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Local Expected Shortfall-Hedging in Discrete Time ⋆

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Abstract. This paper proposes a self-financing trading strategy that minimizes the expected shortfall *locally* when hedging a European contingent claim. A positive shortfall occurs if the hedger is not willing to follow a perfect hedging or a superhedging strategy. In contrast to the classical variance criterion, the *expected shortfall* criterion depends only on undesirable outcomes where the terminal value of the written option exceeds the terminal value of the hedge portfolio. Searching a strategy which minimizes the expected shortfall is equivalent to the iterative solution of linear programs whose number increases exponentially with respect to the number of trading dates. Therefore, we partition this complex overall problem into several one-period problems and minimize the expected shortfall only locally, i.e., only over the next trading period. This approximation is quite accurate and the number of linear programs to be solved increases only linearly with respect to the number of trading dates.

Key words: hedging, self-financing strategies, superhedging, myopic hedging strategies, expected shortfall, coherent risk measures.

JEL Classifications: C61, G10, G12, G13, D81

1. Introduction

In recent years, there was an unprecedented surge in the usage of risk management tools based on the *Value-at-Risk* (VaR). Loosely speaking, VaR can be interpreted as the worst loss over a given time interval under “normal market conditions”. Obviously the shortcomings of the VaR measure stem from its focus on the probability of loss regardless of its magnitude. In contrast to the VaR, the *expected shortfall* takes into account the size of the shortfall and not only the probability of its occurrence. Artzner et al. (1999) state four axioms which should be fulfilled by

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reasonable risk measures. These axioms rule out VaR-based as well as variance-based risk measures. Although the expected shortfall fulfils only three of the four axioms given by Artzner et al. (1999) it is a reasonable risk measure when hedging contingent claims.

Traditional hedging concepts for incomplete markets either minimize the variance of hedging costs or *superhedge* the written contingent claim. The superhedging approach was introduced by Bensaid et al. (1992) in discrete time and El Karoui and Quenez (1995) in continuous time. Recently, Cvitanić and Karatzas (1999) and Föllmer and Leukert (2000) pioneered the *expected shortfall-hedging* approach. While Cvitanić and Karatzas (1999) deals only with complete markets in continuous time, Cvitanić (1998) and Föllmer and Leukert (2000) also examine incomplete markets. But they only show that an optimal solution exists. An explicit solution is not provided.

The purpose of this paper is twofold: First, based on superhedging strategies, we show that searching a strategy which minimizes the expected shortfall is equivalent to the iterative solution of linear programs. Second, after finding that the latter algorithm is very time-consuming, we propose a strategy which minimizes the expected shortfall *locally*. This approximation is quite accurate and its computational time drops significantly compared to the original expected shortfall strategy. We examine the quality of the local expected shortfall strategies and their effects on the total hedging costs.

The paper is organized as follows. Section 2 provides the model framework, recalls the concept of superhedging, and motivates why the expected shortfall is a reasonable risk measure. Section 3 presents a two-step procedure for solving the overall problem of expected shortfall-hedging. Furthermore, this section proposes a numerically efficient algorithm for calculating the corresponding strategies. Section 4 presents the key contribution of the paper, namely the concept of local expected shortfall-hedging and some special cases when it coincides with the corresponding global strategy. A numerical example in Section 5 confirms the quality of this approximation and their influence on the total hedging costs. Section 6 concludes the paper.

2. Model Framework and Reasonable Risk Measures

The model used in this paper is essentially the same as that in Harrison and Pliska (1981). We consider a frictionless market with one *stock* and one riskless *money market account*. For a specified time horizon τ, we assume that securities are traded at $n+1$ *trading dates* $s = 0, \tau/n, 2\tau/n, \dots, \tau$. For simplicity, we use the conventions $t \equiv n \cdot s/\tau$ and $T \equiv n$. The uncertainty about the development of the stock price is described by a fixed probability space $(\Omega, \mathcal{F}, P)$ with $|\Omega| = N < \infty$ and $P(\omega) > 0$ for all $\omega \in \Omega$. The probability measure P reflects the individual viewpoint of an investor concerning the market. The stock price movement is modelled by a stochastic process $S = (S_0, S_1, \dots, S_T)$ with a constant S_0. The money market account price process $B = (B_0, B_1, \ldots, B_T)$ is defined through the *riskless interest rate* r by $B_t = (1 + r)^t$. The stock price process S induces the natural filtration $(\mathcal{F}_t, t = 0, 1, \ldots, T)$ which means that $\mathcal{F}_t$ is the σ-algebra generated by $S_0, \ldots, S_t$. Without loss of generality we assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F} = 2^\Omega$, i.e., $\mathcal{F}_T$ is the σ-algebra of all subsets of Ω. The filtration has the well-known interpretation of the information that becomes available when securities are traded, i.e., $\mathcal{F}_t$ is the set of information which is known up to time t. Furthermore, each $\mathcal{F}_t$ corresponds to a unique partition $\mathcal{P}_t$ of Ω and at time t the investors know which cell of this partition contains the true state of the world. We denote the cells which form the partition $\mathcal{P}_t$ by $A(i, t)$, $i = 1, \ldots, N(t)$ where $N(t) \equiv |\mathcal{P}_t|$. From the fact that the filtration is induced by the stock price process we know that, if we choose an arbitrary trading date $t = 0, 1, \ldots, T$, then the set $\{S_t(\omega) \mid \omega \in A(i, t)\}$ is a singleton for all $i = 1, \ldots, N(t)$. Therefore, we define

$$S_t(i) \equiv S_t(\omega) \text{ with } \omega \in A(i, t). \tag{1}$$

Moreover, each cell $A(i, t + 1) \in \mathcal{P}_{t+1}$ belongs to exactly one cell $A(i, t) \in \mathcal{P}_t$. Therefore, we define for $t = 0, 1, \ldots, T - 1$ and $i = 1, \ldots, N(t)$

$$\mathrm{Succ}(A(i, t)) \equiv \{A(j, t + 1) \mid A(j, t + 1) \subseteq A(i, t)\}.$$

For simplicity, we assume in the following that $|\mathrm{Succ}(A(\cdot, \cdot))|$ is constant (>1) and call $M \equiv |\mathrm{Succ}(A(\cdot, \cdot))|$ the *level of uncertainty* or the constant *splitting-index*. In terms of possible stock prices this means that each possible stock price at time $t = 0, \ldots, T - 1$ is followed by M possible stock prices at $t + 1$ and therefore $N(t) = M^t$ for all $t = 0, 1, \ldots, T$. Typical examples for such stock price models are the binomial model ($M = 2$) and the trinomial model ($M = 3$).

2.1. HEDGING STRATEGIES AND HEDGING COSTS

A trading (or hedging) strategy is a predictable two-dimensional stochastic process. We denote a hedging strategy by $H = \{H_t = (h_t, h_t^0), t = 1, \ldots, T\}$ where h_t (h_t^0) represents the quantity of stocks (money market accounts) held in the hedging portfolio at time t. Predictable means that the components h_t, h_t^0 of a hedging strategy are $\mathcal{F}_{t-1}$ measurable, i.e., the hedger selects his strategy (h_t, h_t^0) after the prices S_{t-1} are observed and holds it until after the announcement of the prices S_t. Because the components h_t and h_t^0 are $\mathcal{F}_{t-1}$-measurable, they must coincide in every cell $A(i, t-1)$ of $\mathcal{P}_{t-1}$, i.e., for $t \in \{1, \ldots, T\}$, $i \in \{1, \ldots, N(t-1)\}$ and ω_1, $\omega_2 \in A(i, t - 1)$ we have $h_t(\omega_1) = h_t(\omega_2)$ as well as $h_t^0(\omega_1) = h_t^0(\omega_2)$. Therefore we define in analogy to (1)

$$h_t(i) \equiv h_t(\omega) \text{ and } h_t^0(i) \equiv h_t^0(\omega) \text{ with } \omega \in A(i, t - 1).$$

The *value process* of a hedging portfolio corresponding to a hedging strategy H is

$$V_t(H) = h_t \cdot S_t + h_t^0 \cdot B_t \quad t = 1, \ldots, T \tag{2}$$

and $V_0(H) = h_1 \cdot S_0 + h_1^0 \cdot B_0$. The *gains process* of a hedging strategy H is

$$G_t(H) = \sum_{i=1}^{t} h_i \cdot \Delta S_i + h_i^0 \cdot \Delta B_i \quad t = 1, \ldots, T$$

where $\Delta S_t \equiv S_t - S_{t-1}$ and $\Delta B_t \equiv B_t - B_{t-1}$ are the price changes of the assets. The gains process has the interpretation of the cumulative gains or losses that result from following the trading strategy H up to time t. We call a strategy *self-financing* if the value of the according hedging portfolio only changes due to gains or losses from trading, i.e., there are no additional cash-flows to or from the hedging portfolio except the initial hedging capital $V_0(H)$ in $t = 0$, i.e,

$$h_t \cdot S_t + h_t^0 \cdot B_t = h_{t+1} \cdot S_t + h_{t+1}^0 \cdot B_t \quad t = 1, \ldots, T-1.$$

Hence, the value of a self-financing strategy H is

$$V_t(H) = V_0(H) + G_t(H) \quad t = 1, \ldots, T.$$

For the remainder of the paper we restrict our attention to self-financing strategies and denote the set of all self-financing strategies by $\mathcal{S}$.

We assume a situation where an investor has written a *European contingent claim* F_T with maturity date T and wants to hedge himself against the occurring risk. A European contingent claim is a (bounded) random variable on $(\Omega, \mathcal{F}, P)$ whose values are not restricted to be nonnegative. If a contingent claim depends only on the terminal stock price S_T, then we call it a *path-independent* contingent claim, otherwise we call it a *path-dependent* contingent claim. Moreover, a contingent claim is called *attainable*, if it can be replicated, i.e., if there is a self- financing strategy H with $V_T(H) = F_T$. A financial market is called a *complete market* if every contingent claim is attainable.[1] Hence, for no-arbitrage reasons the time $t = 0$ value F_0 of the claim must be $V_0(H)$. Thus, in complete markets there is always a strategy such that the *total hedging costs* $C_0 = V_0(H) + (F_T - V_T(H)) \cdot B_T^{-1}$ are constant. Because we only consider self-financing strategies, the total hedging costs consist of only two terms where $V_0(H)$ represents the *initial costs* or the *initial hedging capital* and the second term $(F_T - V_T(H)) \cdot B_T^{-1}$ represents the discounted costs accruing at maturity of the contingent claim (*terminal costs*). Particularly, even if one is not willing or able to invest the initial hedging costs required by the replicating strategy, it is still possible to construct a strategy which leads to constant total hedging costs.

A basic result in discrete market models is that for a stochastic stock price process absence of profitable arbitrage opportunities is equivalent to the existence of an equivalent martingale measure Q such that the discounted price process $S \cdot B^{-1}$

[1] Harrison and Pliska (1981) show that in our setting completeness of the market is equivalent to $M = 2$, i.e., the binomial model is the only complete market model.

becomes a martingale under the measure Q. A martingale measure Q is equivalent to the physical measure P if $P(A) = 0 \Leftrightarrow Q(A) = 0$ for all $A \in \mathcal{F}_T$. Harrison and Pliska (1983) show that a market is complete if and only if there exists a *unique* equivalent martingale measure. Moreover, in an arbitrage-free market every arbitrage-free price of a contingent claim F_T at time t lies in the set (see, e.g., Pliska (1997, p. 27)) $\{E_Q(F_T/B_{T-t} \mid \mathcal{F}_t) \mid Q \in \mathcal{Q}\}$ where $\mathcal{Q}$ is the set of all equivalent martingale measures. Since $\mathcal{Q}$ is not a closed set we also introduce the closure $\bar{\mathcal{Q}}$ which is the smallest closed set containing $\mathcal{Q}$.

In incomplete markets it is not possible to replicate every contingent claim. Nevertheless, given a contingent claim one can avoid any *shortfall risk* by following a so-called *superhedging strategy*, i.e., a strategy H with $V_T(H) \geq F_T$. To the best of our knowledge Bensaid et al. (1992) were the first using the insights of the following proposition when they extended the results of Boyle and Vorst (1991) on optimal hedging strategies with trading frictions.

PROPOSITION 1 (Superhedging). For every contingent claim F_T there exists a superhedging strategy $H^{SH} = \arg\min_{\{H \in \mathcal{S} \mid V_T(H) \geq F_T\}} V_0(H)$ with initial hedging capital $V_0(H^{SH}) = \sup_{Q \in \mathcal{Q}} E_Q(F_T/B_T)$.

The last equality says that the initial price of the superhedging strategy equals the lowest upper price bound for the original contingent claim F_T. When proving the proposition by using linear programming[2] one can also show that the strategy H^{SH} can be determined recursively. The proof uses the (strong) duality relation

$$\min_{\{H \in \mathcal{S} \mid V_T(H) \geq F_T\}} V_0(H) = \max_{Q \in \bar{\mathcal{Q}}} E_Q(F_T/B_T). \tag{3}$$

The recursive procedure starts with defining $Z_T \equiv F_T$ and proceeds with solving the following one-period linear programs for all $t \in \{1, \ldots, T-1\}$ and $j \in \{1, \ldots, N(t-1)\}$:

$$h_t(j) \cdot S_{t-1}(j) + h_t^0(j) \cdot B_{t-1} \longrightarrow \min \quad (\equiv Z_{t-1}(j)). \tag{4}$$

under the constraints

$$h_t(j) \cdot S_t(i) + h_t^0(j) \cdot B_t \geq Z_t(i) \tag{5}$$

for all i such that $A(i,t) \in \text{Succ}(A(j,t-1))$. We call the optimal objective values $Z_{t-1}(j)$ of these one-period problems the *superhedging values* of F_T.

[2] A detailed proof of the proposition can be received on request from the authors.

2.2. EXPECTED SHORTFALL AS A QUASI-COHERENT RISK MEASURE

In contrast to *local risk management* where the risk of only one individual position is considered, the goal of *global risk management* is to control the total risk of a firm or a division and therefore also combinations of risky positions must be considered. Artzner et al. (1999) require four axioms to hold by a reasonable risk measure for local as well as global risk management. They consider a very simple one-period model in a discrete probability space.[3] In multiperiod models an investor must additionally pay attention to market values and risk at each monitoring time. However, for simplicity we measure and monitor risk at only one future date namely the maturity date of the option. Because of our restriction to self-financing strategies this time point is also the only future date where a cash-flow occurs. To be consistent with the notation of Artzner et al. (1999) we denote a risky position by an integrable random variable on $(\Omega, \mathcal{F}, P)$ and a risk measure by ρ. In our case a risky position might be for example the negative total hedging costs or just the negative terminal costs when the initial hedging capital is fixed. With this notation we define the *discounted expected shortfall* (ESD) of a risky position X through

$$\rho(X) = ESD(X) \equiv E_P(\max(-X/B_T; 0)) \equiv E_P(X^-/B_T). \tag{6}$$

We now recall the four axioms for risk measures given by Artzner et al. (1999). For all risky positions X, Y and real numbers α the following relations hold:

AXIOM T (Translation invariance). $\rho(X + \alpha \cdot B_T) = \rho(X) - \alpha$.

AXIOM S (Subadditivity). $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

AXIOM PH (Positive homogeneity). $\rho(\alpha \cdot X) = \alpha \cdot \rho(X)$ when $\alpha \geq 0$.

AXIOM M (Monotonicity). $\rho(Y) \leq \rho(X)$ when $X \leq Y$.

These axioms are not only reasonable for the context of hedging derivatives but are much more general and can therefore be used by insurance companies, banks, regulators, clearing firms, etc. Nevertheless, Artzner et al. (1999) use these axioms only as auxiliary tools, because their fundamental objects are so-called *acceptance sets*. These sets contain all risky positions which are accepted by the management board, supervisor, etc. Artzner et al. (1999) state four axioms for acceptance sets which they consider as more important than the (auxiliary) axioms for risk measures. These axioms are[4]:

AXIOM A1. The acceptance set $\mathcal{A}$ contains the set $\{X \mid X(\omega) \geq 0 \forall \omega \in \Omega\}$.

[3] Artzner et al. (2002) recently extended their work to a multiperiod setting.

[4] For a detailed discussion of these axioms, see Artzner et al. (1999, pp. 206–208).

AXIOM A2. The acceptance set $\mathcal{A}$ does not intersect the set $\{X \mid X(\omega) < 0 \forall \omega \in \Omega\}$.

AXIOM A3. The acceptance set $\mathcal{A}$ is convex.

AXIOM A4. The acceptance set is a positively homogeneous cone.

The convexity of an acceptance set is very important, particularly for a global risk management, because it ensures that the combination of two acceptable risky positions is again acceptable. To get a better interpretation of the three other axioms, Artzner et al. (1999) present a direct relationship between the axioms for acceptance sets and the axioms of risk measures. For that reason they define for each risk measure ρ a corresponding acceptance set through

$$\mathcal{A}(\rho) \equiv \{X \mid \rho(X) \leq 0\}. \tag{7}$$

The key result of Artzner et al. (1999) is that if a risk measure fulfils Axioms T, S, PH, M, then the corresponding acceptance set fulfils the Axioms A1–A4. Therefore they call a risk measure fulfilling Axioms T, S, PH, M a *coherent* risk measure. Using convention (7) the interpretation of the Axioms T, S, PH, M is much more easier.[5] Axiom T states that adding an amount of α to the original position and investing it into the riskless money market account reduces the risk by α. This particularly ensures the relation $\rho(X + \rho(X) \cdot B_T) = 0$ which means that adding the risk $\rho(X)$ to an unacceptable position X makes the position acceptable. Therefore $\rho(X)$ can be interpreted as an *extra capital* or *risk capital*. Axiom S reflects risk aversion, because the extra capital of two individual positions is larger than that of the combined position. Hence, it is possible to decentralize risk management. Axiom PH states that risk does not depend directly on position size and at last Axiom M ensures the reasonable property that a position Y which is in every possible outcome better than a position X should have a lower risk.

There are undesirable consequences for global risk management if especially Axiom S or Axiom M is violated.[6] Artzner et al. (1999) mention that quantile-based risk measures like the Value-at-Risk fail to satisfy Axiom S and variance-based risk measures fail to satisfy Axiom M. They also show that a risk measure ρ is coherent if and only if there exists a family of probability measures (*generalised scenarios*) $\mathcal{P}$ such that $\rho(X) = \sup\{-E_P[X/B_T] \mid P \in \mathcal{P}\}$.[7] However, to determine such a suitable family of probability measures seems to be no easy task and the according risk measure is not very tractable if we cannot find an easier analytical expression for it. Therefore, we now turn to the expected shortfall which is a well-known classical risk-measure. It is easy to verify that the expected shortfall satisfies Axioms S, PH, M but fails to satisfy Axiom T. Nevertheless, the

[5] A more detailed interpretation can be found in Artzner et al. (1999, pp. 208–210).

[6] See Artzner et al. (1999, pp. 209, 216–218) for examples concerning this point.

[7] Artzner et al. (1999), Proposition 4.1.

acceptance set according to the expected shortfall fulfils Axioms A1–A4, because it equals $\{X \mid X(\omega) \geq 0 \forall \omega \in \Omega\}$.[8] This is the main reason for using the expected shortfall criterion. Another reason is that the expected shortfall criterion fulfils Axioms S, PH, and M which have an intuitive interpretation. While this criterion violates Axiom T, i.e., $\rho(X + \alpha \cdot B_T) = \rho(X) - \alpha$, it satisfies the inequality $-B_T^{-1} \cdot E_P(X + \alpha \cdot B_T) \leq ESD(X) - \alpha$. This leads to $E_P(X + ESD(X) \cdot B_T) \geq 0$ such that adding the amount $ESD(X)$ to a risky position and investing it into the riskless money market account leads to a total position which has a positive mean. In contrast, a coherent risk measure would lead to an acceptable position, i.e., a total position with non-positive risk. However, the risk measure $\rho(X) = ESD(X)$ fulfils a slightly modified version of Axiom T:

AXIOM T′. For all risky positions X and all real numbers α we have the inequality $-B_T^{-1} \cdot E_P(X + \alpha \cdot B_T) \leq \rho(X) - \alpha$.

The following example motivates why this modification of Axiom T might be acceptable in the context of global risk management of, e.g., a bank: Suppose a trading room with one supervisor and a lot of (independent) trading desks. Every trading desk reports the expected shortfall to the supervisor. The supervisor now calculates his extra capital by summing all this values and holds back this total sum for potential losses. Because of the assumed large number of trading desks for which the relation $E_P(X + ESD(X)) \geq 0$ holds, the law of large numbers ensures that the total position of the trading room will be essentially positive. Hence, the extra capital hold by the supervisor will essentially compensate the potential losses and therefore there is almost no remaining risk.

In sum, we use the expected shortfall criterion for three main reasons. First, the corresponding acceptance set fulfils Axioms A1–A4. Second, the expected shortfall fulfils Axioms S, PH, M which have a self-contained interpretation independent of acceptance sets. Finally, the expected shortfall fulfils a modification of Axiom T, i.e., Axiom T′, which still allows to implement a reasonable global risk management.

3. Expected Shortfall-Hedging

Hedging the expected shortfall dynamically was first proposed by Cvitanić and Karatzas (1999) and Föllmer and Leukert (2000). Both articles mainly deal with complete markets in continuous time where they find explicit solutions. In incomplete markets this is much more difficult. However, in discrete markets explicit solutions exist even in incomplete markets *and* can be calculated by solving linear programs.

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[8] More generally, for all risk measures $\rho(X) = -E_P(u(X^-))$ with a function $u : \mathbb{R}_+ \to \mathbb{R}_+$, the set $\mathcal{A}(\rho)$ equals $\{X \mid X(\omega) \geq 0 \forall \omega \in \Omega\}$ and fulfils the Axioms A1–A4.

3.1. THE PROBLEM

Hedging the short position in the contingent claim with an arbitrary hedging strategy H leads to a final wealth of $V_T(H) - F_T - V_0(H) \cdot B_T$. If we fix the initial hedging capital $V_0(H)$ to an arbitrary positive constant $\bar{V}_0$, then for risk purposes it suffices to consider only the risky position $V_T(H) - F_T$. The expected shortfall (ES) of this position is $E_P(V_T(H) - F_T)^- = E_P(F_T - V_T(H))^+$. Denoting the set of all self-financing strategies starting with $V_0(H) = \bar{V}_0$ and $F_T - V_T(H) \leq b$, $b \in I\!R_+ \cup \{\infty\}$, by $\mathcal{S}_0^b$ we can now formulate the main problem of this paper:[9]

PROBLEM 1(Expected Shortfall-Hedging). Find a self-financing strategy H^{ES} with $V_0(H^{ES}) = \bar{V}_0$ which minimizes the expected shortfall of the hedging strategy, i.e., find the ES-strategy $H^{ES} = \arg\min_{H \in \mathcal{S}_0^b} E_P[(F_T - V_T(H)]^+$.

Note that the expected shortfall is zero for replicating and superhedging strategies. Furthermore, the probability measure P now appears explicitly in the formula whereas in the case of superhedging we could use every probability measure which is equivalent to P.

3.2. THE TWO-STEP PROCEDURE

Solving Problem 1 is tantamount to find a *dynamic* self-financing trading strategy which solves a *static* optimization problem. The optimization problem is static in the sense that we want to minimize risk which is measured at only one point in time. Therefore it seems natural to decompose Problem 1 in a static and a dynamic part. This two-step procedure was already used in our hedging context by Cvitanić and Karatzas (1999), Pham (1999) and Föllmer and Leukert (2000) and is very similar to the martingale approach of portfolio optimization in continuous time (see, e.g., Pliska (1986), Cox and Huang (1989), Karatzas et al. (1991) and Korn (1997, p. 60)). The following proposition shows that in the objective function of Problem 1 it is possible to replace the terminal value of the (dynamic) trading strategy by an appropriate (static) random variable. However, before we can formulate the proposition, we have to define a feasible set for these random variables.

DEFINITION 1. Let $\mathcal{X}_\infty \equiv \{X \mid X \leq F_T$ and $E_Q(X/B_T) \leq \bar{V}_0$ for all $Q \in \bar{\mathcal{Q}}\}$ denote the set of all *modified contingent claims* for which the price of their superhedging strategy is lower or equal than the initial hedging capital $\bar{V}_0$. Furthermore, for all $b \in I\!R_+$ we denote by $\mathcal{X}_b \equiv \{X F_T - b \leq X \leq F_T$ and $E_Q(X/B_T) \leq \bar{V}_0$ for all $Q \in \bar{\mathcal{Q}}\}$ the set of all modified contingent claims for which b is an upper bound for the shortfall $(V_T(H) - F_T)^- = (F_T - V_T(H))^+$ when H is the superhedging strategy of X.

[9] For simplicity we use the relation $X \leq \infty$ to express that X is an unbounded random variable.

PROPOSITION 2. Let $X^* \in \mathcal{X}_b$ denote a modified contingent claim which is optimal in the sense that $X^* = \arg\min_{X \in \mathcal{X}_b} E_P(F_T - X)$. Then the ES-strategy H^{ES} is the superhedging strategy for the claim X^*:

$$E_P(F_T - V_T(H^{\mathrm{ES}})) = \min_{H \in \mathcal{S}_0^b} E_P(F_T - V_T(H))^+ = E_P(F_T - X^*).$$

Notice that choosing the constant b appropriately is of some importance. If b is too small then the set $\mathcal{X}_b$ might be empty. To avoid such cases we assume in the following that the constant b is chosen such that $b \geq \sup_{Q \in \bar{\mathcal{Q}}} E_Q(F_T) - \bar{V}_0 \cdot B_T$.

Now, Proposition 2 (proved in the appendix) justifies the following *two-step procedure* for solving Problem 1.

STEP 1 (Static optimization problem). Find an optimal modified contingent claim $X^* \in \mathcal{X}_b$ with $X^* = \arg\min_{X \in \mathcal{X}_b} E_P(F_T - X)$.

STEP 2 (Representation problem). Determine a superhedging strategy of X^*.

We first show how to solve step 1 in a complete markets setting to get a better understanding of the problem and its solution. Then we turn to the more complicated case of incomplete markets.

3.3. SOLUTION IN COMPLETE MARKETS

In complete markets it is always possible to implement strategies leading to constant hedging costs. Even if an investor has insufficient hedging capital, by borrowing money, he can follow a hedging strategy which results in constant total hedging costs. Hence, if the sole objective of an investor is to reduce risk, then the replicating strategy which can always be found in a complete markets setting is optimal. Nonetheless, an investor may also consider return or a risk-return relationship as an objective. In this case ES-hedging could be one approach for the investor to find his optimal strategy. As formulated in this paper, ES-hedging gives the investor two degrees of freedom in the sense that he can vary the initial hedging capital as well as the shortfall bound b and then choose the strategy which fits his attitude to risk-return best. Moreover, the way how to solve the problem in a complete markets setting helps to understand the solution in incomplete markets. For example, we show that in complete markets the "*efficient frontier*", i.e., the optimal expected shortfall as a function of the initial hedging capital $\bar{V}_0$, is piecewise linear. Afterwards we see in a numerical example in Section 5 that in incomplete markets the efficient frontier is "almost piecewise linear". Finally, one can use the results of complete markets as building blocks for the solution in incomplete markets (see, e.g., Föllmer and Leukert (1999)).

In complete markets the solution of step 1 is a direct consequence of a slight modification of the fundamental Neyman–Pearson lemma. Particularly, it is possible to determine the optimal modified contingent claim analytically. Föllmer and Leukert (1999, 2000) were the first who used this fundamental statistical lemma in a financial context when solving – in a continuous-time, complete markets setting – the same problem.

PROPOSITION 3. (ES-Hedging with a shortfall bound). In a complete market the modified contingent claim

$$X^*(\omega) = F_T(\omega)1_{\{\frac{P}{Q}(\omega)>c_{\mathrm{ES}}\}} + \gamma 1_{\{\frac{P}{Q}(\omega)=c_{\mathrm{ES}}\}} + (F_T(\omega) - b)1_{\{\frac{P}{Q}(\omega)<c_{\mathrm{ES}}\}} \tag{8}$$

with $c_{\mathrm{ES}} = \arg\min_{c\in\mathbb{R}_+}\{E_Q(F_T 1_{\{P/Q(\omega)>c\}} + (F_T - b)1_{\{P/Q(\omega)\le c\}} \le \bar{V}_0 B_T\}$ and $\gamma = (\bar{V}_0 \cdot B_T - E_Q(F_T 1_{\{P/Q(\omega)>c_{\mathrm{ES}}\}}F_T) - E_Q((F_T - b)1_{\{P/Q(\omega)<c_{\mathrm{ES}}\}}))/(E_Q(1_{\{P/Q(\omega)=c_{\mathrm{ES}}\}}))$ solves the static optimization problem (Step 1) if there is a shortfall bound $b < \infty$. Replicating X^* with a strategy H^{ES} solves the Problem 1.

Proof. See appendix.

PROPOSITION 4 (ES-Hedging without a shortfall bound). In a complete market the modified contingent claim

$$X^*(\omega) = F_T(\omega)1_{\{\frac{P}{Q}(\omega)>c_{\mathrm{ES}}\}} + \gamma 1_{\{\frac{P}{Q}(\omega)=c_{\mathrm{ES}}\}} \tag{9}$$

with $c_{\mathrm{ES}} = \min_{\omega\in\Omega}\{P(\omega)/Q(\omega)\}$ and

$$\gamma = (\bar{V}_0 \cdot B_T - E_Q(1_{\{P/Q>c_{\mathrm{ES}}\}}F_T))/(E_Q(1_{\{P/Q=c_{\mathrm{ES}}\}}))$$

solves the static optimization problem (Step 1) if there is no shortfall bound ($b = \infty$). Replicating X^* with a strategy H^{ES} solves the Problem 1.

The proof is very similar to the one of Proposition 3 given in the appendix. Obviously, the inverse P/Q of the state price density plays a fundamental role in both propositions. If we consider the special case where $b = \infty$ and state price densities are different for different outcomes, then the optimal hedging strategy can be interpreted as follows. Replicate a modified contingent claim which is equal to the original contingent claim except in one state ω^*. This state is used to finance the replication of the other states because w.l.o.g. we can assume that the initial hedging capital is not high enough to replicate the original contingent claim in the whole. Alternatively we can say that the optimal strategy is to sell ($\gamma < 0$) or buy ($\gamma > 0$) exactly γ Arrow–Debreu securities of state ω^* such that there is enough money to buy $F_T(\omega)$ Arrow-Debreu securities of all other states $\omega \in \Omega\backslash\{\omega^*\}$. Therefore, a shortfall occurs only in one state ω^* and the shortfall of the ES-strategy is $F_T(\omega^*) - V_T(H(\omega^*))$. According to Proposition 3 the expected shortfall of the optimal strategy is $(F_T(\omega^*) - \gamma) \cdot P(\omega^*)$ if $\gamma < F_T(\omega^*)$ and zero otherwise. Therefore the efficient frontier is given through $\bar{V}_0 \to (E_Q(F_T) - \bar{V}_0 B_T)^+ \cdot P(\omega^*)/Q(\omega^*)$.

The critical state ω^* can be interpreted as the state for which the inverse of the state price density, i.e., the ratio between the expected payoff from an Arrow–Debreu security $(P(\omega^*))$ and the cost of that security $(Q(\omega^*))$, is worst. In a binomial model (denoting the probability for an up-tick by $P(u)$, resp. $Q(u)$) this is either the state leading to the highest stock price, namely if $P(u)/Q(u) > (1 - P(u))/(1 - Q(u))$, or the state leading to the lowest stock price, otherwise.

Example: Problem 1 in the complete case is illustrated for a two-period binomial model with $S_0 = \$50$ and $S_{t+1} = S_t \cdot Y$ for $t = 0, 1$. The random variable Y has two possible outcomes, namely $Y = 1.1$ (up-tick) with (P-)probability 0.80 and $Y = 0.9$ (down-tick) with (P-)probability 0.20. The outcomes are uniquely determined by the up- and down-ticks, e.g., ud represents the state with an up-tick in the first period and a down-tick in the second one. Furthermore, we assume an investor who tries to hedge a long position in a European call option on the stock with strike $K = 45$ and who is endowed with an initial hedging capital of \$4. Thus, when using the interest rate $r = 0\%$, the fair price of the call is $F_0 = E_Q(F_2/B_2) = \$6.125$.

To determine the optimal strategy with respect to problem 1 we first determine the values of the inverse of the state price density for the possible outcomes:

$$\frac{P(uu)}{Q(uu)} = 2.56 \quad \frac{P(ud)}{Q(ud)} = \frac{P(du)}{Q(du)} = 0.64 \quad \frac{P(dd)}{Q(dd)} = 0.16.$$

Thus, in the case $b = \infty$ the optimal strategy with respect to the expected shortfall is to replicate X^* where $X^*(dd) = \gamma = -8.5$ and $X^* = F_2$ for all other possible outcomes. Of course, the fair value for this (fictitious) modified contingent claim is $X_0^* = E_Q(X^*/B_2) = \$4$. Obviously, in the state dd we have to invest an additional amount of \$8.5. If we want to constraint the shortfall we can impose a shortfall bound by using a real value for b. For example, if we choose $b = 4$, then we find $c_{ES} = 0.64$, $X^*(ud) = X^*(du) = 2.25$ and $X^*(dd) = -4$. Hence, there is no state where we have to spend more than \$4 at maturity.

Notice that in multiperiod binomial trees expected shortfall strategies are always path-independent. This results from the fact that replicating strategies are path-independent as long as the contingent claim to be hedged is path-independent and that in binomial models the optimal modified contingent claim is always path-independent because paths which lead to the same final state also lead to the same value of the state price density.

3.4. SOLUTION IN INCOMPLETE MARKETS

Expected shortfall-hedging in incomplete markets is much more complicated than in complete markets. This results from the fact that in incomplete markets the set of equivalent martingale measures is no longer a singleton and even of infinite size. Cvitanić (1998) and Föllmer and Leukert (1999/2000) examine the incomplete case in continuous time models. But they only show that an optimal solution exists. An explicit solution or an algorithm to calculate an explicit solution is not provided. For the discrete model under consideration we are able to devise such an algorithm. Remember that Step 1 of the two-step procedure minimizes $E_P(F_T - X)$ or maximizes $E_P(X)$ under the constraints

$$F_T - b \leq X \leq F_T \text{ and } \max_{Q \in \bar{\mathcal{Q}}} E_Q(X/B_T) \leq \bar{V}_0. \tag{10}$$

Although the set of martingale measures is of infinite size it suffices to consider only a finite number of martingale measures for the optimization. This results from the fact that the set of martingale measures is a convex polyhedron and the constraints (10) form again a linear program.[10]

PROPOSITION 5. For every contingent claim F_T there exist a finite number of martingale measures $Q_1, \ldots, Q_L \in \bar{\mathcal{Q}}$ such that the static optimization problem (Step 1) is equivalent to maximize $E_P(X)$ under the constraints

$$F_T - b \leq X \leq F_T \text{ and } \max_{i=1,\ldots,L} E_{Q_i}(X/B_T) \leq \bar{V}_0 \tag{11}$$

where L denotes the number of extreme points of the convex polyhedron $\bar{\mathcal{Q}}$.

Now we present a procedure which finds the extreme points of the convex polyhedron and hence solves the static optimization problem in incomplete markets.

ALGORITHM FOR CALCULATING EXPECTED SHORTFALL STRATEGIES

(S0) Initialization: Set $i \equiv 1$ and define $Q_1 \equiv \arg\max_{Q \in \bar{\mathcal{Q}}} E_Q(F_T)$.

(S1) Maximize $E_P(X_i)$ under the constraints $F_T - b \leq X_i \leq F_T$ and $\max_{j=1,\ldots,i} E_{Q_j}(X_i/B_T) \leq \bar{V}_0$.

(S2) If $\max_{Q \in \bar{\mathcal{Q}}} E_Q(X_i^*/B_T) \leq \bar{V}_0$ holds for the optimal solution X_i^* of step (S1), then it is optimal in the sense of the static optimization problem and the algorithm terminates.
Otherwise, define $Q_{i+1} \equiv \arg\max_{Q \in \bar{\mathcal{Q}}} E_Q(X_i^*/B_T)$, increase $i = i + 1$ and return to step (S1)

Note that due to Proposition 5, the algorithm stops after a finite number of steps. Moreover, the algorithm solves Step 2 of the two-step procedure concurrently due to the duality relation (3). Furthermore, remember that all probability measures

[10] In a trinominal model this convex polyhedron is described by 3^n variables, 3^n inequalities and $1 + (3^n - 1)/2$ equalities. The variables correspond to the martingale probabilities at each node at time T of the multiperiod tree. The inequalities ensure that all these 3^n values are positive and the $(3^n - 1)/2$ equalities ensure that the measure has the martingale property. Finally, we need one more equality to make the measure a probability measure. In a recombining tree we need only $\sum_{i=0}^{n-1} (i+1) \cdot (i+2)/2$ equalities to describe the martingale property.

and contingent claims can be represented by vectors of length N and therefore steps (S1) and (S2) can be solved by linear programming. Step (S1) is a standard linear programming problem and in step (S2) the lowest upper bound for arbitrage-free prices of a contingent claim is searched. Due to relation (3) the latter problem can be solved by linear programming as well.

The optimal ES-strategy is in general path-dependent, and even worse, the algorithm needs a lot of computational time. This can be seen when having a closer look at the different steps of the algorithm. In step (S0) we determine the superhedging strategy of F_T as described in Section 2.1. Every iteration of step (S1) solves a linear program with a growing number of constraints, and in each iteration step (S2) a superhedging strategy is determined. The linear optimization of step (S1) is very fast at the beginning of the algorithm but becomes slower and slower when the number of constraints increases. Because of the path-dependency of the optimal ES-strategy, computational time increases substantially with respect to the number of periods. In the numerical example in Section 5, we present some explicit results about the computational effort of the above algorithm.

4. Local Expected Shortfall-Hedging

4.1. THE PROBLEM

As mentioned in the foregoing section, solving Problem 1 needs a lot of computational time. To overcome this drawback, we now focus on *myopic* hedging strategies. An investor behaves myopically if his sequence of decisions is obtained as a series of single-period decisions (starting with the first period), where each period is treated as if it were the last one. We call this simplified procedure *Local Expected Shortfall-Hedging* (LES-Hedging).

PROBLEM 2 (LES-Hedging). Let F_T be a European contingent claim and $F_t^{\mathrm{SH}} \equiv Z_t$ the corresponding superhedging values, and let $\mathcal{G}_t = \sigma(H_1^{\mathrm{LES}}, \ldots, H_t^{\mathrm{LES}})$ denote the σ-field generated by the LES-hedging strategy until time t. Then, find sequentially a self-financing strategy $H^{\mathrm{LES}} = (H_1^{\mathrm{LES}}, \ldots, H_T^{\mathrm{LES}})$ with $V_0(H^{\mathrm{LES}}) = \bar{V}_0$ whose components H_t^{LES} minimize the *(local) expected shortfall* $E_P[(F_t^{\mathrm{SH}} - V_t(H))^+ \mid \mathcal{F}_{t-1} \vee \mathcal{G}_{t-1}]$ for $t = 1, \ldots, T$.[11]

For simplicity, we restrict ourselves in the theoretical part of this section to the case $b = \infty$. However, it is no problem to impose a bound $b < \infty$ as in the foregoing section. In later examples we will make use of this possibility.

[11] Recall that according to convention (2) $V_t(H)$ depends only on the hedging strategy until time t.

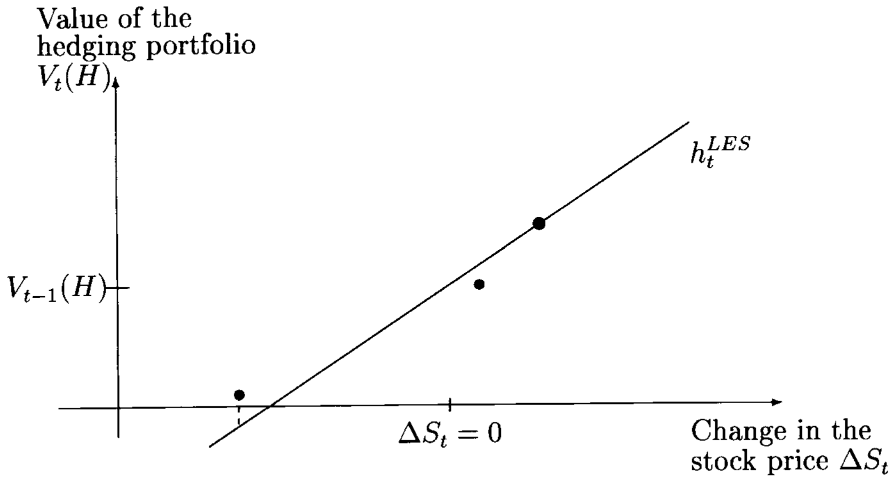

Figure 1. Minimizing the local expected shortfall. This figure illustrates the LES-strategy in a trinomial tree with zero interest rate and for a fixed trading date $t \in \{1, \ldots, T\}$. At the end of the tth period, the values of the hedging portfolio $V_t(H)$ should be as close as possible to the superhedging values F_t^{SH}. With the knowledge of $\mathcal{F}_{t-1}$ there are three possible superhedging values at t, marked by the dots. Because of the zero interest rate, the change in the value of the hedging portfolio is $V_t(H) - V_{t-1}(H) = h_t \cdot \Delta S_t$. Hence, the future values of the hedging portfolio are represented by a line $V_t(H) = V_{t-1}(H) + h_t \cdot \Delta S_t$ whose slope corresponds to the hedge ratio h_t. The latter minimizes the local expected shortfall $E_P((F_t^{\mathrm{SH}} - V_t(H))^+ \mid \mathcal{F}_{t-1})$ if the sum of the probability-weighted *positive* differences between the lines and the dots is minimal. The different weights of the possible states are visualized through the size of the dots.

4.2. SOME PROPERTIES OF THE LES-STRATEGY

LEMMA 1. In the one-period case ($T = 1$), a strategy which minimizes the local expected shortfall also minimizes the expected shortfall, i.e., $H^{\mathrm{ES}} = H^{\mathrm{LES}}$.

Proof. In the one-period case we have $F_1^{\mathrm{SH}} = F_1$ by definition. Hence the two problems coincide. □

The lemma shows that one can solve Problem 2 iteratively by solving Problem 1 in one-period settings. In contrast to the recursive procedure for solving Problem 1 in a multiperiod setting this iterative procedure is much faster.

LEMMA 2. Let H^{SH} be a superhedging strategy of F_T. If $\bar{V}_0 \geq F_0^{\mathrm{SH}} = V_0(H^{\mathrm{SH}})$, then the ES-strategy and the LES-strategy coincide, i.e., $H^{\mathrm{ES}} = H^{\mathrm{LES}}$.

Proof. By definition, a superhedging strategy fulfils $V_T(H^{\mathrm{SH}}) \geq F_T$ and $V_t(H^{\mathrm{SH}}) = F_t^{\mathrm{SH}}$. Moreover $V_t(H^{\mathrm{SH}}) \geq F_t^{\mathrm{SH}}$ holds for all $t = 0, \ldots, T-1$. Because $\bar{V}_0 \geq V_0(H^{\mathrm{SH}})$, we can find a self-financing strategy $H^{\mathrm{SH}*}$ such that

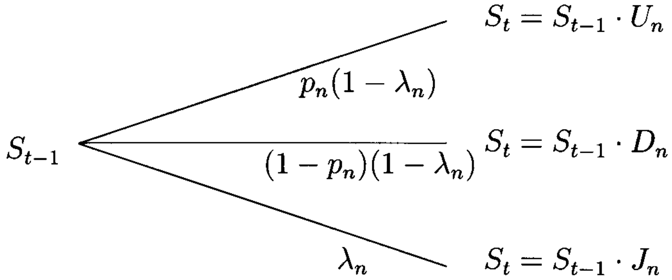

Figure 2. Evolution of the stock price process in a subtree of a trinomial model.

$V_0(H^{\text{SH}*}) = \bar{V}_0$ and $V_T(H^{\text{SH}*}) \geq F_T$ as well as $V_t(H^{\text{SH}*}) \geq F_t^{\text{SH}}$ for all $t = 1, \ldots, T-1$. Therefore, when using $H^{\text{SH}*}$ the expected shortfall as well as the local expected shortfall is zero and hence $H^{\text{SH}*}$ is optimal for Problems 1 and 2. □

PROPOSITION 6. The LES-strategy coincides with the ES-strategy if the market is complete and volatility is constant.

Proof. See appendix.

4.3. TWO-PERIOD EXAMPLES

We consider the simplest version of an incomplete market: a trinomial model. Starting with a time horizon τ, we therefore assume that there are three possible successors for each stock price at each trading date. Hence, there are also three possible returns in each trading period of length τ/n: U_n, D_n and J_n. We interpret these values as returns from "normal" up- (U_n) and down-movements (D_n) and from a "rare" event which we interpret as a jump (J_n) in the stock price process. The probability for this rare event is denoted by λ_n whereas the probability for an up-movement conditional that there is no jump is p_n. The stock price change can therefore be written as $S_t = S_{t-1} \cdot Y$, $t = 1, \ldots, T$ where Y is a random variable which takes the values U_n, D_n and J_n with probability $p_n \cdot (1-\lambda_n)$, $(1-p_n) \cdot (1-\lambda_n)$ and λ_n. Figure 2 visualizes the evolution of the stock price process in one subtree.

If λ_n is zero then we neglect the rare outcome J_n and the trinomial tree reduces to a binomial tree. If furthermore $U_n = \exp\{\alpha\tau/n + \sigma\sqrt{\tau}/\sqrt{n}\}$, $D_n = \exp\{\alpha\tau/n - \sigma\sqrt{\tau}/\sqrt{n}\}$ and $p_n = (e^{\alpha\tau/n} - D_n)/(U_n - D_n)$, then one can show (see, e.g., Duffie (1992, p. 198)) that this binomial tree converges in distribution for $n \to \infty$ to the continuous-time stock price process of the Black and Scholes (1973) model with an annual volatility σ and an expected rate of return α.

Now, if $\lambda_n > 0$ then an additional jump component with a constant jump size is present. Thus, we can interpret the trinomial model under consideration as a discrete version of a jump-diffusion process with constant jump size. Such continuous-time models including jumps are very popular in finance theory and are used – often in a more general form – for example by Merton (1976), Cox and

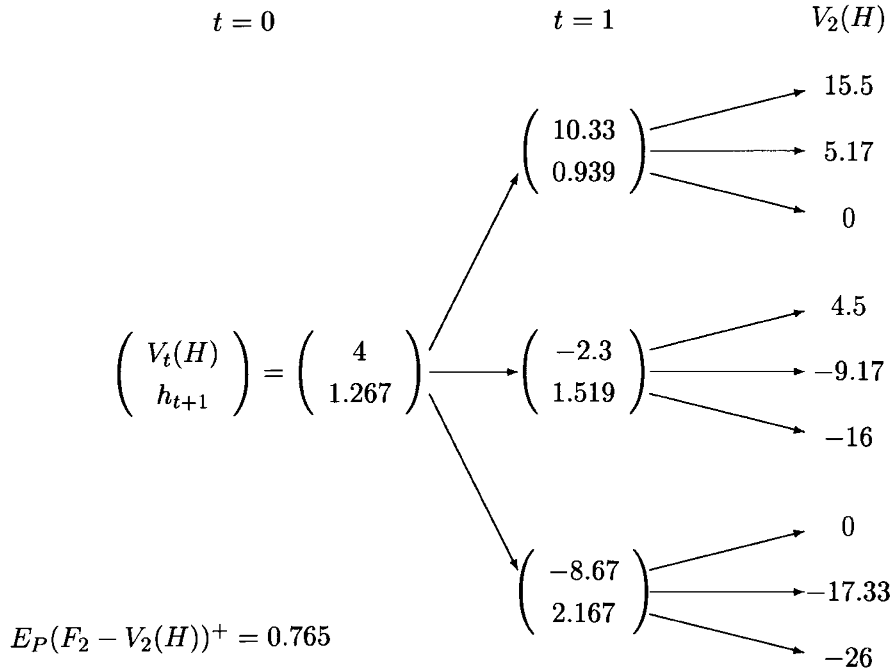

Figure 3. LES-strategy without a shortfall bound ($b = \infty$).

Ross (1976), and Jarrow and Madan (1995). Moreover, Amin (1993) presents a discretization of Merton's (1976) jump-diffusion model where the jump size is log-normally distributed. This would also be an appropriate model to examine here, but for simplicity and especially to be able to calculate the ES-strategy in reasonable time, we use the trinomial model as presented above. Finally notice that the number of jumps is binomially distributed in the interval $[0, \tau]$ and hence the expected number of jumps in this interval is $n \cdot \lambda_n$. Therefore, $\lambda \equiv n \cdot \lambda_n / \tau$ is the expected number of jumps per year.[12]

In the following example we consider essentially the same parameters as in the binomial example of Section 3. For the "jump-state" j we assume the gross rate of return $J = 0.8$. The probability for this state is $p(j) = 0.05$ and for the other states $p(u) = 0.75$ and $p(d) = 0.2$. Figures 3 and 4 illustrate the behaviour of the LES-strategy in a two-period example without and with a shortfall bound, respectively. It is interesting to observe that in the case without a shortfall bound (Figure 3) the hedge ratio h_2 increases if the stock price S_1 decreases. This follows from the fact that in this example the probability of a jump is apparently small enough to compensate the shortfall due to the jump component. More precisely, the

[12] In the following examples we calculate λ_n from a given expected number of jumps per year. Thereby we have to be careful, because this only makes sense for large n or small τ. Hence, when calculating the ES-strategy which needs a lot of computational time already for more than 4 periods, we choose a short time horizon τ.

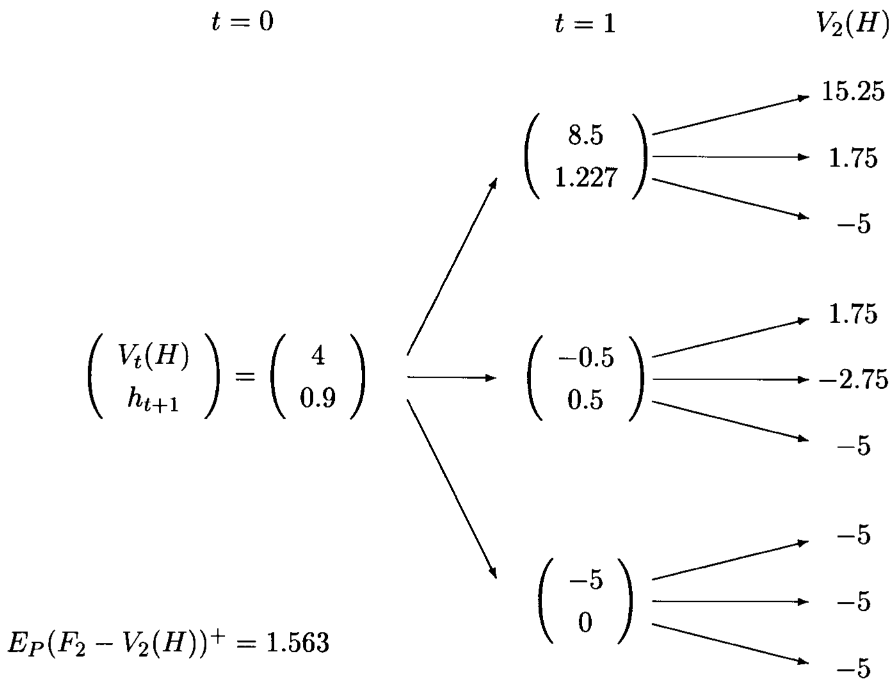

Figure 4. LES-strategy with shortfall bound $b = 5$.

relatively high hedge ratio $h_2(j) = 2.167$ results since the expected shortfall from the jump-states ju, jd and jj is a linear and decreasing function of $h_2(j)$ as long as $h_2(j) \leq 2.167$. This follows from $V_2(H, j\cdot) = -8,67 + h_2(j) \cdot (S_2(j\cdot) - 40)$ leading to the expected shortfall $0.05 \cdot (8.67 + h_2(j) \cdot (-4 \cdot 0.75 + 4 \cdot 0.2 + 8 \cdot 0.05)) = 0.05 \cdot (8.67 - 1.8h_2(j))$ for $h_2(j) \leq 2.167$. Unfortunately, this LES-strategy without a shortfall bound results in a relatively high shortfall especially in state jj (26). Although this happens only with probability 0.0025, it may be to risky for an investor because the amount to add to his portfolio is more than six times his initial hedging capital. Figure 4 represents the case where an investor restricts his shortfall to $b = 5$. In this case we get a completely different strategy with the hedge ratio h_2 decreasing with respect to S_1. The expected shortfall of this strategy is now higher (1.563) as in the case without a shortfall bound (0.765), but the shortfall does now not exceed 5 which may give additional safety to the investor.

5. Numerical Comparison

In this section we use the same model as in the previous section. But now, to be more realistic we use a higher number of periods and therefore another parameter set. We therefore consider a stock with a current price of \$50. The annualized volatility of the "normal" return of the stock, i.e., the return neglecting the jump part is assumed to be $\sigma = 20\%$, the expected rate of the "normal" return is $\alpha = 15\%$ and the riskless interest rate is $r = 5\%$. Moreover, we assume that in the case of a jump the return of the stock is $J_n = \exp\{\alpha\tau/n - 2\sigma\sqrt{\tau}/\sqrt{n}\}$. The contingent claim to be hedged is a European call option with strike price $K = \$47$.

Table 1 presents the expected shortfall resulting from the ES-strategy, the LES-strategy, and the trivial strategy of not hedging at all ($h_t \equiv 0$). The absolute deviation between the expected shortfall of the LES-strategy and the ES-strategy is quite small. Only when the initial hedging capital approaches the capital needed for superhedging there are larger percentage deviations. Moreover, remember that the expected shortfall of both strategies coincide if the expected number of jumps is zero or if the initial hedging capital exceeds the initial hedging capital needed for a superhedging strategy.

Figure 5 demonstrates that the piecewise linear structure as observed in the complete market holds for the ES- as well as for the LES-strategy. We also see that the efficient frontier of the approximative strategy fits the efficient frontier of the ES-strategy very well.

Table 2 presents the number of linear programs to be solved when calculating the ES-strategy and the LES-strategy, respectively. Panel A contains the theoretical values including the number of linear programs needed for the initialization step of the algorithm. The latter one is $\sum_{t=0}^{n-1} 3^t = (3^n - 1)/2$ for the ES-strategy if the original contingent claim is path-dependent and $\sum_{t=0}^{n-1}(t+1)(t+2)/2$ if the original contingent claim is path-independent. For the LES-strategy we have n more linear programs to solve in each case since for the LES-strategy the (one-period) ES-algorithm is initiated n times. The number of iterations which equals the number of modified contingent claims to be determined cannot be specified exactly. But in the special case of a one-period situation where the set of martingale measure has only two extreme points at most two iterations can occur. For the ES-strategy we observed that $3^n \cdot (n-1)$ is a reasonable approximation for the number of iterations. When calculating the total number of linear programs needed to determine the strategies we see that this number increases *exponentially* for the ES-strategy while for the LES-strategy it increases only *linearly*. Panel B contains the values for a concrete example grouped by the number of constraints. Remarkable is the fact that most of the linear programs have two constraints because these are essentially the linear programs whose solutions are needed to solve the superhedging problems recursively.

Table 3 presents some statistical values of the distribution of the total hedging costs for different initial hedging capital. We now use a time horizon $\tau = 0.25$ and a sample of $n = 10$ trading periods which leads to 59,049 different paths. Most of the observations are very close to the initial hedging capital. Moreover, the mean of the total hedging costs decreases when the initial hedging costs decrease. But the standard deviation as well as the 90%, 95% and 99% quantile increases when the initial hedging capital decreases. Particularly, the maximum of the total hedging

Table I. ES-strategy versus the LES-strategy: Magnitude of the expected shortfall

Initial hedging capital $\bar{V}_0$	Number of Periods $n = 2$			$n = 3$			$n = 4$		
	ES	LES	No hedge	ES	LES	No hedge	ES	LES	No hedge
Expected number of jumps = 1 per year									
0	3.2630	3.2630	3.4770	3.1618	3.2369	3.5618	3.0205	3.1589	3.5626
1	2.2976	2.2976	2.7556	2.2363	2.3114	2.7268	2.1282	2.2666	2.7127
2	1.3322	1.3322	2.0500	1.3107	1.3858	1.9199	1.2359	1.3743	2.0498
3	0.3668	0.3668	1.3679	0.3852	0.4603	1.4510	0.3436	0.4821	1.4107
4	0.0000	0.0000	0.8637	0.0000	0.0000	0.9894	0.0000	0.0000	0.9103
Expected number of jumps = 3 per year									
0	2.9360	2.9423	3.0181	3.0381	3.0554	3.1380	3.0787	3.1043	3.1965
1	2.0975	2.1038	2.3419	2.1581	2.1754	2.3770	2.1771	2.2096	2.4099
2	1.2590	1.2653	1.7090	1.2782	1.3121	1.6657	1.2781	1.3367	1.7840
3	0.4473	0.5289	1.1404	0.4042	0.4669	1.2376	0.3840	0.4792	1.2207
4	0.0000	0.0000	0.7200	0.0000	0.0000	0.8293	0.0000	0.0000	0.7833
Expected number of jumps = 5 per year									
0	2.4283	2.4283	2.5890	2.5380	2.5386	2.7441	2.5805	2.5889	2.8533
1	1.7633	1.7633	1.9613	1.8118	1.8123	2.0560	1.8309	1.8445	2.1280
2	1.0983	1.0983	1.3990	1.0860	1.0951	1.4336	1.0883	1.1194	1.5433
3	0.4538	0.4583	0.9335	0.3740	0.3879	1.0461	0.3567	0.4034	1.0498
4	0.0000	0.0000	0.5894	0.0000	0.0000	0.6875	0.0000	0.0000	0.6698

This table illustrates the expected shortfall resulting from the ES-strategy versus the LES-strategy compared with the alternative strategy of not hedging at all. The period numbers are chosen very small such that it is possible to calculate the exact solutions in passable time. Parameter values: initial stock price = \$50; annual interest rate (r) = 5%; annual volatility of the "normal" stock price return (σ) = 20%; annual expected rate of the "normal" return of the stock (α) = 15%; time to maturity of the option (τ) = 1/12; strike price of the option (K) = \$47.

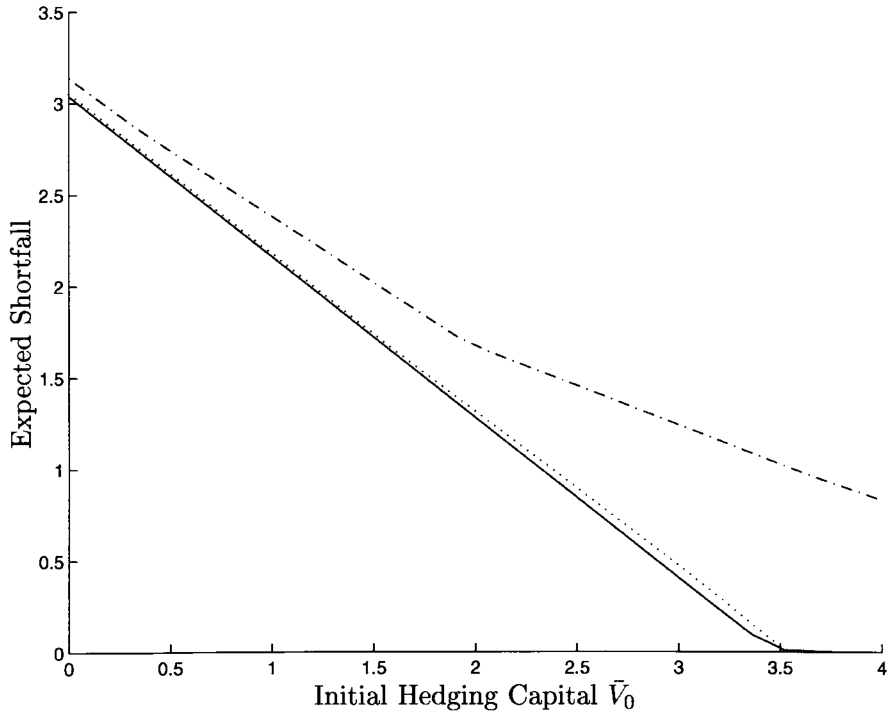

Figure 5. Efficient frontier. This figure presents the efficient frontier, i.e., the optimal expected shortfall values as a function of the initial hedging capital. The efficient frontier of the ES-strategy is represented by the solid line whereas the dotted line visualizes the efficient frontier of the LES-strategy. For comparison the efficient frontier of the trivial hedging strategy is represented by the dashed line.
Parameter values: initial stock price = \$50; annual interest rate (r) = 5%; annual volatility of the "normal" stock price return (σ) = 20%; annual expected rate of the "normal" return of the stock (α) = 15%; time to maturity of the option (τ) = 1/12; strike price of the option (K) = \$47; expected number of jumps (λ) = 3 per year; number of trading periods (n) = 3.

costs is very large for a low initial hedging capital. Of course, these extremely high values only occur with low probability.

In Table 4 it is shown how the distribution of the total hedging costs changes when we impose an upper bound b_c for these costs. Such an upper bound can be easily implemented by using $b = (b_c - \bar{V}_0) \cdot B_T$ as upper bound for the shortfall. We can observe that the mean as well as lower quantiles decrease when the upper bound for the costs increase. In contrast to this, the standard deviation and the upper quantiles increase with increasing b_c. Not surprisingly, the more restrictive the bounds for the total hedging costs are, the more observations are very close to that bound.

Table II. ES-strategy versus LES-strategy: Number of linear programs to be solved

Panel A: Number of LP's to be solved in a n-period trinominal model

	Main loops	Iterations per main loop (= no. of mcc)	No. of LP's per iteration	Total number of LP's (incl. number of LP's to be solved for initilization)
ES-strategy	1	$\sim 3^n \cdot (n-1)$	$1 + \frac{3^n - 1}{2}$	$\sim 3^n \cdot (n-1) \cdot (1 + \frac{3^n-1}{2})$ $+ \sum_{t=0}^{n-1} \binom{t+2}{2}$
		(average value)		(average value)
LES-strategy	n	≤ 2	$1 + \frac{3^1-1}{2} = 2$	$\leq 4n + n + \sum_{t=0}^{n-1} \binom{t+2}{2}$

Panel B: Number of LP's to be solved in the example under consideration

Number of constraints in linear programs	Number of periods							
	$n = 2$		$n = 3$		$n = 4$		$n = 5$	
	ES	LES	ES	LES	ES	LES	ES	LES
1	1	2	1	3	1	4	1	5
2	29	12	443	20	5,881	34	97,406	53
3	1	0	1	0	1	0	1	0
≥ 4	3	0	30	0	143	0	801	0
Total	34	14	475	23	6,026	38	98,209	58

This table presents the number of linear programs which have to be solved when calculating the ES-strategy resp. the LES-strategy for a path-independent option. Panel A presents the theoretical values including the linear programs needed for initialization. Panel B presents the number of linear programs needed to solve in a concrete example where the parameters are chosen as in Figure 6 and the initial hedging capital is $\bar{V}_0 = 2$. Thereby, linear problems with more than 3 constraints are grouped together.

6. Summary and Extensions

This paper presents hedging strategies in an incomplete, discrete financial markets setting. We justify the expected shortfall as a suitable risk measure and we present a two-step procedure to minimize the *expected shortfall* for a given capital constraint by solving linear programs. This two-step procedure goes back to Cvitanić and Karatzas (1999), Pham (1999) and Föllmer and Leukert (1999, 2000) and makes use of the *superhedging* concept. We calculate expected shortfall strategies analytically in complete markets and provide an algorithm for calculating such strategies even in incomplete markets. Moreover, we show that it is possible to impose a flexible ex-ante bound for the shortfall in these strategies. Since the algorithm is very time-consuming, we propose a strategy which minimizes the expected shortfall *locally*. The latter approximation is quite accurate and its computational time drops significantly compared to the original ES-strategy.

Table III. Distribution of the total hedging costs

	Initial hedging capital					
	$\bar{V}_0 = 5$	$\bar{V}_0 = 4$	$\bar{V}_0 = 3$	$\bar{V}_0 = 2$	$\bar{V}_0 = 1$	$\bar{V}_0 = 0$
Mean	4.31	4.07	3.78	3.48	3.18	2.87
Std. dev.	0.40	2.00	3.97	5.96	7.94	9.94
Minimum	3.43	2.43	1.43	0.43	−0.56	−1.56
5% quantile	3.54	2.85	1.90	0.90	−0.10	−1.10
50% quantile	4.34	3.61	2.72	1.73	0.88	−0.12
75% quantile	4.60	3.97	2.97	1.99	1.14	0.31
90% quantile	4.80	4.42	4.75	4.87	4.98	5.10
95% quantile	4.85	5.84	8.44	10.90	13.35	15.82
99% quantile	4.97	13.94	23.63	33.33	43.03	52.73
Maximum	5.00	106.95	208.58	310.22	411.86	513.50

This table presents some statistical values of the distribution of the total hedging costs when the initial hedging capital varies. The initial hedging capital of $5 corresponds to a superhedging strategy.
Parameter values: initial stock price = $50; annual interest rate (r) = 5%; annual volatility of the stock price return (σ) = 20%; annual expected rate of return of the stock (α) = 15%; time to maturity of the option (τ) = 0.25; strike price of the option (K) = $47; expected number of jumps (λ) = 3 per year; number of trading periods (n) = 10.

Table IV. Distribution of the total hedging costs

	Upper bound for the total hedging costs					
	$b_c = 6$	$b_c = 8$	$b_c = 10$	$b_c = 15$	$b_c = 20$	$b_c = 25$
Mean	4.17	4.08	4.01	3.91	3.85	3.81
Std. Dev.	2.02	2.99	3.64	4.63	5.29	5.78
Minimum	0.59	0.44	0.44	0.44	0.44	0.44
5% quantile	1.31	1.00	0.95	0.90	0.90	0.90
50% quantile	5.42	1.97	1.94	1.82	1.78	1.78
75% quantile	5.98	7.95	9.16	2.71	2.14	2.09
90% quantile	5.98	7.95	9.97	15.00	10.82	5.40
95% quantile	5.98	7.95	9.97	15.00	20.00	25.00
99% quantile	5.98	7.95	9.97	15.00	20.00	25.00
Maximum	6.00	8.00	10.00	15.00	20.00	25.00

This table presents some statistical values of the distribution of the total hedging costs when we impose an upper bound for the costs. The initial hedging capital used in this example is $2.
Parameter values: initial stock price = $50; annual interest rate (r) = 5%; annual volatility of the stock price return (σ) = 20%; annual expected rate of return of the stock (α) = 15%; time to maturity of the option (τ) = 0.25; strike price of the option (K) = $47; expected number of jumps (λ) = 3 per year; number of trading periods (n) = 10.

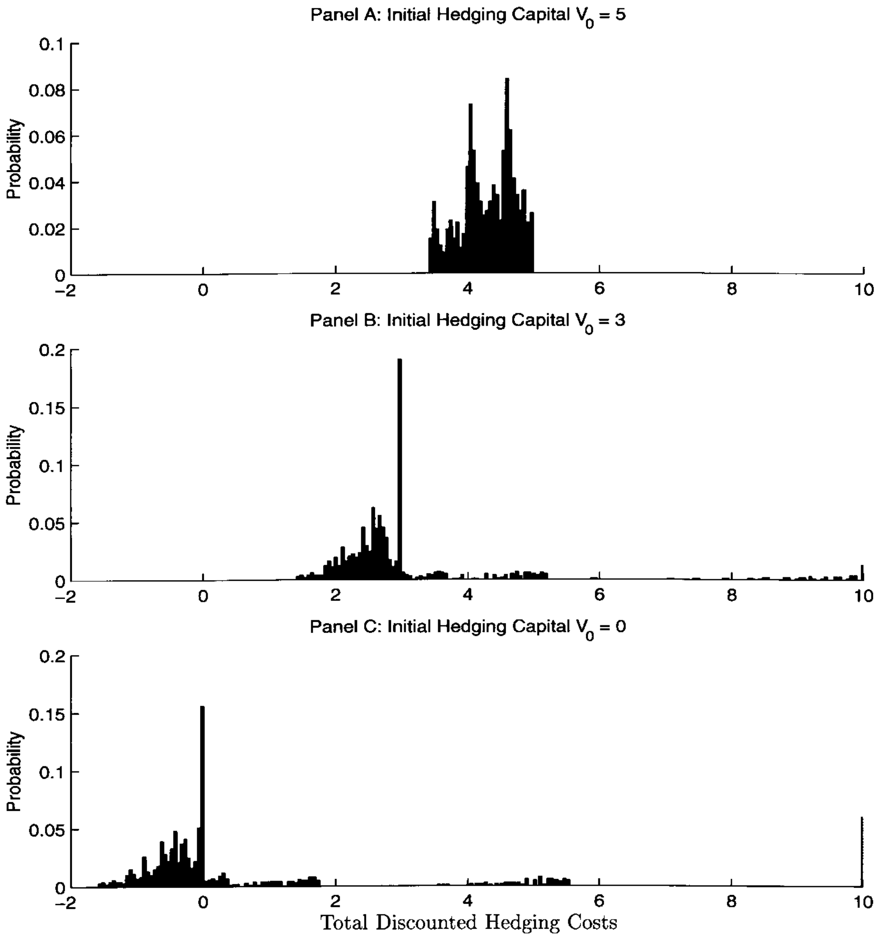

Figure 6. Distribution of the total hedging costs of the local expected shortfall strategy. This figure visualizes the distribution of the total hedging costs of the LES-strategy when the initial hedging capital varies. Panel A presents the case of a superhedging strategy, i.e., a strategy with initial hedging capital $\bar{V}_0 = 5$. Panel B is based on $\bar{V}_0 = 3$ whereas in Panel C the strategy needs no initial investment ($\bar{V}_0 = 0$). All values larger than ten are collected in the rightmost bar. Parameter values: initial stock price = \$50; annual interest rate (r) = 5%; annual volatility of the "normal" stock price return (σ) = 20%; annual expected rate of the "normal" return of the stock (α) = 15%; time to maturity of the option (τ) = 0.25; strike price of the option (K) = \$47; expected number of jumps (λ) = 3 per year; number of trading periods (n) = 10.

In the numerical example we restrict ourselves to a trinomial model. However, it would be interesting to examine multinomial models, as for example the one provided by Amin (1993). Amin's discrete model converges to the continuous-time jump-diffusion model presented by Merton (1976) and hence one may try to find an approximation for an ES-strategy when the stock price process follows a jump-diffusion with lognormally distributed jumps.

Appendix

Proof of Proposition 2

The following proof is in the spirit of similar proofs by Föllmer and Leukert (2000) in a continuous-time setting and by Pham (1999) in a discrete time-setting. We only prove the case $b = \infty$ because the proof for $b < \infty$ is very similar to that.

(i) First, we show $J \equiv \inf_{H \in \mathcal{S}_0^b} E_P(F_T - V_T(H))^+ = \inf_{X \in \mathcal{X}} E_P(F_T - X) \equiv \bar{J}$.

Consider an arbitrary self-financing strategy $H \in \mathcal{S}_0^b$ and define $X \equiv F_T - (F_T - V_T(H))^+ = \min(F_T, V_T(H))$ which leads to $X \leq F_T$ and $X \leq V_T(H)$. Moreover, note that the discounted value process $V_t(H)/B_t$ is a martingale under every martingale measure $Q \in \bar{\mathcal{Q}}$. Thus, we obtain for all $Q\imath\bar{\mathcal{Q}}$ the relation $E_Q(X/B_T) \leq E_Q(V_T(H)/B_T) = \bar{V}_0$. Hence, $X \in \mathcal{X}$. Additionally, we have $E_P(F_T - V_T(H))^+ = E(F_T - X) \geq \bar{J}$ and thus particularly $J \geq \bar{J}$ holds.

Consider now an arbitrary $X \in \mathcal{X}$. Then we have $\max_{Q \in \bar{\mathcal{Q}}} E_Q(X/B_T) \leq \bar{V}_0$. From Proposition 1 we know that there exists a superhedging strategy $H^{\mathrm{SH}} \in \mathcal{S}_0^b$ of X with $V_T(H^{\mathrm{SH}}) \geq X$ and $V_0(H^{\mathrm{SH}}) = \bar{V}_0$. Because of $X \leq F_T$ and $X \leq V_T(H^{\mathrm{SH}})$ this strategy fulfils $(F_T - V_T(H^{\mathrm{SH}}))^+ \leq F_T - X$ and we therefore get $J \leq E_P(F_T - V_T(H^{\mathrm{SH}}))^+ \leq E_P(F_T - X) = \bar{J}(X)$ for all $X \in \mathcal{X}$. Hence, we have $J \leq \bar{J}$ and with the above result $J \geq \bar{J}$ finally $J = \bar{J}$.

(ii) Second, from Proposition 1 we know that there exists a superhedging strategy $H^{\mathrm{ES}} \in \mathcal{S}_0^b$ for X^*. Similar to (i) we get $(F_T - V_T(H^{\mathrm{ES}}))^+ \leq F_T - X^*$. Hence, $J \leq E_P(F_T - V_T(H^{\mathrm{ES}}))^+ \leq E_P(F_T - X^*) = \bar{J} = J$ and therefore $E_P(F_T - V_T(H^{\mathrm{ES}}))^+ = J$. □

Proof of Proposition 3

The following proof follows closely the proof of the fundamental Neyman–Pearson lemma as given in Witting (1985, p. 192) or Ferguson (1969, p. 200). First, notice that the constant c_{ES} is well-defined because of the before made assumption $b \geq \sup_{Q \in \bar{\mathcal{Q}}}(F_T) - \bar{V}_0 B_T$.

Remember that the problem is to solve $\max_{X \in \mathcal{X}_b} E_P(X)$, or more explicitly

$$\max_{F_T - b \leq X \leq F_T} E_P(X) \tag{A1}$$

under the constraint

$$E_Q(X/B_T) \leq \bar{V}_0. \tag{A2}$$

In the standard formulation of the Neyman–Pearson lemma the boundedness of X by $0 \leq X \leq 1$ is required. Nevertheless, as we show in the following, it is possible to prove the slight extension $F_T - b \leq X \leq F_T$ similarly to the proof of the fundamental Neyman-Pearson lemma.

Define for an arbitrary modified contingent claim $X \in \mathcal{X}$ and with c_{ES} as defined in the proposition:

$$f(X) \equiv E_P(X)$$

$$g \equiv \sup_{F_T - b \leq X \leq F_T} (f(X) + c_{\text{ES}} \cdot (\bar{V}_0 \cdot B_T - E_Q(X))).$$

We can rewrite these expressions to

$$\begin{aligned} g &= c_{\text{ES}} \cdot \bar{V}_0 \cdot B_T + \sup_{F_T - b \leq X \leq F_T} (E_P(X) - c_{\text{ES}} \cdot E_P(X \cdot Q/P)) \\ &= c_{\text{ES}} \cdot \bar{V}_0 \cdot B_T + \sup_{F_T - b \leq X \leq F_T} E_P(X(1 - c_{\text{ES}} \cdot Q/P)) \\ &= c_{\text{ES}} \cdot \bar{V}_0 \cdot B_T + \sup_{F_T - b \leq X \leq F_T} E_P(X(1 - c_{\text{ES}} \cdot Q/P)^+ - X(1 - c_{ES} \cdot Q/P)^-) \\ &= c_{\text{ES}} \cdot \bar{V}_0 \cdot B_T + E_P(F_T(1 - c_{\text{ES}} \cdot Q/P)^+ - (F_T - b)(1 - c_{\text{ES}} \cdot Q/P)^-) \\ f(X) &= E_P(X) \\ &= E_P(X\,(1 - c_{\text{ES}} \cdot Q/P + c_{\text{ES}} \cdot Q/P)) \\ &= E_P(X\,(1 - c_{\text{ES}} \cdot Q/P)^+ - X\,(1 - c_{\text{ES}} \cdot Q/P)^-) + c_{\text{ES}} \cdot E_Q(X). \end{aligned}$$

Hence the difference between g and $f(X)$ is:

$$\begin{aligned} g - f(X) \;=\; & c_{\text{ES}} \cdot (\bar{V}_0 \cdot B_T - E_Q(X)) + E_P((F_T - X)\,(1 - c_{\text{ES}} \cdot Q/P)^+ \\ & + E_P((X - F_T + b)\,(1 - c_{\text{ES}} \cdot Q/P)^-). \end{aligned}$$

The three terms on the right hand side of the last equation are positive because c_{ES} is positive and $F_T - b \leq X \leq F_T$ as well as $E_Q(X) \leq \bar{V}_0 B_T$ holds. Therefore, we have $g \geq f(X)$ and thus, g is an upper bound for f. If for a $X^* \in \mathcal{X}_b$ the three terms are zero, then we have $g = f(X^*)$ and hence X^* is optimal in that case. Using X^* as defined in Proposition 3, all three terms become zero because γ is chosen such that $E_Q(X^*) = \bar{V}_0 B_T$ and $P/Q(\omega) \gtrless c_{\text{ES}}$ is equivalent to $1 - c_{\text{ES}} \cdot Q/P(\omega) \gtrless 0$: □

Proof of Proposition 6

Notice that in the complete case F_t^{SH} corresponds to the values of the replicating portfolio of F_T. Hence, due to Lemma 2 we can assume w.l.o.g. $\bar{V}_0 < F_0^S$. Using the notation from page 12 we furthermore assume w.l.o.g. $P(u)/Q(u) > (1 - P(u))/(1 - Q(u))$.

Now, we determine the first part of the approximative strategy. At $t = 0$ we have to decide about H_1^{LES} which minimizes $E_P(F_1^{\text{SH}} - V_1(H^{\text{LES}}))^+$ under the constraint $V_0(H^{\text{LES}}) = \bar{V}_0$. From Lemma 1 we know that we can use Proposition 3 to solve this problem. Thus, H_1^{LES} replicates a modified contingent claim $X^{*,1}$ which is equal to F_1^{SH} if there is an up-tick in the first period and which is equal to a constant γ^{LES} if there is a down-tick in the first period. Hence, H_1^{LES} is the unique solution of the following system of linear equations:

$$F_1^{SH}(u) = h_1^{\text{LES}} \cdot S_1(u) + h_1^{0,\text{LES}} \cdot B_1 \tag{A3}$$

$$\bar{V}_0 = h_1^{\mathrm{LES}} \cdot S_0 + h_1^{0,\mathrm{LES}} \cdot B_0 \tag{A4}$$

Now we turn to the ES-strategy. Because $(1 - P(u))^T/(1 - Q(u))^T < (P(u)/Q(u))^i \cdot ((1 - P(u))/(1 - Q(u)))^{T-i}$ for all $i = 1, \ldots, T$ we know from Proposition 3 that the ES-strategy H^{ES} also replicates a modified claim X^*. This modified claim is equal to F_T if there is at least one up-tick until maturity and is equal to a constant γ^{ES} if there is no up-tick at all. Therefore, we know that the first part of the strategy (H_1^{ES}) replicates a modified contingent claim which is equal F_1^{SH} if there is an up-tick in the first period and which is equal to a constant γ_1^{ES} if there is a down-tick in the first period. Hence, H_1^{ES} also solves the equations (A3) and (A4) and therefore $H_1^{\mathrm{LES}} = H_1^{\mathrm{ES}}$. Moreover, we conclude that the constants γ^{LES} and γ_1^{ES} are equal. Now consider the next period. If there is an up-tick in the first period, both strategies H^{ES} and H^{LES} are worth F_1^S at $t = 1$ and hence the rest is just to replicate the contingent claim. If there is a down-tick in the first period then the same arguments as above with γ^{LES} taking the role of $\bar{V}_0$ also shows that the strategies coincide. Thus we finally get $H^{\mathrm{LES}} = H^{\mathrm{ES}}$. □

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