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Optimal control of option portfolios and applications*

Optimale Steuerung von Optionsportefeuilles und Anwendungen

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Abstract. We present an expected utility maximisation framework for optimally controlling a portfolio of options. By combining the replication approach to option pricing with ideas of the martingale approach to (stock) portfolio optimisation we arrive at an explicit solution of the option portfolio problem. Its characteristics are illustrated by some specific examples. As an application, we calculate an optimal option and consumption strategy for an investor who is obliged to hold a stock position until the time horizon.

Zusammenfassung. Wir präsentieren einen Erwartungsnutzenansatz zur optimalen Steuerung eines aus Optionen bestehenden Portefeuilles. Die Kombination des Replikationsansatzes der klassischen Optionsbewertungstheorie mit dem Martingalansatz zur Portefeuilleoptimierung führt zu expliziten Lösungen des Optionsportefeuilleproblems. Wesentliche Lösungseigenschaften werden anhand von Beispielen erhellt. Wir benutzen diese Lösungsmethode zur Bestimmung einer optimalen Portefeuille- und Konsumstrategie eines Investors, der verpflichtet ist, eine vorgegebene Aktienposition bis zum Planungshorizont zu halten.

Key words: Portfolio optimisation – Continuous trading – Option pricing – Complete markets – trading constraints

Schlüsselwörter: Portefeuilleoptimierung – Zeitstetiger Handel – Optionsbewertung – Vollständige Märkte – Handelsbeschränkungen

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1 Introduction

The trading in derivative securities such as options, futures or swaps has become increasingly popular over recent years. Originally introduced for hedging purposes, they are nowadays also used for speculation. Of course, there are some good reasons for that. Derivative markets are usually more liquid than the markets for the underlying stocks. Compared to trading in stocks, transaction costs for trading in derivatives are usually lower. Also, the pricing and hedging of derivatives is by now well understood due to the acceptance of the Black-Scholes formula and its various variants. However, there is still a lack of research concerning the determination of optimal portfolios of derivatives. If a portfolio of options is hold for speculative reasons then the trading philosophy of the investor must be different from that of a hedger. The control of the Greeks is no longer the most important task. Instead, the investor tries to maximise his expected utility from terminal wealth of the portfolio of options. Therefore, in this paper, we set up a corresponding portfolio problem for option trading under the assumption that the underlying stock market is complete. Our main contributions and insights to this problem are the following:

- an explicit solution of the option portfolio problem consisting of the optimal terminal wealth and the optimal trading strategies is given,
- the amount of computational work to determine the optimal strategy for the option problem roughly equals that for computing the optimal strategy in a corresponding stock portfolio problem,
- frequency and volume of the rebalancing actions are comparable to those of a corresponding (dynamic) stock portfolio problem,
- the form of the optimal payoff for the option portfolio problem only depends on the choice of the utility function and *not* on the payoff of the derivatives that enter the portfolios,
- the form of the optimal trading strategies can differ significantly from those of a corresponding stock portfolio problem *and* this form is highly sensitive with respect to the payoff of the derivatives that enter the portfolios, i.e. the non-linearity of the derivatives prices only affects the way how the optimal terminal wealth is generated, not its explicit form!

All these findings are illustrated by specific examples.

Mathematically, we combine results of the replication approach of option pricing and of continuous-time portfolio optimisation for stock portfolios (see Korn, 1997, for an overview to this subject). While a direct application of the stochastic control approach of portfolio optimisation to our option portfolio problem seems to be hopeless (due to the highly non-linear form of the Hamilton-Jacobi-Bellman equation), we can make use of the separation principle of the martingale approach. This will directly yield the explicit form of the optimal terminal wealth and the optimal consumption process. To obtain the optimal option strategies, we can combine results of the replication approach of option pricing and of continuous-time portfolio optimisation. Therefore, Section 2 contains the description of the market model and some basics needed from the theory of option pricing. Section 3 reviews some results of portfolio optimisation while in Section 4 we determine the solution to the option portfolio problem with the help of the results presented in the two foregoing sections. An optimal consumption problem under trading constraints is studied in Section 5.

2 Some basics of option pricing in a complete market model

We look at a financial market consisting of one riskless asset ("bond") and n risky assets ("stocks"). Let their prices $P_i(t)$, $i = 0, ..., n$, be governed by the equations

$$dP_0(t) = P_0(t)r(t)dt\,, \quad P_0(0) = 1\,, \tag{2.1}$$

$$dP_i(t) = P_i(t)\left(b_i\,(t)\,dt + \sum_{j=1}^{n} \sigma_{ij}\,(t)\;dW_j\,(t)\right)\,,$$

$$P_i(0) = p_i\,, \quad i = 1, \ldots, n \tag{2.2}$$

where $W(t)$ is an n-dimensional Brownian motion on a complete, filtered probability space (Ω, F, P) with $\{F_t\}_{t\in[0,T]}$, $T < \infty$, the corresponding Brownian filtration. The market coefficients $r(t)$, $b(t)$, and $\sigma(t)$ are assumed to be progressively measurable and uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$. Furthermore, $\sigma(t)\sigma(t)'$ is required to be uniformly positive definite.

If the $(n + 1)$-vector $\varphi(t)$ denotes the number of shares of different securities held by an investor at time t then we will call

$$X(t) := \sum_{i=0}^{n} \varphi_i\,(t)\,P_i\,(t)$$

the corresponding wealth process.

Definition 2.1. *i) A trading strategy is an $\mathbf{R}^{n+1}$-valued, progressively measurable process $\varphi(t)$, $t \in [0, T]$, with*

$$\int_0^T |\varphi_0\,(t)|\,dt < \infty \text{ a.s.}, \; \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^T \left(\varphi_i\,(t)\,P_i\,(t)\,\sigma_{ij}\,(t)\right)^2 dt < \infty \text{ a.s.}.$$

ii) A non-negative, progressively measurable process $c(t)$, $t \in [0, T]$, with

$$\int_0^T c(t)dt < \infty \quad \text{a.s.}$$

is called a consumption rate process (for brevity: consumption process).

iii) A pair (φ, c) *of a trading strategy* φ *and a consumption process* c *is called self-financing if the wealth process* $X(t)$ *corresponding to* φ *satisfies*

$$X(t) = X(0) + \sum_{i=0}^{n} \int_0^t \varphi_i(s)\, dP_i(s) - \int_0^t c(s)\, ds \quad \forall t \in [0, T] \ .$$

It is called admissible if it is self-financing and its corresponding wealth process $X(t)$ *is non-negative. Let* $A(x)$ *denote the set of admissible pairs with* $X(0) = x$.

Definition 2.2. *i) A contingent claim is a non-negative,* F_T*-measurable random variable* B*, such that*

$$E(B^{\mu}) < \infty \tag{2.3}$$

for some $\mu > 1$.
ii) $(\varphi, c) \in A(x)$ *is called a replicating strategy for the contingent claim* B *if the wealth process* $X(t)$ *corresponding to* (φ, c) *satisfies*

$$X(T) = B \quad \textit{a.s.}\ .$$

Let $D(x) \subseteq A(x)$ *be the set of replicating strategies for* B*. The real number*

$$p := \inf \{x > 0 \mid D(x) \neq \varnothing\}$$

is called the fair price of the contingent claim B.

Remark 2.3. a) The most popular examples of contingent claims are the European call option and the European put option whose terminal payouts are given by

$$B_{\text{call}} = (P_1(T) - K)^+ \ , \quad B_{\text{put}} = (K - P_1(T))^+ \ ,$$

respectively (where x^+ denotes the positive part of the real number x). Here, the positive constant K denotes the exercise price, i.e. the fixed price for which the holder of a call can buy one share of stock from the seller of the call at time T and for which the holder of a put can sell one share of stock to the seller of the call at time T. Further examples of contingent claims such as Asian options, look backs, digitals, etc. can be found in e.g. Jarrow and Turnbull (1996), Wilmott et al. (1994) or Baxter and Rennie (1996).
b) The main result of option pricing in complete markets is summarised in Theorem 2.4 below. The idea behind it is that every contingent claim should have a value equal to the cost of setting up its cheapest replicating strategy. Otherwise there would be a possibility of making arbitrage gains. The existence of such a cheapest replicating strategy for every contingent claim is the defining feature of a complete market.

Theorem 2.4. *Let*

$$H(t) = \exp\left(-\int_0^t \left(r\,(s) + \frac{1}{2}\,\|\theta\,(s)\|^2\right) ds - \int_0^t \theta\,(s)'\,dW\,(s)\right)$$

denote the stochastic deflator process with $\theta(t) = \sigma^{-1}\,(t)\,(b(t) - r(t)\underline{1})$, $t \in [0, T]$. *Then, the fair price* p *of a contingent claim* B *equals*

$$p = E(H(T)B)\ ,$$

and there exists a unique replicating strategy $(\varphi^*, c^*) \in D(p)$ *with* $c^*(t) = 0$ *for all* $t \in [0, T]$ *a.s. Its corresponding wealth process,* $X^*(t)$*, (the "valuation process for* B*") admits the representation*

$$X^*(t) = \frac{1}{H\,(t)} E\,(H(T)B|F_t)\ .$$

Remark 2.5. a) For a proof of Theorem 2.4 see Karatzas (1989) or Section 2.4 of the monograph Korn (1997).
b) One can rephrase Theorem 2.4 by the introduction of the unique equivalent martingale measure $\tilde{P}$ for the stock prices (i.e. the probability measure with respect to which all discounted stock prices $P_i(t)/P_0(t)$ are martingales). It is defined as

$$\tilde{P}(A) := E\,(Z(T)1_A) \quad \forall A \in F_T$$

with

$$Z(t) := \exp\left(-\int_0^t \theta\,(s)'\,dW\,(s) - \frac{1}{2}\int_0^t \|\theta\,(s)\|^2\,ds\right)\ , \quad t \in [0, T]\ .$$

Noting that we have

$$\tilde{E}\left(\exp\left(-\int_0^T r\,(s)\,ds\right) B\right) = E\,(H\,(T)\,B)$$

(where $\tilde{E}$ denotes expectation with respect to $\tilde{P}$) we see that option pricing in a complete market as presented above consists of calculating the discounted expectation of the payoff with respect to the equivalent martingale measure.

One can get an explicit form of the replicating strategy by imposing additional assumptions on the option price process.

Theorem 2.6. *Assume that the price of an option at time* t *can be written as a* $C^{1,2}$*-function* $f(t, p_1, \ldots, p_n)$ *of time and underlying stock prices.*
i) Then the replicating strategy ψ^* *is given by*

$$\psi_i^*(t) = f_{p_i}\,(t, P_1\,(t)\,, \ldots, P_n\,(t))\ , \quad i = 1, \ldots, n\ ,$$
$$\psi_0^*(t) = \frac{(f\,(t, P_1(t), \ldots, P_n(t)) - \sum_{i=1}^n f_{p_i}\,(t, P_1\,(t)\,, \ldots, P_n\,(t))\,P_i\,(t))}{P_0(t)},$$

and the function $f(t, p_1, \ldots, p_n)$ is a solution of the partial differential equation

$$f_t + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} p_i p_j f_{p_i p_j} + \sum_{i=1}^{n} r p_i f_{p_i} - rf = 0 \; .$$

Here, we have set $a(t) := \sigma(t)\sigma(t)'$ and where the subscripts $t, p_1, \ldots, p_n$ mean partial derivative with respect to the corresponding variable.
ii) The price process $f(t, P_1(t), \ldots, P_n(t))$ obeys the stochastic differential equation

$$\begin{aligned} & df\left(t, P_1(t), \ldots, P_n(t)\right) \\ & = \Bigg(rf\left(t, P_1\left(t\right), \ldots, P_n\left(t\right)\right) \\ & \quad + \sum_{i=1}^{n} f_{p_i}\left(t, P_1\left(t\right), \ldots, P_n\left(t\right)\right) P_i\left(t\right)\left(b_i - r\right)\Bigg) dt \\ & \quad + \left(\sum_{i=1}^{n} f_{p_i}\left(t, P_1\left(t\right), \ldots, P_n\left(t\right)\right) P_i\left(t\right) \sum_{j=1}^{n} \sigma_{ij}\left(t\right) dW_j\left(t\right)\right) \end{aligned} \tag{2.4}$$

Proof. i) is a well-known result which can e.g. be found in Karatzas (1989). The form of the stochastic differential equation in ii) follows by application of Itô's rule to the process $f\left(t, P_1(t), \ldots, P_n(t)\right)$ and using the fact that f satisfies the partial differential equation given in part i). □

Example 2.7 (Black-Scholes Formula). If we assume a market with $n = 1$ and constant market coefficients r, b, σ then explicit computation for the case of the European call yields the fair price $f^{\text{call}}(t, p)$ (if the stock price at time t equals p_1) as

$$f^{\text{call}}(t, p_1) = p_1 \Phi(d_1(t)) - K e^{-r(T-t)} \Phi\left(d_2\left(t\right)\right)$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution and where we have set

$$d_1(t) = \frac{\ln\left(\frac{p_1}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\left(T - t\right)}{\sigma\sqrt{T - t}} \; , \quad d_2(t) = d_1(t) - \sigma\sqrt{T - t} \; .$$

The replicating strategy ψ^* for the European call is given by

$$\psi_0^*(t) = -K e^{-rT} \Phi\left(d_2(t)\right), \quad \psi_1^*(t) = \Phi\left(d_1(t)\right) \; .$$

If we introduce the portfolio process $\pi^*(t)$ corresponding to $\psi^*(t)$ as the fraction of the investor's wealth held in the stock at time t then we obtain

$$\pi^*(t) = \frac{(\Phi(d_1(t))\,p_1)}{f^{\text{call}}(t, p_1)} \; .$$

Note especially that $\pi^*(t)$ is always bigger than one. By analogy, for the European put we obtain

$$f^{\text{put}}(t, p_1) = Ke^{-r(T-t)}\Phi(-d_2(t)) - p_1\Phi(-d_1(t))$$

and similar forms for the replicating strategy.

3 The continuous-time portfolio problem

The continuous-time portfolio problem (P) consists of maximising total expected utility of consumption over the trading interval $[0, T]$ and/or of terminal wealth $X(T)$, i.e. of finding an admissible pair (φ, c) that maximises

$$J(x; \varphi, c) := E\left(\int_0^T U_1(t, c(t))\,dt + U_2\left(X^{x,\varphi,c}(T)\right)\right) \tag{3.1}$$

where $X^{x,\varphi,c}(t)$ is the wealth process of an investor endowed with an initial wealth of $x > 0$ and following an admissible strategy (φ, c). We require that the utility functions $U_1(t, \cdot)$ and $U_2(\cdot)$ are C^1, strictly concave and satisfy

$$U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty \; , \quad U'(\infty) := \lim_{x \downarrow \infty} U'(x) = 0 \; . \tag{3.2}$$

Further, $U_1(\cdot, \cdot)$ should be continuous. Typical examples are $U_1(t, x) = \exp(-\beta t)\ln(x)$ or $U_2(x) = \frac{1}{\alpha}x^\alpha$ for $\alpha \in (0, 1)$.

To ensure the existence of the expected value in (3.1), we have to restrict the class of admissible pairs to the following sub-class of $A(x)$:

$$A'(x) := \left\{(\varphi, c) \in A(x) \,\middle|\, E\left(\int_0^T U_1(t, c(t))^- \,dt + U_2\left(X^{x,\varphi,c}(T)\right)^-\right) < \infty\right\} . \tag{3.3}$$

Note that by introducing this class we do not exclude strategies that will possibly lead to an infinite utility. We only require that the expected value over the negative parts of the utility functions is finite. Clearly, if both utility functions are non-negative then $A(x)$ and $A'(x)$ coincide.

The central idea of the martingale approach of portfolio optimisation is a decomposition of the portfolio problem into a static optimisation problem ("Determination of optimal cash flows, i.e. consumption $c(t)$ and/or terminal wealth B") and a representation problem ("Find a strategy that yields the (already determined!) optimal cash flow”). This two-step procedure heavily relies on the completeness of the market model. It is described in e.g. Pliska (1986), Karatzas e.a. (1987) or Korn and Trautmann (1995). To formulate its main result we introduce the function $X : (0,\infty) \to \mathbf{R}$ by

$$X(y) := E\left(H\,(T)\, I_2\,(yH\,(T)) + \int_0^T H\,(t)\, I_1\,(t, yH\,(t))\, dt \right) \quad \forall\, y > 0$$

where $I_1(t,\cdot)$, $I_2(\cdot)$ are the inverse functions of $U_1'(t,\cdot)$, $U_2'(\cdot)$, respectively. This function is strictly decreasing, continuous and possesses an inverse function $Y(x)$. With its help a complete solution of problem (P) is given in Theorem 3.1 (see Section 3.4 of Korn (1997) for a proof):

Theorem 3.1. *Let $x > 0$. Under the assumption of*

$$X(y) < \infty \quad \forall\, y \in (0,\infty)$$

the optimal terminal wealth B^* and the optimal consumption process $c^*(t)$, $t \in [0,T]$, for problem (P) are given by

$$B^* := I_2\,(Y\,(x)\, H\,(T))\ , \tag{3.4}$$

$$c^*(t) := I_1\,(t, Y\,(x)\, H\,(t))\ . \tag{3.5}$$

Moreover, there exists a trading strategy $\xi(t)$, $t \in [0,T]$, such that we have

$$(\xi, c^*) \in A'(x)\ , \quad X^{x,\xi,c^*}(T) = B^* \quad \textit{a.s.}\ ,$$
$$J(x;\xi,c^*) = \max_{(\varphi,c)\in A'(x)} J(x;\varphi,c)\ ,$$

i.e. (ξ, c^*) solves the portfolio problem (P). If we consider the pure consumption problem by setting $U_2(x) \equiv 0$ in (P) then the optimal consumption process is still given by representation (3.5) and the optimal terminal wealth B^* equals zero in this case. Contrary, in the pure terminal wealth problem (obtained from (P) by setting $U_1(t,c) \equiv 0$ for all $t \in [0,T]$), we still have representation (3.4) for B^* with $c^*(t) \equiv 0$ for all $t \in [0,T]$).

Remark 3.2. a) In the logarithmic case

$$U_1(t,x) = U_2(x) = \ln(x) \quad \forall\, t \in [0,T]$$

application of Theorem 3.1 and explicit computations yield (see Example 3.19 of Korn (1997) for the details)

$$c^*(t) = \frac{x}{T+1}\frac{1}{H\,(t)} \quad \forall\, t \in [0,T]\ , \quad B^* = \frac{x}{T+1}\frac{1}{H\,(T)}\ .$$

If we define the portfolio process $\pi^*(t)$ as the n-vector of the fractions of the optimal wealth invested in the n stocks then we have

$$\pi^*(t) = \left(\sigma(t)\sigma(t)'\right)^{-1}(b(t) - r(t)\underline{1}) \quad \forall\, t \in [0,T]\ .$$

Hence, the optimal trading strategy $\xi(t)$ is given by

$$\begin{aligned}\xi_i(t) &= \pi_i^*(t)X^*(t)/P_i(t)\ , \quad i = 1, \ldots, n,\\ \xi_0(t) &= \left(1 - \sum_{i=1}^{n} \pi_i^*(t)\right) X^*(t)/P_0(t)\end{aligned}$$

where $X^*(t)$ denotes the corresponding wealth process.
b) The computation of the portfolio process $\pi^*(t)$ or of the trading strategy $\xi(t)$ is the more difficult part of the solution of problem (P). We will only give a result for the case where the optimal wealth process has a special structure (see Section 3.4 of Korn (1997)). More general results can be found in Ocone and Karatzas (1991). If we assume that the optimal wealth process $X(t)$ can be represented in the form

$$\begin{aligned}X(t) &= \frac{1}{H(t)} E\left(\int_t^T H(s)\, c^*(s)\, ds + H(T)\, B^* | F_t\right)\\ &= f\left(t, W_1(t), \ldots, W_n(t)\right) \qquad (3.6)\end{aligned}$$

for a non-negative function $f \in C^{1,2}\left([0,T) \times \mathbf{R}^n\right)$ with $f(0,\ldots,0) = x$ and where B^*, c^* are the optimal terminal wealth and optimal consumption as given in Theorem 3.1, then the optimal trading strategy $\xi(t)$, $t \in [0,T]$, is given by

$$\begin{aligned}\xi_i(t) &= \frac{1}{P_i(t)} \left(\sigma(t)^{-1} \nabla_x f\left(t, W_1(t), \ldots, W_n(t)\right)\right)_i\ , \quad i = 1, \ldots, n\ ,\\ \xi_0(t) &= \left(X^*(t) - \sum_{i=1}^{n} \xi_i(t)\, P_i(t)\right) \Big/ P_0(t)\end{aligned}$$

where $X^*(t)$ is the optimal wealth process of Theorem 3.1 (provided that $\xi(t)$ meets the requirements of Definition 2.1). $\nabla_x f(\cdot)$ denotes the gradient of f with respect to the last n variables i.e. the ones that represent the components of the Brownian motion.

4 Optimal control of an option portfolio

In this section we seek a closed form solution to the portfolio problem if instead of the n stocks, we are only trading in options on these stocks. To be able to do so, we assume that the n options in our market satisfy the assumptions of Theorem 2.6. In particular, their price processes are then given by the stochastic differential equation (2.4). If we define an admissible trading strategy in this market as an $\mathbf{R}^{n+1}$-valued progressively measurable process $\varphi(t)$ such that the corresponding wealth process $X(t)$ satisfies

$$\begin{aligned} X(t) &= \varphi_0(t) P_0(t) + \sum_{i=1}^{n} \varphi_i(t) f^{(i)}(t, P_1(t), \ldots, P_n(t)) \qquad (4.1) \\ &= x + \int_0^t \varphi_0(s)\, dP_0(s) \\ &\quad + \sum_{i=1}^{n} \int_0^t \varphi_i(s)\, df^{(i)}(s, P_1(s), \ldots, P_n(s)) - \int_0^t c(s) ds \end{aligned}$$

where $c(t)$ is a consumption rate process, then the problem to solve is

$$\max_{\varphi,c} E\left(\int_0^T U_1(t, c(t))\, dt + U_2(X(T))\right) . \qquad (4.2)$$

In view of the martingale approach to portfolio optimisation as described in the preceeding section, intuition suggests the following: As long as the option prices carry the same information as the stock prices (mathematically: as long as they generate the same filtration) then the optimal terminal wealth of the option portfolio problem (4.2) and that of the *stock* portfolio (P) should coincide. Then, it will be possible to reconstruct the corresponding trading strategy from the optimal one of the stock portfolio problem via an "inversion" of option replication.

Theorem 4.1. (Optimal solution of the option portfolio problem) *Assume that the option prices satisfy the requirements of Theorem 2.6 and that for every $t \in [0,T)$ the matrix $\psi(t) = (\psi_{ij}(t)), i, j = 1, \ldots, n$, with*

$$\psi_{ij}(t) := f^{(i)}_{p_j}(t, P_1(t), \ldots, P_n(t)) \qquad (4.3)$$

is regular (for all $\omega \in \Omega$).
i) Under the assumptions of Theorem 3.1 the optimal terminal wealth B^* and the optimal consumption process $c^*(t)$ for the option portfolio problem (4.2) are given by the representations (3.4) and (3.5).
ii) Let $\xi(t)$ be the optimal trading strategy of the corresponding stock portfolio problem. An optimal trading strategy φ (i.e. one such that the corresponding wealth process satisfies $X(T) = B^*$ a.s.) is given by

$$\begin{aligned} \bar{\varphi}(t) &= \left(\psi'(t)\right)^{-1} \bar{\xi}(t), \qquad (4.4) \\ \varphi_0(t) &= \left(X(t) - \sum_{i=1}^{n} \bar{\varphi}_i(t) f^{(i)}(t, P_1(t), \ldots, P_n(t))\right) \Big/ P_0(t) \end{aligned}$$

where $\bar{\varphi}(t)$ and $\bar{\xi}(t)$ are the last n components of $\varphi(t)$ and $\xi(t)$, and $X(t)$ is the wealth process.

Proof. i) Note that due to Theorem 2.6 we have

$$f^{(i)}\left(t, P_1(t), \dots, P_n(t)\right) = \sum_{j=0}^{n} \psi_{ij}(t)\, P_j(t)$$

where $\psi_{i0}(t)$ equals $\psi_0^*(t)$ of Theorem 2.6 ii). As the strategy $(\psi_{i0}(t), \dots, \psi_{in}(t))$ is self-financing, we also have

$$df^{(i)}\left(t, P_1(t), \dots, P_n(t)\right) = \sum_{j=0}^{n} \psi_{ij}(t)\, dP_j(t) \ . \tag{4.5}$$

Let now

$$X(t) = \varphi_0(t)P_0(t) + \sum_{i=1}^{n} \varphi_i(t)\, f^{(i)}\left(t, P_1(t), \dots, P_n(t)\right)$$

be the wealth process corresponding to a trading strategy $\varphi(t)$ which is admissible for the option portfolio problem. Using the above representations of the option prices, we obtain

$$\begin{aligned} X(t) \;=\; & \left(\varphi_0(t) + \sum_{i=1}^{n} \varphi_i(t)\, \psi_{i0}(t)\right) P_0(t) \\ & + \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \varphi_i(t)\, \psi_{ij}(t)\right) P_j(t) \\ =:\; & \zeta_0(t)P_0(t) + \sum_{j=1}^{n} \zeta_j(t)\, P_j(t) \end{aligned}$$

and

$$dX(t) = \zeta_0(t)dP_0(t) + \sum_{j=1}^{n} \zeta_j(t)\, dP_j(t) - c(t)dt$$

where $c(t)$ is a consumption process. Due to our assumptions on $(\varphi(t), c(t))$, the corresponding pair (ζ, c) is an admissible strategy for the stock portfolio problem. Thus, part a) of Theorem 2.7 in Korn (1997) is applicable to the wealth process of every pair of trading strategy and consumption process $(\varphi(t), c(t))$ which is admissible for the option portfolio problem, too. With this fact the above assertion in part i) follows by exactly repeating the proof

of Theorem 3.16 in Korn (1997) if we can prove that there exists a replicating strategy for B^* and $c^*(t)$. But this is ensured by part ii) of the above Theorem 4.1 which will be proved below.
ii) Let $\xi(t)$ be the optimal trading strategy of the corresponding stock portfolio problem with corresponding wealth process $X(t)$, i.e. we have

$$\begin{aligned} X(T) &= B^* \quad \text{a.s.}\ , \\ dX(t) &= \xi_0(t)dP_0(t) + \sum_{i=1}^{n} \xi_i\,(t)\,dP_i\,(t) - c^*(t)dt \\ &= \left(\xi_0\,(t)\,P_0\,(t)\,r + \sum_{i=1}^{n} \xi_i\,(t)\,P_i\,(t)\,b_i\,(t) - c^*\,(t)\right) dt \\ &\quad + \sum_{i,j=1}^{n} \xi_i\,(t)\,P_i\,(t)\,\sigma_{ij}\,(t)\,dW_j\,(t)\ . \end{aligned}$$

If on the other hand there exists a trading strategy $\varphi(t)$ in the options market with a corresponding wealth process of $X(t)$ (and a consumption process of $c^*(t)$) then it must have the following representation as in the proof of part i):

$$\begin{aligned} &dX(t) \\ &= \left(\varphi_0\,(t) + \sum_{i=1}^{n} \varphi_i\,(t)\,\psi_{i0}\,(t)\right) dP_0(t) \\ &\quad + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \varphi_i\,(t)\,\psi_{ik}\,(t)\right) dP_k\,(t) - c^*(t)dt \\ &= \left(\varphi_0\,(t)\,rP_0\,(t)\right. \\ &\quad \left. + \sum_{i=1}^{n} \varphi_i\,(t) \left(\psi_{i0}\,(t)\,rP_0\,(t) + \sum_{j=1}^{n} \psi_{ij}\,(t)\,b_j P_j\,(t)\right) - c^*\,(t)\right) dt \\ &\quad + \sum_{i,j,k=1}^{n} \varphi_i\,(t)\,\psi_{ik}\,(t)\,P_k\,(t)\,\sigma_{kj}\,(t)\,dW_j\,(t)\ . \end{aligned}$$

Comparison of the coefficients of the dW-terms of both representations of $X(t)$ yields the desired form of the last n components of $\varphi(t)$. The form of $\varphi_0(t)$ is an immediate consequence of the self-financing condition. To show that the strategy $\varphi(t)$ is admissible, it suffices to show that the stochastic integrals

$$\sum_{i=1}^{n}\int_0^t \varphi_i\left(s\right) df^{(i)}\left(s, P_1\left(s\right), \ldots, P_n\left(s\right)\right)$$

in equation (4.1) are defined. Using the representation (4.5) for the df-terms, the explicit form (4.4) of $\varphi(t)$, and the admissibility of the strategy $\xi(t)$ in the stock portfolio problem yields the admissibility of $\varphi(t)$ for the option portfolio problem. □

Remark 4.2. a) As already announced, part i) of Theorem 4.1 demonstrates that the optimal terminal wealth and consumption only depend on the choice of the utility functions. The different choice of the risky assets (compared to the stock portfolio problem) only manifests itself in the form of the trading strategy that generates the optimal consumption and the optimal terminal wealth (see part ii) of Theorem 4.1). The examples below will further highlight these facts.
b) The first two of the following figures give a graphical representation of the solution of both the option valuation and the *stock* portfolio problem. The third figure shows how these two solution methods are combined to yield our solution method of the *option* portfolio problem. In either figure the upper part denotes the problem while the lower part shows the way it is solved. For ease of exposition, we omit consumption in the optimisation problems, or better, restrict ourselves to pure maximisation problems of expected utility of terminal wealth. In view of Theorems 2.4, 3.1 and 4.1 Figures 1–3 are self-explaining.
c) The regularity condition on the "delta-matrix" $\psi(t)$ in Theorem 4.1 ensures that the market made up of the bond and the n options is still a complete

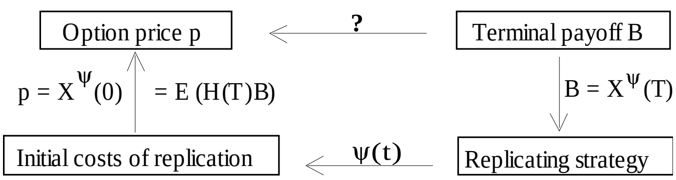

Fig. 1. Option valuation via Theorem 2.4

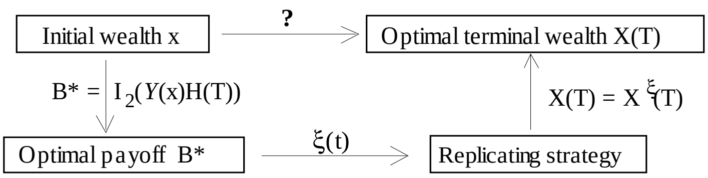

Fig. 2. Stock portfolio optimistaion via Theorem 3.1 (without consumption)

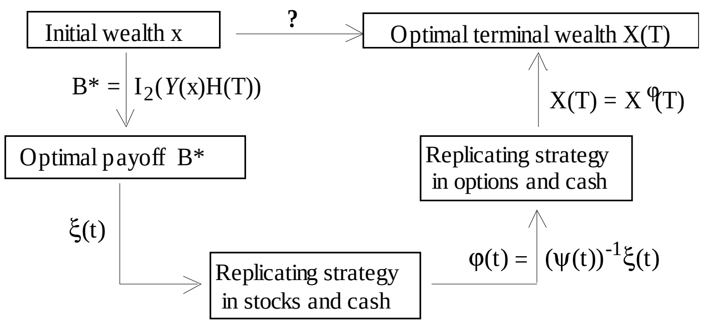

Fig. 3. Option portfolio optimisation via Theorem 4.1 (without consumption)

one. At every time instant t, there should be an impact of *all* n components of the Brownian motion on the n option prices. Without this kind of non-degeneracy assumption we are not able to reduce the "option portfolio problem" to the portfolio problem in the assets underlying the options.

Example 4.3 (Logarithmic utility). We look at the case of the logarithmic utility function as in Remark 3.2 a). For simplicity, let us consider the case of $n = 1$ and constant market coefficients. Then, the optimal portfolio process $\pi(t)$ is a constant π^* equal to $(b - r)/\sigma^2$. By the definition of the portfolio process we obtain

$$\xi_1(t) = \frac{\pi^*(t)\,X(t)}{P_1(t)} = \frac{b-r}{\sigma^2}\frac{X(t)}{P_1(t)}$$

as the optimal number of shares in the stock portfolio problem. Application of Theorem 4.1 yields

$$\varphi_1(t) = \frac{b-r}{\sigma^2}\frac{X(t)}{\psi_1(t)\,P_1(t)}\ ,$$

and the bond component $\varphi_0(t)$ is automatically determined by the self-financing requirement, i.e. we have

$$\begin{aligned}\varphi_0(t) &= (X(t) - \varphi_1(t) f(t, P_1(t)))/\,P_0(t)\\ &= \left(\frac{\sigma^2\psi_1(t)\,P_1(t) - (b-r)\,f(t, P_1(t))}{\sigma^2\psi_1(t)\,P_1(t)}\right)\frac{X(t)}{P_0(t)}\ .\end{aligned}$$

With this representation we can further compute the portfolio process corresponding to the option problem $\pi_{\text{opt}}(t)$ as

$$\pi_{\text{opt}}(t) := \frac{\varphi_1(t)\,f(t, P_1(t))}{X(t)} = \frac{b-r}{\sigma^2}\frac{f(t, P_1(t))}{f_p(t, P_1(t))\,P_1(t)}\ . \qquad (4.6)$$

Note in particular that this portfolio process is only constant for $f(t, P_1(t)) = P_1(t)$, i.e. if the option is equal to the underlying. Thus, for all non-artificial options we have a non-constant portfolio process which depends on both time and the current stock price. It must be pointed out that following the portfolio process $\pi_{\text{opt}}(t)$ requires no additional computations as both the option price $f(t, P_1(t))$ and the option's delta $f_p(t, P_1(t))$ are numbers that someone trading in this option would automatically compute. Although $\pi_{\text{opt}}(t)$ is non-linear in the stock price $P_1(t)$, following the constant portfolio process π^* in the stock portfolio problem also requires rebalancing of the risky position at every time instant t. Thus, regarding the trading frequency, there is no difference between the optimal stock and the optimal option portfolio strategy. To verify that $\pi_{\text{opt}}(t)$ is indeed the optimal portfolio process, we only have to check the regularity assumption on $f_p(t, p)$. To do so, we need more information on the explicit form of the option. As an example, we specialise to the case of a European call where we know that $f_p(t, p)$ is always positive. Further, the last quotient appearing in $\pi_{\text{opt}}(t)$ is always positive but smaller than one with a non-vanishing denominator. This directly implies that we have

$$0 < \pi_{\text{opt}}(t) < \pi^* = (b - r)/\sigma^2 \ .$$

Hence, the fraction of money invested in the risky asset is always *strictly smaller* if an investor follows the optimal option strategy than if he would follow the optimal stock strategy. As holding a European call option is more risky than holding the underlying stock, one can say that the seemingly reduced risk of having a smaller fraction of money invested in the risky asset is compensated by the fact that this amount bears a higher risk than holding the same amount in the stock. We also give a graphical comparison of the two portfolio processes $\pi_{\text{opt}}(t)$ and π^* as functions of the underlying stock price in Fig. 4 (where we have chosen $r = 0$, $b = 0.05$, $\sigma = 0.25$, $T = 1$, $t = 0$, $K = 100$). It clearly highlights the fact that for high stock prices (compared to the strike price) the option portfolio process tends to the stock portfolio process while for small stock prices the option portfolio tends to zero.

Hence, the more risky the call option is – that is, the more the option is out of the money – the smaller is the fraction of wealth $\pi_{\text{opt}}(0)$ invested in the risky security.

In the case of a European put we can parallel the foregoing discussion. Furthermore, we obtain analogous results when using other members of the HARA class as utility functions.

Example 4.4 (Exponential utility). In contrast to the logarithmic utility case where the optimal *fraction* of wealth invested in the stocks is constant, it is

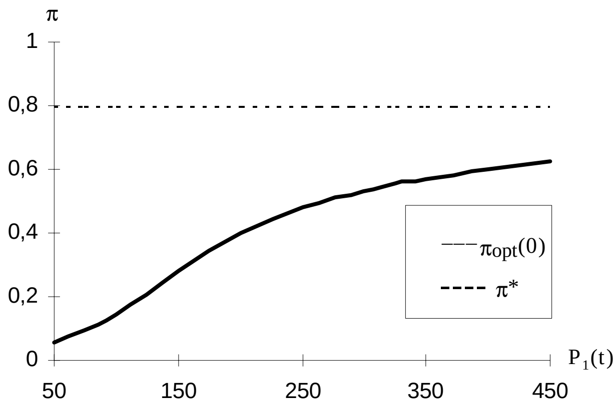

Fig. 4. π^* and $\pi_{\rm opt}(0)$ as functions of the stock price

the optimal *amount* of money invested in the stock that is constant if

$$U(x) = 1 - e^{-\lambda x} \quad (\lambda > 0 \text{ fixed})$$

(if we further choose $r = 0$ and do not allow for consumption). This choice of utility function does not satisfy the requirements (3.2), but it can be treated with similar methods. In particular, the implications of Theorems 3.1 and 4.1 remain true. The optimal number of shares of stock in the stock portfolio problem is given as (see Pliska (1986) or Section 3.5 of Korn (1997))

$$\xi_1(t) = \frac{b}{\lambda\sigma^2}\frac{1}{P_1(t)}$$

which, as a consequence of Theorem 4.1, leads to the optimal number of option contracts in the option portfolio problem of

$$\varphi_1(t) = \frac{b}{\lambda\sigma^2}\frac{1}{f_p(t, P_1(t))\, P_1(t)} \ .$$

If we compare the optimal amounts of money invested in the corresponding risky asset in both problems,

$$\xi_1(t)P_1(t) = \frac{b}{\lambda\sigma^2} \quad \text{and} \quad \varphi_1(t)f(t, P_1(t)) = \frac{b}{\lambda\sigma^2}\frac{f(t, P_1(t))}{f_p(t, P_1(t))\, P_1(t)} \ ,$$

we arrive at similar results as in the case of the European call in Example 4.3: we always invest a positive amount of money in the corresponding risky asset, but always the amount of money invested in the stock in the stock portfolio problem exceeds the amount invested in the option in the option portfolio problem.

Example 4.5 (A multi-dimensional, mixed stock and option problem). Assume the case of the logarithmic utility function and that we have $n = 2$. The trader wants to maximise the terminal wealth from investing in stock number 1 and in call options on stock 2. As the pure stock can be regarded as a call with zero strike price this problem falls into the range of Theorem 4.1 (as all stocks can be regarded as appropriate calls with zero strike, the stock portfolio problem is a special case of the option portfolio problem!). The matrix $\psi(t)$ of Theorem 4.1 is now given as

$$\psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & \Phi\left(d_1\left(t\right)\right) \end{pmatrix}$$

where in the definition of $d_1(t)$ of Example 2.7 we have to replace σ by

$$\sqrt{\sigma_{21}^2 + \sigma_{22}^2}\ .$$

To see this last claim, note that the market with two stocks and a driving two-dimensional Brownian motion is a complete one. Hence, computing the call price means computing discounted expectation of the terminal payoff with respect to the equivalent martingale measure $\tilde{P}$. This can be done in the following way. If $\tilde{W}(t) := \left(\tilde{W}_1(t), \tilde{W}_2(t)\right)'$ is a $\tilde{P}$-Brownian motion then the sum

$$\sigma_{12}\tilde{W}_1(t) + \sigma_{22}\tilde{W}_2(t)$$

occurring in the computation of the above expectation can be replaced by

$$\sqrt{\sigma_{21}^2 + \sigma_{22}^2} Z(t)$$

where $Z(t)$ is normally distributed with zero expectation and variance t under $\tilde{P}$. To compute the call price and the replicating strategy we can then proceed as in the usual one-dimensional setting if we replace σ by the square root expression above.

An application of Theorem 4.1 and analogous calculations as in Example 4.3 will show that in the current setting the optimal fraction of wealth invested in the risky assets will always be bigger than in the setting where we trade in a call option on the first stock (with non-vanishing strike) instead of trading in the first stock. On the other hand, it will be smaller than in the pure two-dimensional stock portfolio problem.

Example 4.6 (Options with non-monotonic payoffs). We assume the setting of Example 4.4. As the denominator of the optimal option strategy $\varphi_1(t)$ already indicates, if $f_p(t, p)$ vanishes for a finite positive p then the number of option contracts in the optimal portfolio will tend to infinity if the stock

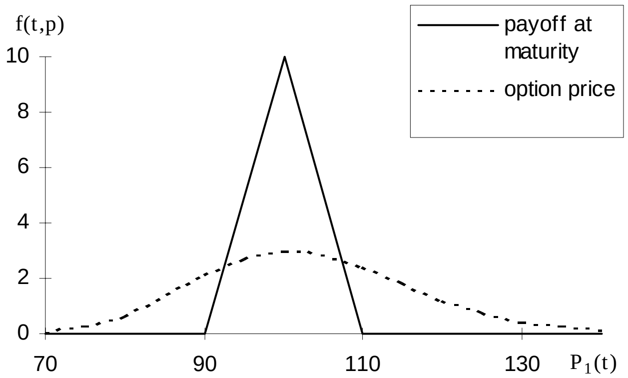

Fig. 5. Option price and payoff

price is close to that value p. Especially, $\varphi_1(t)$ is not defined if this specific value p is attained by the stock price. We will demonstrate this feature by looking at an option with a payoff as given by the diagram in Fig. 5, a so called butterfly spread whose payoff is the same as that of having a long position in one call with strike 90 and one with strike 110 and a short position in two calls with strike 100. Before doing so in detail, we should point out that if we change the optimal option trading strategy of Theorem 4.1 only on a set of zero measure with respect to $L[0, T] \otimes P$ then the resulting strategy will still be admissible and optimal for the option problem. We use this fact to define $\varphi_1(t)$ to be zero whenever $P_1(t)$ reaches the unique value p such that we have $f_p(t, p) = 0$. It is not hard but quite cumbersome to show that for every $t \in [0, T)$ this point p is unique. To highlight the *dangers and risks* occurring in such a situation we refer to Figs. 5–7. There, we have given some relevant graphs for the choice of $r = 0$, $b = 0.05$, $\sigma = 0.25$, $\lambda = 0.1$ and $T = 0.25$. Figure 5 depicts the price of our butterfly spread with 3 months time to maturity as function of the underlying stock price together with its payoff.

In Fig. 6 we see the already indicated behaviour of the optimal strategy for the option portfolio problem. The number of options must be increased (decreased) dramatically if the stock price is close to the value $p^* \approx 100.5$ where the derivative of the option price with respect to p vanishes. The reason for this is the much smaller diffusion part of the option price compared to the stock price. However, the corresponding wealth processes in both the stock and the option portfolio problem have to coincide. Thus, it needs a huge number of options to generate the diffusion component of this wealth process. In particular, the option position changes its sign around this value of the stock price. This fact will become even more dramatic if we look at Fig. 7 where we plot the optimal amount of money invested in the option

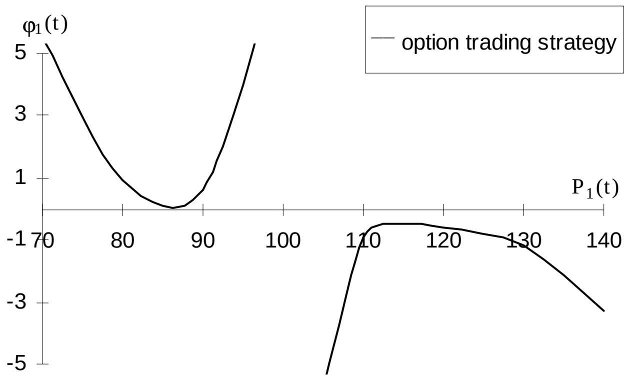

Fig. 6. Optimal strategy for the option portfolio problem

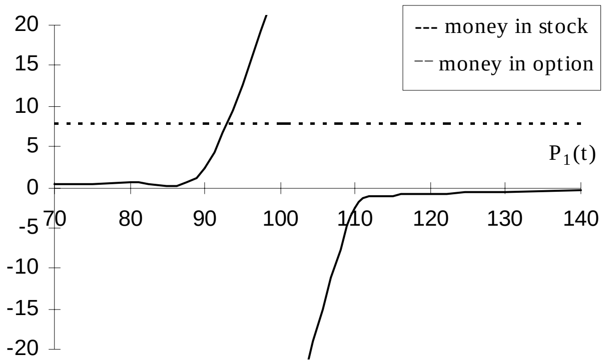

Fig. 7. Optimal amount of money invested in the stock and in the option

against the (constant) optimal amount of money invested in the stock in the stock portfolio problem.

While outside the region of a positive option payoff we always invest less money in the option than in the stock in the corresponding stock portfolio problem the situation completely changes if the stock price is close to p^*. We have to hold huge positive amounts (if the stock price is below p^*) or negative amounts of money (if the stock price is above p^*) in the option, respectively. In particular, there will be a tremendous change in the option position if the stock price crosses the value p^*.

This is a common feature of all options with non-monotonic payoff. The delta neutrality in the point p^* is a desirable feature for hedging purposes or if the goal of the investor is to control the Greeks of an option portfolio. However, if the goal is utility maximisation such points of delta-neutrality of an option are critical. They require huge option positions which are also extremely sensitive with respect to price changes of the underlying. In our one-dimensional example, one could informally say that it is the change from zero – which corresponds to the stock position in the replicating strategy for the option if the stock price equals p^* – to "$1/0$" which corresponds to the option position in the optimal trading strategy of Theorem 4.1 if the stock price equals p^*.

In practise however, such huge positions cannot be hold due to limited supply and demand in the relevant market. A strategy of just keeping the option position constant until the price of the underlying has moved away sufficiently far from p^* and then following the optimal strategy again should be an acceptable approximation. Although such a behaviour is not optimal it avoids the dangers of building up extremely high and very volatile option positions.

5 Optimal consumption under trading constraints

As an application of the methodology developed in the foregoing section, we will look at a realistic problem. An investor is required to hold the position $(\zeta_1(t), \ldots, \zeta_n(t))$ in the stock market where $(\zeta_1(t), \ldots, \zeta_n(t))$ are the last n components of an admissible stock trading strategy ζ with a zero-consumption process. We also assume that the investor wants to consume parts of his wealth in an optimal way throughout $[0, T]$ but is only allowed to trade in options on the stocks. In this situation, his wealth process has the form

$$\begin{aligned} X(t) &= \varphi_0(t)P_0(t) + \sum_{i=1}^{n} \varphi_i(t)\, f^{(i)}(t, P_1(t), \ldots, P_n(t)) \\ &\quad + \sum_{i=1}^{n} \zeta_i(t)\, P_i(t) \qquad (5.1) \\ &= x + \int_0^t \varphi_0(s)\, dP_0(s) \\ &\quad + \sum_{i=1}^{n} \int_0^t \varphi_i(s)\, df^{(i)}(s, P_1(s), \ldots, P_n(s)) \\ &\quad + \sum_{j=1}^{n} \int_0^t \zeta_j(s)\, dP_j(s) - \int_0^t c(s)\, ds \end{aligned}$$

where x is his given initial wealth and $c(t)$ is a consumption process such that with the option trading strategy $\varphi(t)$ the pair (φ, c) is admissible. We insert this form for the wealth process into the option portfolio problem (4.2) of Section 4 and call it the *modified option portfolio problem with constraint* ζ. This situation is covered by a generalised version of Theorem 4.1:

Theorem 5.1. *Given the assumptions of Theorem 4.1, assume that an investor wants to solve the modified option portfolio problem with constraint* ζ.
i) Under the assumptions of Theorem 3.1, the optimal terminal wealth B^* *and the optimal consumption process* $c^*(t)$ *for the modified option portfolio problem with constraint* ζ *are given by the representations (3.4) and (3.5).*
ii) Let $\xi(t)$ *be the optimal trading strategy of the corresponding stock portfolio problem. Then, an optimal trading strategy* φ *for the modified option portfolio problem with constraint* ζ *(i.e. one such that the corresponding wealth process satisfies* $X(T) = B^*$ *a.s.) is given by*

$$\bar{\varphi}(t) = \left(\psi'(t)\right)^{-1} \left(\bar{\xi} - \bar{\zeta}\right)(t) \,, \tag{5.2}$$

$$\varphi_0(t) = \tag{5.3}$$

$$\left(X\left(t\right) - \sum_{i=1}^{n} \left(\varphi_i\left(t\right) f^{(i)}\left(t, P_1\left(t\right), \dots, P_n\left(t\right)\right) + \zeta_i\left(t\right) P_i\left(t\right) \right) \right) \Big/ P_0(t)$$

where $\bar{\varphi}(t)$, $\bar{\zeta}\,(t)$ *and* $\bar{\xi}(t)$ *are the last* n *components of* $\varphi(t)$, $\zeta(t)$ *and* $\xi(t)$, *and* $X(t)$ *is the wealth process.*

Proof. The proof is an immediate consequence of the proof of Theorem 4.1. Simply note that to replicate the optimal terminal wealth and consumption of the stock portfolio problem the investor has to hold the option positions according to representation (5.2) and (5.3) in addition to the required stock positions given by the constraint ζ. □

Remark 5.2. It is also possible to prove a similar theorem where the constraints are given by an options position or by mixed stock/option constraints. The only necessary requirement is that the remaining "unconstrained" securities generate the same complete market as in the unconstrained case. In this situation, all constraints can be dealt with by the method given in the proof of Theorem 4.1 with the modifications indicated in the proof of Theorem 5.1 above.

Example 5.3 (Optimal consumption in the presence of a fixed stock position). As indicated in the section headline, our main application of Theorem 5.1 is the optimal consumption of a package of shares of stock on $[0, T]$ under the requirement to hold a constant number of shares of this stock throughout. More precisely, we are obliged to follow a constant, positive trading strategy ζ_1 in the stock. We are only allowed to trade in the bond and a derivative with price $f(t, P_1(t))$. Our goal will be to maximise the expected utility of consumption on $[0, T]$ in the case of the power utility function. That is, we solve the problem

$$\max_{\varphi, c} E\left(\int_0^T e^{-\beta t} \left(c\left(t\right)\right)^{\delta} dt\right) \tag{5.4}$$

where the pairs (φ, c) are admissible, and where the constants β and δ satisfy

$$0 < \beta \, , \quad 0 < \delta < 1 \, .$$

Further, we assume constant market coefficients. As there are no additional endowments, the investor's initial wealth x is given by

$$x = \zeta_1 P_1 (0) \ \ .$$

As reported in Fleming and Rishel (1975, pp. 160–161), the optimal consumption process $c^*(t)$ and the optimal portfolio process $\pi^*(t)$ for the corresponding stock portfolio problem are given by

$$c^*(t) = \gamma(t) X(t) \, , \quad \pi^*(t) = \frac{b-r}{(1-\delta)\,\sigma^2}$$

with

$$\gamma(t) = \frac{1}{1-\delta} \left(\frac{\beta - \nu\delta}{1 - e^{-\frac{(\beta-r)(T-t)}{1-\delta}}} \right) \, ,$$
$$\nu = r + \frac{(b-r)^2}{2\sigma^2 (1-\delta)} \, , \quad X(T) = 0 \quad \text{a.s} \, .$$

in the case of $\beta \neq \nu\delta$. Using these results and Theorem 5.1 leads to the optimal option trading strategy

$$\varphi_1(t) = \frac{\xi_1 (t) - \zeta_1}{\psi (t)} = \frac{1}{f_p (t, P_1 (t))} \left(\frac{b-r}{(1-\delta)\,\sigma^2} \frac{X (t)}{P_1 (t)} - \zeta_1 \right) \, . \tag{5.5}$$

Note in particular that due to the presence of the constant position in the stock the number of options in the optimal portfolio can change its sign depending on the stock price and the wealth process. Further, note that the optimal option strategy consists of zero holdings if the optimal number of shares in the stock problem $\xi_1(t)$ equals ζ_1, a result which is of course not surprising. If however the optimal stock strategy does not equal ζ_1 – which is the case $L[0,T] \otimes P$-almost surely – then the optimal consumption $c^*(t)$ can only be realised with the help of trading in options. There still remains the question in which type of options the above investor should trade. From a theoretical point of view all available options with a non-vanishing delta are equally suited. From a more practical point of view, the remarks made in Example 4.3 about extremely high strikes and the danger to suffer from the volatility smile effect are valid here, too.

6 Conclusion

We have given closed form solutions for the problem of maximising the expected utility from consumption and terminal wealth if the tradable assets are options and not stocks as in the usual formulation of the portfolio problem. Solving this problem with classical stochastic control methods seems to be highly complicated as the option price is given by the non-linear stochastic differential equation (2.4). By combining the martingale approach to portfolio optimisation with the technique of replication of options in complete markets, we could avoid the complexity due to the non-linearity of the stochastic differential equation (2.4) entering the optimisation problem. The results and techniques given above raise a lot of further interesting and open questions. One interesting aspect is that of being able to treat additional constraints in a portfolio problem by introducing redundant securities. We could e.g. look at an option portfolio problem where we are allowed in trading both a put and a call option on the same underlying stock. As we could solve the corresponding problem with either the call or the put alone there will be no uniqueness in the optimal trading strategy. This fact can be used to deal with further constraints such as a maximum number of calls bought or the desire to find a combination of both the call and put such that a minimum amount of money is invested in the risky securities while still obtaining the optimal utility. Another possibility for future research is to extend the results to incomplete markets. We could look at markets where it is possible to complete them by the introduction of a finite number of options and look at the resulting increase in the optimal utility by additionally trading in options. Also, the impact of the pricing rule (which is no longer unique in incomplete markets) on both the optimal trading strategy and the optimal utility can be examined. Finally, let us mention a paper on non-linear portfolios, Cvitanic (1997). There, the non-linearity is introduced via more complicated price processes, and hedging and portfolio problems are treated with the help of forward-backward stochastic differential equations. It could be promising to apply these techniques to our task of computing optimal option portfolios.

References

Baxter M, Rennie A (1996) Financial calculus. Cambridge University Press, Cambridge

Black F, Scholes M (1973) The pricing of options and corporate liabilities. Journal of Political Economics 81: 637–659

Cvitanic J (1997) Nonlinear financial markets: hedging and portfolio optimization. In: Dempster MHA, Pliska S (eds) Mathematics of financial derivatives, pp 227–254. Cambridge University Press, Cambridge

Duffie D (1992) Dynamic asset pricing theory. Princeton University Press, Princeton

Fleming WH, Rishel RW (1975) Deterministic and stochastic optimal control. Springer, Berlin Heidelberg New York

Jarrow RA, Turnbull S (1996) Derivative securities. South-Western College Publishing, Cincinnati, Ohio

Karatzas I (1989) Optimization problems in continuous trading. SIAM Journal on Control and Optimization 27: 1221–1259

Karatzas I, Lehoczky JP, Shreve SE (1987) Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. SIAM Journal on Control and Optimization 27: 1157–1186

Korn R (1997) Optimal portfolios – stochastic models for optimal investment and risk management in continuous time. World Scientific Publishing, Singapore

Korn R, Trautmann S (1995) Continuous-time portfolio optimization under terminal wealth constraints. ZOR 42(1): 69–93

Ocone D, Karatzas I (1991) A generalized Clark representation formula, with application to optimal portfolios. Stochastics and Stochastics Reports 34: 187–228

Pliska SR (1986) A stochastic calculus model of continuous trading: Optimal portfolios. Mathematics of Operations Research 11: 371–382

Wilmott P, Dewynne JN, Howison SD (1993) Option pricing: mathematical models and computation. Oxford Financial Press, Oxford