frac{B_t}{B_s} dC_s}{S_{t-}} \, d\left\langle \frac{S}{B}, \frac{S}{B} \right\rangle_t ,
\end{aligned}
$$

while for the remaining terms, we have

$$
\begin{aligned}
d\left\langle (1-H) \frac{g^C}{B}, \frac{S}{B} \right\rangle_t &= \frac{g_t^C}{S_{t-}} \, d\left\langle \frac{S}{B}, \frac{S}{B} \right\rangle_t , \\
d\left\langle (1-H) \frac{g^F}{B}, \frac{S}{B} \right\rangle_t &= \frac{g_t^F}{S_{t-}} \, d\left\langle \frac{S}{B}, \frac{S}{B} \right\rangle_t , \\
d\left\langle (1-H) \frac{g^Z}{B}, \frac{S}{B} \right\rangle_t &= \frac{g_t^Z}{S_{t-}} \, d\left\langle \frac{S}{B}, \frac{S}{B} \right\rangle_t ,
\end{aligned}
$$

respectively. Altogether, by (A3),

$$
d\left\langle \frac{V^F}{B}, \frac{S}{B} \right\rangle_t = \left( -\frac{\mu^Z(t)}{S_{t-}} + \frac{g_t^C}{S_{t-}} + \frac{g_t^F}{S_{t-}} + \frac{g_t^Z}{S_{t-}} \right) d\left\langle \frac{S}{B}, \frac{S}{B} \right\rangle_t ,
$$

so the locally risk-minimizing hedge ratio is given by

$$
h_t^S = -\frac{\mu^Z(t)}{S_{t-}} + \frac{g_t^C}{S_{t-}} + \frac{g_t^F}{S_{t-}} + \frac{g_t^Z}{S_{t-}}
$$

Since $H_0 = 0$ and $H_0^Z = 0$,

$$
F_0 = V_0^F = \widehat{E}\left[ \frac{F_T}{B_T} \middle| \mathcal{G}_0 \right] = g_0^C + g_0^F + g_0^Z , \tag{A4}
$$

by equation (A1).

In the following we verify that $L^F/B$ is a square-integrable martingale with $L_0^F = 0$ which is $P$-orthogonal to $M$.

Since $L_0 = 0$, $\sup_{t\in[0,T]} \sigma^Z(t) < \infty$ by assumption and

$$
\begin{aligned}
& \mathrm{E}[L_s/B_s|\mathcal{G}_t] \\
= \; & \mathrm{E}\left[\mathbb{1}_{\{\tau\leq s\}}\frac{1}{B_\tau}\left(Z_\tau - \mu^Z(\tau)\right)\Big|\mathcal{G}_t\right] \\
= \; & \mathrm{E}\left[\left(\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\leq s\}}\right)\frac{1}{B_\tau}\left(Z_\tau - \mu^Z(\tau)\right)\Big|\mathcal{G}_t\right] \\
= \; & \mathbb{1}_{\{\tau<t\}}\frac{1}{B_\tau}\left(Z_\tau - \mu^Z(\tau)\right) \\
& + \int_t^s \frac{1}{B_\tau}\exp\left\{-\int_t^u \widehat{\lambda}(v)dv\right\}\widehat{\lambda}(u)\left(\mu^Z(u) - \mu^Z(u)\right)\, du \\
= \; & \mathbb{1}_{\{\tau\leq t\}}\frac{1}{B_\tau}\left(Z_\tau - \mu^Z(\tau)\right) + 0 \\
= \; & L_t/B_t,
\end{aligned}
$$

for $s \geq t$, $L/B$ is a $(\mathcal{G})$-martingale.

$L$ is stronly $P$-orthogonal to $M$ since

$$
\begin{aligned}
& \mathrm{E}[L_s/B_s \cdot M_s|\mathcal{G}_t] \\
= \; & \mathrm{E}\left[\left(\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\leq s\}}\right)\frac{1}{B_\tau}\left(Z_\tau - \mu^Z(\tau)\right)M_s\Big|\mathcal{G}_t\right] \\
= \; & L_t \cdot \mathrm{E}[M_s|F_t] \\
& + \int_t^s \frac{1}{B_u}\left(\mu^Z(u) - \mu^Z(u)\right)\exp\left\{-\int_t^u \widehat{\lambda}(v)dv\right\}\widehat{\lambda}(u)\, \mathrm{E}[M_s|\mathcal{G}_t]\, du \\
= \; & L_t/B_t \cdot M_t
\end{aligned}
$$

for any $s \geq t$. $\square$

**Proof of Lemma 2.** From equation (23) in Bielecki, Jeanblanc and Rutkowski (2008), it follows that the discounted cumulative value of the senior bond $(Z, C, F)$ follows the dynamics

$$\frac{dV_t^F}{B_t} = (1 - H_t)G_t^{-1}dm_t + \frac{Z(t) - (g_t^C + g_t^F + g_t^Z)}{B_t}d\widetilde{H}_t,$$

hence

$$
\begin{aligned}
&\frac{F_T}{B_T} \\
= \quad & F_0 + \int_0^T (1-H_t) G_t^{-1} dm_t \\
& \qquad + \int_0^T \left( \frac{Z(t) - g_t^C + g_t^F + g_t^Z}{B_t} \right) d\widetilde{H}_t \\
= \quad & F_0 + \int_0^T (1-H_t)\, G_t^{-1}\, \frac{\xi_t^m}{\xi_t^S}\, dm_t^S + \int_0^T \frac{Z(t) - \mu^Z(t)}{B_t}\, d\widetilde{H}_t \\
& \qquad + \int_0^T \left( \frac{\mu^Z(t) - g_t^C + g_t^F + g_t^Z}{B_t} \right) d\widetilde{H}_t \\
= \quad & F_0 + \int_0^T \Bigg[ (1-H_t)\, G_t^{-1}\, \frac{\xi_t^m}{\xi_t^S} \\
& \qquad + \frac{g_t^C + g_t^F + g_t^Z}{S_{t-}} - \frac{\mu^Z(t)}{S_{t-}} \Bigg]\, d\left( \frac{S_t}{B_t} \right) + \int_0^T \frac{Z(t) - \mu^Z(t)}{B_t}\, d\widetilde{H}_t,
\end{aligned}
$$

is the FS-decomposition of the senior bond in case of a non-trivial reference filtration $(\mathcal{F}_t)$. In particular, the locally risk-minimizing hedge ratio is given by

$$
h_t^S = (1-H_t)\, G_t^{-1}\, \frac{\xi_t^m}{\xi_t^S} + \frac{g_t^C + g_t^F + g_t^Z}{S_{t-}} - \frac{\mu^Z(t)}{S_{t-}}.
$$

□

**Proof of Lemma 3.**

$$
\begin{aligned}
\frac{F_T}{B_T} \quad = \quad & F_0 + \int_0^T (1-H_t) G_t^{-1}\, dm_t + \int_0^T \left( \frac{Z(t) - g_t^C + g_t^F + g_t^Z}{B_t} \right) d\widetilde{H}_t \\
= \quad & F_0 + \int_0^T (1-H_t)\, G_t^{-1}\, \frac{\xi_t^{m,r} + \xi_t^{m,\widehat{\lambda}}}{\xi_t^{S,r} + \xi_t^{S,\widehat{\lambda}}}\, dm_t^S + \int_0^T \frac{Z(t) - \mu^Z(t)}{B_t}\, d\widetilde{H}_t \\
& \qquad + \int_0^T \left( \frac{\mu^Z(t) - g_t^C + g_t^F + g_t^Z}{B_t} \right) d\widetilde{H}_t \\
= \quad & F_0 + \int_0^T \left[ (1-H_t)\, \frac{\xi_t^m}{\xi_t^S} + \frac{g_t^C + g_t^F + g_t^Z - \mu^Z(t)}{S_{t-}} \right] d\left( \frac{S_t}{B_t} \right) \\
& \qquad + \int_0^T \frac{Z(t) - \mu^Z(t)}{B_t}\, d\widetilde{H}_t.
\end{aligned}
$$

□

The crucial step in deriving hedge ratios in continuous-time models usually is to calculate the predictable process appearing in the martingale representation of some payoff. In our setting, we are interested in the process $m$ with martingale representation 14. Bruti-Liberati and Platen (2010, p. 591ff.) considered the problem of finding explicit integral representations of general derivatives' payoff structures. For the reader's convenience, we state these results, which are basically due to Heath (1995), in the two propositions below.

We first consider a market driven by a single state variable, a stochastic process $Y$ with dynamics[14]

$$dY_t = \alpha(t,Y_t)\;dt + \sigma(t,Y_t)\;dW_t.$$

Consider a European contingent claim $Z$ whose payoff at maturity $T$ depends on the evolution of the state variable, i.e.

$$Z = Z(\overline{Y}_T),$$

where $\overline{Y}_t = \{Y_s : s \leq t\}$ for all $t$. In particular, we have $\mathcal{F}_t = \sigma(W_s : s \leq t) = \sigma(\overline{Y}_t)$ and, for an $(\mathcal{F}_t)$-martingale $m$, the martingale representation writes

$$m_t = m_0 + \int_0^t \xi_s^m dW_s. \tag{A5}$$

We then have the following result which follows from Bruti-Liberati and Platen (2010, p. 597).

**Proposition A.1 (Explicit Hedge Ratio)**
*Define the martingale $m$ by $m_t = E[Z|\mathcal{F}_t]$ for all $t$. Suppose there exists a deterministic function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ of class[15] $C^{1,3}$ such that*

$$u(t,y) = E[Z|\mathcal{F}_t]$$

*for any $t$ and $y$. Then, the process $\xi^m$ in (A5) is given by*

$$\xi_s^m = \sigma(s,Y_s) \cdot \frac{\partial}{\partial y} u(s,Y_s).$$

[14] In our applications, this corresponds to the case of either the interest rate or the default rate being stochastic. In this case, we have $Y_t = r_t$ respectively $Y_t = \widehat{\lambda}_t$ for all $t$.

[15] A function $u : [0,T] \times \mathbb{R} \to \mathbb{R},(t,y) \mapsto u(t,y)$ is of class $C^{1,3}$, if $u$ is continuously differentiable with respect to $t$ and three times continuously differentiable with respect to $y$.

We now consider a market driven by two state variables, i.e. a two-dimensional stochastic process $Y = (Y^1,Y^2)$ with dynamics[16]

$$dY_t^i = \alpha^i(t,Y_t)\ dt + \sum_{j=1}^{2} \sigma^{i,j}(t,Y_t)\ dW_t^i. \tag{A6}$$

for $i = 1,2$. Consider a European contingent claim $Z$ whose payoff at maturity $T$ depends on the evolution of the two state variables, i.e.

$$Z = Z(\overline{Y}_T^1,\overline{Y}_T^2),$$

where $\overline{Y}_t^i = \{Y_s^i : s \leq t\}$ for all $t$, $i = 1,2$. In this case, the martingale representation writes

$$m_t = m_0 + \int_0^t \xi_s^{m,1} dW_s^1 + \int_0^t \xi_s^{m,2} dW_s^2. \tag{A7}$$

We now state the explicit formula for the processes $\xi_s^{m,i}$, $i = 1,2$, in case the state variable $Y^i$ only depends on $W^i$, $i = 1,2$. In particular, we then have

$$\sigma^{i,j} = \delta^{i,j} \cdot \sigma^{i,i} \tag{A8}$$

in (A6), where $\delta$ denotes the Kronecker delta. The following result is a direct consequence from Bruti-Liberati and Platen (2010, p. 605).

**Proposition A.2 (Explicit Hedge Ratio, Case of a Two-Dimensional BM)** *Define the martingale $m$ by $m_t = E[Z|\mathcal{F}_t]$ for all $t$. Suppose there exists a deterministic function $u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ of class $C^{1,3}$ such that*

$$u(t,y^1,y^2) = E[Z|\mathcal{F}_t]$$

*for any $t$ and $y$. Then, the processes $\xi^{m,i}$, $i = 1,2$, in (A7) are given by*

$$\begin{aligned} \xi_s^{m,1} &= \sigma^{1,1}(s,Y_s) \cdot \frac{\partial}{\partial y^1} u(s,Y_s), \\ \xi_s^{m,2} &= \sigma^{2,2}(s,Y_s) \cdot \frac{\partial}{\partial y^2} u(s,Y_s). \end{aligned}$$

[16] In our applications, this corresponds to the case of both the interest rate and the default rate being stochastic. In this case, we have $Y_t^1 = r_t$ and $Y_t^2 = \widehat{\lambda}_t$ for all $t$.

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