A Technical Note on the Paper "Robustness of option-like warrant valuation" by Schulz and Trautmann (1994)

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ABSTRACT

In their 1994 paper Schulz and Trautmann present a special firm value model, where the unobserved parameters of the asset value V and its volatility σ are calculated from the observable current stock price S and its volatility σ_S with respect to a system of two nonlinear equations. Unfortunately, the proof of a unique solution (V, σ) is not provided. To complete the considerations in the paper of Schulz and Trautmann a detailed analysis of the underlying nonlinear equations and their unique solution is presented in this note. For the proof only elementary calculus is used.

KEY WORDS. Black&Scholes option pricing, firm value model, system of nonlinear equations.

1. INTRODUCTION AND DESCRIPTION OF THE PROBLEM

Schulz and Trautmann (1994) consider a firm value model, where for the financing of the company only shares of outstanding common stock and warrants are used. Each warrant entitles the owner to receive one share of stock upon the payment of a certain exercise price. For our purposes we will use the same notation.

V	$\equiv$ current value of the firm at time t
V_T	$\equiv$ value of the firm at time T
σ	$\equiv$ asset volatility
S	$\equiv$ price per share of common stock at time t
S_T	$\equiv$ price per share of common stock at time T
σ_S	$\equiv$ stock volatility
$\varepsilon_{S,V}$	$\equiv$ elasticity of the stock price with respect to the value of the firm
N	$\equiv$ number of outstanding shares of common stock
τ	$\equiv$ time until maturity of the outstanding warrants $\tau = T - t$
r	$\equiv$ riskless interest rate
$W(V, \sigma)$	$\equiv$ price per warrant at time t
W_T	$\equiv$ price per warrant at time T
K	$\equiv$ exercise price of the warrants at time T
n	$\equiv$ number of outstanding warrants
λ	$\equiv$ dilution factor $\left(\lambda = \dfrac{n}{N}\right)$
$\mathcal{N}$	$\equiv$ standardized cumulative normal distribution function

Hence, the current value V of the company is given by

$$V = NS + nW.$$

It is known that the warrants have the familiar payoff profile $W_T = (S_T - K)^+$ at time T. Evidently the condition $S_T > K$ is eqivalent to $V_T > NK$, such that the payoff profile can be immediately rewritten by

$$W_T = \frac{1}{1+\lambda}\left(\frac{V_T}{N} - K\right)^+ .$$

Since Schulz and Trautmann (1994) assume that the value of the firm follows a constant variance diffusion process the price of the warrant with payoff profile W_T can be calculated by applying the familiar pricing formula of Black and Scholes [cf. Black and Scholes (1973), Merton (1973)].

With respect to fact that the current stock price S may be observable and the current stock volatility σ_S may be estimated, Schulz and Trautmann arrive at the following system of two nonlinear equations dependent on the unknown variables V and σ,

$$S = \frac{V}{N} - \frac{n}{N} W(V, \sigma), \tag{1}$$

$$\sigma_S = \sigma \cdot \varepsilon_{S,V}, \tag{2}$$

where the value of a warrant $W(V, \sigma)$ and the elasticity $\varepsilon_{S,V}$ are given by

$$W(V, \sigma) = \frac{1}{1+\lambda} \pi_{BS}\left(\frac{V_T}{N} - K\right)^+ (V, \sigma, \tau, r) \tag{3}$$

and by

$$\varepsilon_{S,V} = \frac{\partial S}{\partial V}\frac{V}{S} = \left(\frac{1}{N} - \frac{n}{N}\frac{\partial}{\partial V} W(V, \sigma)\right)\frac{V}{S}. \tag{4}$$

With $\pi_{BS}\left(\frac{V_T}{N} - K\right)^+ (V, \sigma, \tau, r)$ we denote the familiar price of a call with strike K and time τ until maturity in the well-known Black&Scholes option pricing model, i.e.

$$\begin{aligned} \pi_{BS}\left(\frac{V_T}{N} - K\right)^+ (V, \sigma, \tau, r) &= \frac{V}{N}\mathcal{N}(d_1) - Ke^{-r\tau}\mathcal{N}(d_2), \\ d_1 &= \frac{\log\frac{V}{NK} + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \\ d_2 &= d_1 - \sigma\sqrt{\tau}. \end{aligned} \tag{5}$$

For observed value S and estimated volatility σ_S the corresponding unknown underlying firm value V and its volatility σ are computed by a numerical routine in the paper of Schulz and Trautmann (1994). However, the proof of the non-trivial fact that there always exists a unique solution (V, σ) for each given pair (S, σ_S) is missing.

2. PROOF OF EXISTENCE OF A UNIQUE SOLUTION

Using eq. (5) and the relation $\frac{1}{1+\lambda} = \frac{N}{N+n}$ we obtain the following representation of eq. (3),

$$W(V,\sigma) = \frac{N}{N+n}\left(\frac{V}{N}\mathcal{N}(d_1) - Ke^{-r\tau}\mathcal{N}(d_2)\right). \tag{6}$$

Since eq. (6) is only a slight modification of the classical Black&Scholes formula, the following relation holds,

$$\frac{\partial}{\partial V}W(V,\sigma) = \frac{N}{N+n}\frac{1}{N}\mathcal{N}(d_1). \tag{7}$$

Eqs. (6) and (7) enable us to rewrite eq. (1) by

$$S = \frac{V}{N} - \frac{n}{N+n}\left(\frac{V}{N}\mathcal{N}(d_1) - Ke^{-r\tau}\mathcal{N}(d_2)\right) \tag{8}$$

and eq. (2) by

$$\sigma_S S = \sigma\left(\frac{V}{N} - \frac{n}{N+n}\frac{V}{N}\mathcal{N}(d_1)\right). \tag{9}$$

From eq. (8) we notice

$$\frac{V}{N} - \frac{n}{N+n}\frac{V}{N}\mathcal{N}(d_1) = S - \frac{n}{N+n}Ke^{-r\tau}\mathcal{N}(d_2)$$

and thus we arrive at

$$\sigma_S S = \sigma\left(S - \frac{n}{N+n}Ke^{-r\tau}\mathcal{N}(d_2)\right). \tag{10}$$

Evidently eq. (10) provides the characteristic inequality

$$\sigma_S \le \sigma. \tag{11}$$

Moreover we obtain directly from eq. (10) a representation of V as a function of σ as our first main result in this proof,

$$V = NKe^{-r\tau}\exp\left\{\sigma\sqrt{\tau}\mathcal{N}^{-1}\left(\frac{N+n}{n}\left(1-\frac{\sigma_S}{\sigma}\right)\frac{S}{K}e^{r\tau}\right) + \frac{\sigma^2}{2}\tau\right\}. \tag{12}$$

Here, $\mathcal{N}^{-1}$ denotes the inverse function of $\mathcal{N}$ and for convenience we define

$$w\left(\sigma\sqrt{\tau}\right) := \mathcal{N}^{-1}\left(\frac{N+n}{n}\left(1-\frac{\sigma_S\sqrt{\tau}}{\sigma\sqrt{\tau}}\right)\frac{S}{K}e^{r\tau}\right).$$

Replacing V in eq. (8) by the representation provided in eq. (12) leads to one remaining equation in σ of which we have to prove unique solvability,

$$\sigma_S\sqrt{\tau}\frac{S}{K}e^{r\tau} = \sigma\sqrt{\tau}\exp\left\{\sigma\sqrt{\tau}w\left(\sigma\sqrt{\tau}\right) + \frac{\sigma^2}{2}\tau\right\}\left(1 - \frac{n}{N+n}\mathcal{N}\left(w\left(\sigma\sqrt{\tau}\right) + \sigma\sqrt{\tau}\right)\right) \tag{13}$$

Since eq. (13) is not very comfortable to investigate, we will use the substitutions $x := \sigma\sqrt{\tau}, \nu := \frac{n}{N+n}, \kappa := \sigma_S\sqrt{\tau}\frac{S}{K}e^{r\tau}, \beta := \frac{N+n}{n}\frac{S}{K}e^{r\tau}$ and obtain with

$$w(x) = \mathcal{N}^{-1}\left(\beta - \frac{\kappa}{\nu x}\right), \frac{\kappa}{\nu\beta} < x < \frac{\kappa}{\nu(\beta-1)^+},$$

the following identical equation in x,

$$\kappa = x\exp\left\{xw(x) + \frac{x^2}{2}\right\}(1 - \nu\mathcal{N}(w(x) + x)). \tag{14}$$

Now, eq. (14) is equivalent to

$$M(x) = 0, \tag{15}$$

where the function $M : \left(\frac{\kappa}{\nu\beta}; \frac{\kappa}{\nu(\beta-1)^+}\right) \to \mathbf{R}$ is defined by

$$M(x) := 1 - \nu\mathcal{N}(w(x) + x) - \frac{\kappa}{x}\exp\left\{-\frac{1}{2}\left(x^2 + 2xw(x)\right)\right\}. \tag{16}$$

To complete our proof it is sufficient to demonstrate that there exists exactly one zero of M. Since M is a continuous function, the existence of at least one zero of M is easy to prove. With $w(x) \to -\infty$ we obtain $M(x) \to -\infty$ for $x \to \frac{\kappa}{\nu\beta}$, and with $w(x) \to \mathcal{N}^{-1}(\min\{\beta; 1\})$ we obtain $M(x) \to 1 - \nu (> 0)$ for $x \to \frac{\kappa}{\nu(\beta-1)^+}$. Thus at least one zero of M exists. With respect to the first derivative $\frac{d}{dx}M(x)$ of M we use the well-known relation

$$\frac{d}{dy}f^{-1}(y) = \frac{1}{\frac{d}{dx}f\left(f^{-1}(y)\right)}$$

for the derivative of the inverse function f^{-1} of a certain function f. Thus we obtain with

$$\frac{d}{dx}w(x) = \sqrt{2\pi}\frac{\kappa}{\nu x^2}\exp\left\{\frac{1}{2}w^2(x)\right\}$$

the relation

$$\frac{d}{dx}M(x) = p(x)\exp\left\{-\frac{1}{2}\left(x^2 + 2xw(x)\right)\right\}, \frac{\kappa}{\nu\beta} < x < \frac{\kappa}{\nu(\beta-1)^+},$$

with

$$p(x) := -\frac{\nu}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}w^2(x)\right\} + \kappa + \frac{\kappa}{x}w(x) + \sqrt{2\pi}\frac{\kappa}{\nu x^2}\exp\left\{\frac{1}{2}w^2(x)\right\}. \tag{17}$$

When investigating the properties of $\frac{d}{dx}M(x)$ and respectively of $p(x)$, we pay attention to the first derivative $\frac{d}{dx}p(x)$ of p,

$$\frac{d}{dx}p(x) = \sqrt{2\pi}\frac{\kappa^2}{\nu x^3}\exp\left\{\frac{1}{2}w^2(x)\right\}\left(\sqrt{2\pi}\frac{\kappa}{\nu x}w(x)\exp\left\{\frac{1}{2}w^2(x)\right\} - 1\right).$$

For local extrema $z \in \left(\frac{\kappa}{\nu\beta}; \frac{\kappa}{\nu(\beta-1)^+}\right)$ of p the necessary condition $\frac{d}{dx}p(z) = 0$, i.e.

$$\frac{\kappa}{z}w(z) = \frac{\nu}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}w^2(z)\right\} \tag{18}$$

must hold.

Now we can close our considerations. For $\beta > 1$ we have $p(x) \to \infty$ as for $x \to \frac{\kappa}{\nu\beta}$ as for $x \to \frac{\kappa}{\nu(\beta-1)^+}$ $(< \infty)$. Evidently, p has a global minimum $z_0 \in \left(\frac{\kappa}{\nu\beta}; \frac{\kappa}{\nu(\beta-1)^+}\right)$. For z_0 eq. (18) holds and we notice that

$$p(z_0) = \kappa + \sqrt{2\pi}\frac{\kappa}{\nu z_0^2}\exp\left\{\frac{1}{2}w^2(z_0)\right\} > 0.$$

Thus p and $\frac{d}{dx}M$ are functions with positive values on $\left(\frac{\kappa}{\nu\beta}; \frac{\kappa}{\nu(\beta-1)^+}\right)$, i.e. M is strictly increasing. As a strictly increasing continuous function there exists exactly one zero x_0 of M. For $\beta \le 1$ we have $p(x) \to \infty$ for $x \to \frac{\kappa}{\nu\beta}$, but $p(x) \to \kappa - \frac{\nu}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\mathcal{N}^{-1}(\beta)\right)^2\right\}$ for $x \to \frac{\kappa}{\nu(\beta-1)^+}(= \infty)$. If $\lim_{x\to\infty} p(x) \ge 0$ we obtain by a similar argument as before that p and $\frac{d}{dx}M$ are strictly positive, and thus the existence of exactly one zero x_0 of M is proved. However, if $\lim_{x\to\infty} p(x) < 0$, then there exists at least one zero $v_0 \in \left(\frac{\kappa}{\nu\beta}; \frac{\kappa}{\nu(\beta-1)^+}\right)$ of p. Moreover v_0 must be the single zero of p, for otherwise we would find a local minimum z_0 of p with $p(z_0) \le 0$, which is impossible by eq. (18). Since v_0 is a single zero with change of sign of p, v_0 is the single local maximum of M and thus its global maximum with $M(v_0) > 1 - \nu$. Then M is strictly increasing on $\left(\frac{\kappa}{\nu\beta}; v_0\right)$ and strictly decreasing and positive on $\left(v_0; \frac{\kappa}{\nu(\beta-1)^+}\right)$. Thus there exists exactly one zero x_0 of M, and the relation $x_0 \in \left(\frac{\kappa}{\nu\beta}; v_0\right)$ holds.

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