Option Hedging in the Presence of Jump Risk

by

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Abstract

We examine Schweizer's (1991) *locally risk-minimizing* (LRM) hedge approach for hedging a European call in the case when the stock price follows a Poisson jump diffusion process with lognormally distributed jump sizes. In contrast to Merton's (1976) hedging strategy where diffusion risk is perfectly hedged while jump risk remains un-hedged, the locally risk-minimizing strategy hedges both diffusion risk as well as jump risk *partly*. The hedge ratio consists of a diffusion component and a jump component. The value of the LRM hedge portfolio is equal to the (discounted) expected terminal payoff of the option weighted with a so-called *minimal martingale density*. It is a weighted sum of Black/Scholes values where some weights may be negative. The latter property is due to our result that the minimal martingale density is negative with positive probability if, e.g., the market price of risk is positive. However, introducing a single call does not admit arbitrage opportunities if its value is smaller than the underlying stock price and larger than the Black/Scholes value based on the diffusion volatility.

We relate the LRM approach to the so-called locally variance-minimizing (LVM) hedging strategy in Bates' (1991) systematic jump risk model. By numerical analysis we find that the LRM and LVM hedge ratios are less sensitive to changes in the stock price than delta hedging strategies in the models of Merton, Black/Scholes, and Bates. If the expected jump size is significantly different from zero and positive (negative), then the LRM and LVM hedge ratios are substantially larger (smaller) than e.g. Merton's for out-of-the-money (in-the-money) calls. Moreover, the worst case behaviour of the LRM strategy and LVM strategy are substantially better: The 99%-, 95%-, as well as the 90%-quantile of the total hedging costs are significantly lower than for the alternative strategies.

1 Introduction

In the complete market model of Black and Scholes (1973) the terminal payoff of a European option can be duplicated by a self-financing portfolio consisting of the underlying stock and a money market account. Thus, one can hedge the risk of a short position in the option perfectly by buying the duplicating portfolio. Unfortunately, the Black/Scholes model does not allow for jumps in the stock price process. Modelling jumps is, however, economically very appealing not only because of market crashes (as in 1987 or 1989) but also because of the ability to explain skewness and kurtosis observed in stock return data. However, modelling jumps with stochastic amplitude leads to incompleteness of the market model. Thus, a self-financing duplicating portfolio no longer exists: pricing by arbitrage and perfectly hedging the option are no longer possible. Option pricing requires general equilibrium arguments, whereas in order to hedge the option (beside the initial investment) additional cash infusions or withdrawals during the lifetime of the option or at the expiration date are necessary. Since these additional costs are stochastic they determine the risk of the hedging strategy.

The main purpose of this paper is not valuation but *hedging* options in the presence of *systematic* jump risk. We assume that the stock price moves according to a Poisson jump diffusion process with constant parameters and lognormally distributed jump sizes, as it was first studied in the context of option valuation by Merton (1976). We particularly focus on the *locally risk-minimizing* hedging strategy. Föllmer and Sondermann (1986) pioneered this approach in the special case where the discounted actual stock price follows a *martingale*. At each point in time they require that the risk, defined as the expected quadratic hedging error, is minimized. However, in semimartingale models a risk-minimizing strategy does not always exist. Therefore, Schweizer (1991) introduces a locally risk-minimizing strategy (LRM) and shows that – under certain assumptions – a strategy is *locally* risk-minimizing if the cost process is a martingale which is orthogonal to the martingale part of the stock price process. The value of the hedge portfolio is then the discounted expected terminal payoff of the option under the so-called minimal equivalent martingale measure. Colwell and Elliott (1993) present a general formula for the minimal equivalent martingale measure and the LRM strategy in jump diffusion models. However, their assumptions only allow for *bounded* jumps. If jump sizes are lognormally distributed, as assumed in this paper, a minimal *equivalent* martingale measure does *not* exist since the corresponding density process (the minimal martingale density) is no longer strictly positive. We repeat the analysis of Colwell and Elliott (1993) for lognormally distributed jumps. In contrast to Colwell and Elliott (1993), we emphasize the fact that the minimal martingale density is not strictly positive and calculate *explicitly* both the value of the hedge portfolio and the hedge ratio. The value of the hedge portfolio can be written as a weighted sum of Black/Scholes values where some of the weights may be *negative*. However, we derive arbitrage boundaries for a European call in a jump diffusion model, and all LRM hedge portfolio values computed lie within these boundaries. The hedge ratio consists of a diffusion as well as a jump component. The strategy is *mean-self-financing*, that is at each point in time the expected sum of discounted cash infusions or withdrawals until maturity is zero.

We compare the LRM approach with the following strategies: If an investor (erroneously) assumes jump risk to be diversifiable he will apply Merton's (1976) hedging strategy. If on the other hand jump risk is not even perceived, a Black/Scholes strategy (based on the diffusion volatility or the total volatility) would seem appropriate. However, if jump risk is perceived and deemed to be systematic, Bates' (1991) equilibrium model offers a way to price but not to hedge the option. Assuming Bates' price to be correct, we introduce a so-called locally *variance* minimizing (LVM) strategy (a modification of the LRM strategy) and compare it with a delta-hedging strategy. In contrast to the LRM strategy where the hedge portfolio value coincides with Bates' option value only at expiration, these latter two strategies track Bates' prices at all times. We calculate the LRM strategy for representative parameters. We find that the LRM and LVM hedge ratios are less sensitive to changes in the stock price than delta hedging strategies in the models of Merton, Black/Scholes or Bates. This is a very desirable property because it leads to less adjustments of the hedge portfolio. In addition, the worst case behaviour of the LRM and LVM strategies is considerably better: The 99%-, 95%-, and 90% quantiles of the total hedging costs are substantially lower.

The paper is organized as follows. Section 2 presents the relevant stochastic processes. Section 3 discusses especially the LRM hedging strategy and compares it with the alternative strategies in Merton's, the Black/Scholes, and Bates' model. Finally, arbitrage boundaries in the jump diffusion model are presented. In Section 4 we analyse how the LRM hedge reacts to changes in the parameter values. A comparison of sensitivities and total hedging costs is made with respect to the other strategies. Section 5 summarizes the main results.

2 Model Framework

We consider a frictionless market with one stock and a riskless security. Trading is continuous in time on the interval $[0,T]$. The riskless interest rate, r, is constant, and therefore the value of the money market account at time t is given through $B_t = \exp(rt)$.

2.1 Stock Price Dynamics

The stock price follows a Poisson jump diffusion process with lognormally distributed jump sizes. It satisfies the following stochastic differential equation:

$$dS_t = \alpha S_{t^-} dt + \sigma_D S_{t^-} dW_t + \int_{-\infty}^{\infty} y S_{t^-} (\nu(dy,dt) - H(dy)dt) \tag{1}$$

where

α	$\equiv$	the constant instantaneous drift of the total process,
W	$\equiv$	a standard Brownian Motion, σ_D is the constant volatility of the diffusion,
ν	$\equiv$	a homogeneous Poisson measure with deterministic compensator $H(dy)dt$ (see, e.g., Jacod/Shiryaev (1987 p. 70)). The measure ν counts the number of jumps of size $x \in [y, y+dy]$ which occur at time $s \in [t, t+dt]$. We denote by $\lambda \equiv E(\int_0^1 \int d\nu)$ the mean number of jumps in unit time.
H	$\equiv$	the λ-fold of the distribution function of a random variable L_i where $ln(1+L_i)$ is normally distributed with mean $\alpha_J - \frac{1}{2}\sigma_J^2$ and variance σ_J^2. We write $\ln(1+L_i) \sim \mathcal{N}(\alpha_J - \frac{1}{2}\sigma_J^2, \sigma_J^2)$. Moreover, $k \equiv \frac{1}{\lambda}\int yH(dy) = E(L_i) = e^{\alpha_J} - 1$ denotes the expected percentage jump size in the stock price.

Alternatively, the integral expression in (1) can be written in the following way: $N_t = \int_0^t \int \nu(dy, ds)$ is a Poisson-counting process with parameter λ and arrival time $T_i = \inf\{t | N_t = i\}$. Furthermore, $(L_i = \int_{T_{i-1}}^{T_i} \int y\nu(dy,dt))_i$ is a sequence of independent identically distributed random variables with $ln(1+L_i) \sim \mathcal{N}(\alpha_J - \frac{1}{2}\sigma_J^2, \sigma_J^2)$. Then

$$\nu(\omega; dy, dt) = \sum_i \mathbf{1}_{[T_i(\omega)]} \, \delta_{(L_i(\omega), T_i(\omega))}(dy, dt)\,,$$

where δ_a denotes the Dirac measure in point a and $\mathbf{1}$ the indicator function. With $I = \sum_i L_i \mathbf{1}_{]T_{i-1}, T_i]}$ follows

$$\int_{-\infty}^{\infty} y S_{t^-} (\nu(dy,dt) - H(dy)dt) = S_{t^-}(I_t dN_t - \lambda k dt)\,.$$

The process is defined on a probability space $(\Omega, \mathcal{F}_T, P)$ endowed with the right-continuous, P-complete filtration, $(\mathcal{F}_t)_t$, generated by S (see e.g. Jacod/Shiryaev (1987, p. 2)). With the exponential formula for semimartingales (see, e.g., Protter (1990, p. 77)) follows:

$$S_t = S_0 \exp\left\{(\alpha - \frac{1}{2}\sigma_D^2 - \lambda k)t + \sigma_D W_t + \sum_{i=1}^{N_t} \ln(1+L_i)\right\} .$$

S_T is square-integrable. We will also work with discounted processes since it is often convenient not having to regard the interest rate explicitly. That gives us the following stochastic differential equations for the discounted price processes of the money market account, β, and the stock price, Z:

$$\begin{aligned} d\beta &= 0 \\ dZ_t &= (\alpha - r)Z_{t^-}dt + \sigma_D Z_{t^-}dW_t + \int_{-\infty}^{\infty} yZ_{t^-}(\nu(dy,dt) - H(dy)dt) . \end{aligned} \qquad (2)$$

$(\alpha - r)$ equals the expected excess return of the stock. Z is a *special semimartingale*. Therefore, there exists a unique decomposition of Z into a predictable process with finite variation A and a local martingale M. The latter is the sum of a continuous local martingale $M^c \equiv Z^c$ and a purely discontinuous local martingale $M^d \equiv Z^d$ (see e.g. Jacod/Shiryaev (1987, p. 43)):

$$Z = Z_0 + \underbrace{\int Z(\alpha - r)dt}_{A} + \underbrace{\underbrace{\int Z\sigma_D dW}_{M^c} + \underbrace{\int\int yZ(\nu(dy,dt) - H(dy)dt)}_{M^d}}_{M} \quad . \qquad (3)$$

We denote by $\sigma_{jump}^2 = \lambda E(L_i^2) = \lambda((k+1)^2 e^{\sigma_J^2} - 2k - 1)$ the instantaneous variance of the jump component of the price process and by $\sigma_{tot}^2 = \sigma_D^2 + \sigma_{jump}^2$ the instantaneous variance of the total price process.[1] Then $\gamma = \sigma_{jump}^2 / \sigma_{tot}^2$ represents the percentage of the total stock variance explained by the jump component.

[1] Contrary to most empirical papers (e.g. Bates (1991), Beinert/Trautmann (1991)) our process parameters refer to the price process and not the return process. The instantaneous variance of the stock price equals (see, e.g., Duffie (1992, p. 80))

$$\sigma_{tot}^2 = \frac{d}{d\tau} var\left(\frac{Z_{t+\tau}}{Z_{t^-}} | \mathcal{F}_{t^-}\right)\Bigg|_{\tau=0} .$$

Replacing Z by Z^d and Z by Z^c we get σ_{jump}^2 and σ_D^2, respectively.

2.2 Density Processes

The value of a European option corresponds to the expected value of its discounted terminal payoff with respect to a not necessarily unique equivalent martingale measure.[2] Put differently, the value of the option equals the expected value of the discounted payoff weighted with the Radon-Nikodym-density of the martingale measure. We determine the value of a hedge portfolio in a similar way. Therefore, we look at density processes ζ in a jump diffusion model such that $Z\zeta$ becomes a martingale under P. We want to consider a Markovian model where density processes are given through the following differential equation:

$$d\zeta_t = \zeta_{t^-}\, g(t, Z_{t^-})dW_t + \int_{-\infty}^{\infty} \zeta_{t^-}\,(h(t, Z_{t^-}, y) - 1)\ (\nu(dy, dt) - H(dy)dt) \qquad (4)$$

with $\zeta_0 = 1$. We assume functions g and h such that ζ is P-square-integrable. This guarantees that $S\zeta$ is integrable. Moreover, ζ is a martingale and $\zeta_t = E(\zeta_T|\mathcal{F}_t)$. The function g is responsible for the transformation of the diffusion part of the price process whereas h transforms the jump part. In order to be able to consider a Black/Scholes hedge as well as the locally risk-minimizing hedge, we do not – contrary to Colwell and Elliott (1993) – assume ζ to be strictly positive. Hence the corresponding measure $dP^{g,h} = \zeta_T dP$ is not necessarily equivalent to P. It is even signed when the density process becomes negative with positive probability. To allow for the non-positivity of ζ as well as the martingale property of ζZ we recall the following definition of Schweizer (1992).

Definition 1 *A local P-martingale ζ with $\zeta_0 = 1$ is called a martingale density for Z if the process ζZ is a local P-martingale. ζ is called a strict martingale density if, in addition, ζ is strictly positive.*

It follows:

Lemma 1 *The density process ζ is a martingale density if*

$$\{\ (\alpha - r) + \sigma_D g + \int (h-1) y H(dy)\ \} = 0 \qquad (5)$$

[2]Recall that an equivalent martingale measure P^* is a probability measure such that on the one hand Z is a martingale with respect to P^* and on the other hand for $A \in \mathcal{F}_T$ we have $P(A) > 0$ if and only if $P^*(A) > 0$. This measure defines a unique Radon-Nikodym-density $\zeta_T \equiv dP^*/dP$ and a density process $\zeta_t = E(\zeta_T|\mathcal{F}_t)$ such that $Z\zeta$ is a (local) martingale under P.

PROOF: see appendix or Colwell/Elliott (1993).

In the following we only consider martingale densities. The existence of a strictly positive martingale density also implies that the jump diffusion model contains no arbitrage opportunities: If we choose a constant function $h > 0$ and determine the function g according to equation (5), the corresponding density process is strictly positive:

$$\zeta_t = \exp\left(gW_t - \frac{g^2}{2}t + \ln(h)N_t + (1-h)\lambda t\right) > 0\,. \tag{6}$$

Therefore $P^{g,h}$ is an equivalent martingale measure implying no arbitrage (see, e.g., Duffie (1992)). Since h is constant but otherwise arbitrarily chosen there exist many equivalent martingale measures and thus the market is incomplete.

2.3 Value of the Hedge Portfolio

Consider a European path-independent option which pays off $c(S_T)$ at time T for a suitable function c. Then – as mentioned above – we can describe the value of the option at time t as the discounted expected value of the payoff (conditional on $\mathcal{F}_t$) with respect to some equivalent martingale measure. By analogy, we want to define the value of the *hedge portfolio*, F, as the discounted expected payoff of the option weighted with the martingale density. We get[3]

$$F(t, S_t) \equiv \frac{B_t}{B_T} E\left(c(S_T)\frac{\zeta_T}{\zeta_t}|\mathcal{F}_t\right)\,. \tag{7}$$

However, since ζ may not be strictly positive, we have to be cautious about interpreting F as the value of the call in an arbitrage-free market. We only consider functions $c(S_T)$ such that $F(t,s)$ is once continuously differentiable in t and and twice continuously differentiable in s for $t < T$.[4] In addition, $c(S_T)$ should be P-square-integrable and thus $c(S_T)\zeta$ integrable. The discounted value process is given

[3]To be correct we would have to write ζ_{tT} instead of ζ_T/ζ_t where $\zeta_{tT} = 1 + \int_t^T d\zeta_{ts}$ (see, e.g., Colwell/Elliott (1990)). That means that we forget the history of ζ for calculating $F(t)$. This is relevant when ζ_t is zero with positive probability as with the Black/Scholes hedging strategy discussed in section 3.

[4]That is particularly the case for the European call and the martingale densities considered in section 3. However, the value function $F(t,s)$ is usually not differentiable with respect to s for $t = T$.

by

$$\begin{aligned} V(t, Z_t) &\equiv \frac{1}{B_t} F(t, S_t) \\ &= \frac{1}{B_t} F(t, B_t Z_t) \\ &= E\left(\frac{1}{B_T} c(B_T Z_T) \frac{\zeta_T}{\zeta_t} | \mathcal{F}_t \right) . \end{aligned} \tag{8}$$

The following Proposition 1 confirms the results of Colwell and Elliott (1993) with respect to the differential representation of the value process of the hedge portfolio.

Proposition 1 *The discounted value function $V(t, z)$ satisfies the deterministic partial differential equation*

$$\begin{aligned} 0 &= V_t - \int V_z yzhH(dy) + \frac{1}{2} V_{zz} z^2 \sigma_D^2 \\ &\quad + \int \Big(V(z(1+y)) - V(z) \Big) hH(dy) \,, \end{aligned} \tag{9}$$

and V(t,Z(t)) satisfies the stochastic differential equation

$$\begin{aligned} dV &= V_z Z(\alpha - r)dt + V_z Z \int (h-1) y H(dy) dt + V_z \sigma_D Z dW \\ &\quad + \int \Big(V(Z(1+y)) - V(Z) \Big) \Big(\nu(dy, dt) - hH(dy)dt \Big) . \end{aligned} \tag{10}$$

Accordingly, we have the following partial differential equation for $F(t, s)$:

$$\begin{aligned} 0 &= F_t - rF + rF_s s - \int F_s yshH(dy) + \frac{1}{2} F_{ss} s^2 \sigma_D^2 \\ &\quad + \int \Big(F(s(1+y)) - F(s) \Big) hH(dy) . \end{aligned} \tag{11}$$

The subscripts denote partial derivatives. As in the following the integrands of the stochastic differential equations are evaluated at time t^-.

PROOF: see appendix or Colwell/Elliott (1993).

This partial differential equation is known for special functions h. For instance, for $h = 1$ we get Merton's (1976) differential equation:

$$F_t - rF + (r - \lambda k) F_s s + \frac{1}{2} F_{ss} s^2 \sigma_D^2 + \lambda E_L(\Delta F) = 0 \,, \tag{12}$$

where $E_L(\Delta F) = E_L(F(s(1+L_i)) - F(s))$ denotes the expectation with respect to the jump size L_i. For $h = 0$ we derive the Black/Scholes (1973) differential equation:

$$F_t - rF + rsF_s + \frac{1}{2}F_{ss}s^2\sigma_D^2 = 0\,. \tag{13}$$

If we choose a positive and constant function h with $h \neq 1$ we obtain Merton's equation with a transformed jump parameter $\hat{\lambda} = \lambda \cdot h$.

2.4 Hedge Ratio and Hedging Error

In the following we choose a martingale density ζ in order to specify a hedge plan for the option. The value of the hedge portfolio F is then given through (7) and thus the value of the portfolio at the expiration date, F_T, is equal to the exercise value of the option $c(S_T)$. We also choose a predictable hedge ratio ϕ with $E(\int \phi^2 d\langle Z, Z\rangle) < \infty$.[5] It determines the number of shares of the stock in the hedge portfolio. The number of shares in the money market account, η, is determined such that the portfolio value equals F, i.e. $\eta = (F - \phi S)/B$. Since the market is incomplete we can not perfectly hedge the option with a self-financing strategy. Therefore, we need additional cash infusions (or withdrawals) in order to finance the hedge portfolio. The costs C induced by the hedging strategy then consist of the initial cost of the hedge portfolio, $C_0 = F_0$, and the additional cash flows during the life of the option, necessary to maintain the hedge portfolio. We also call C hedging error or tracking error (although, strictly speaking, the hedging (tracking) error only consists of the additional costs, $C - C_0$). Thus, the changes in the value of the hedge portfolio are due to the additional costs as well as to gains in the stock position and in the money market account, $dF = \phi dS + \eta dB + dC$. It follows that[6]

$$C(t) \equiv F(t) - \int_0^t \phi dS - \int_0^t \eta dB\,. \tag{14}$$

The discounted hedging error, Γ, with $\Gamma_0 = C_0$ is then equal to

$$\Gamma(t) \equiv \Gamma_0 + \int_0^t \frac{1}{B} dC = V(t) - \int_0^t \phi dZ\,. \tag{15}$$

Using equations (2) and (10) and rearranging terms we get the following stochastic differential equation for Γ.[7]

[5] $\langle \cdot, \cdot \rangle$ denotes the predictable (or conditional) quadratic covariation process (see e.g. Protter (1990, p. 98)).

[6] It is more intuitive to evaluate the second term for η_- since the hedging strategy should be predictable. However, note that $\int \eta_- dB = \int \eta dB$.

[7] Replacing V with F and Z with S yields the stochastic differential equation for C.

$$
\begin{aligned}
d\Gamma &= dV - \phi dZ \\
&= V_z \sigma_D Z dW + \int \Big(V(Z(1+y)) - V(Z) \Big) \Big(\nu(dy,dt) - H(dy)dt \Big) \\
&\quad -\phi \sigma_D Z dW - \int \phi y Z (\nu(dy,dt) - H(dy)dt) \qquad (16) \\
&\quad + \Big\{ V_z Z(\alpha - r) + V_z Z \int (h-1) y H(dy) \\
&\quad + \int \Big(V(Z(1+y)) - V(Z))(1-h) H(dy) - \phi(\alpha - r) Z \Big\} dt\,. \\
&= \int (V(Z(1+y)) - V(Z) - \phi y Z)\, \nu(dy,dt) \\
&\quad + (V_z \sigma_D Z - \phi \sigma_D Z)\, dW \qquad (17) \\
&\quad + \Big\{ V_z Z(\alpha - r) + V_z Z \int (h-1) y H(dy) \\
&\quad - \int \Big(V(Z(1+y)) - V(Z) \Big) h H(dy) + \int \phi y Z H(dy) \\
&\qquad - \phi(\alpha - r) Z \Big\} dt\,.
\end{aligned}
$$

The (discounted) hedging error Γ consists of a jump component, a diffusion component, and a time component. A strategy is called *mean-self-financing* if the discounted cost process forms a martingale, that means that for each t the remaining discounted costs have zero expectation.

3 Alternative Hedging Strategies

So far we calculated the value of the hedge portfolio in the assumed incomplete market setting by means of a martingale density. We now examine the locally risk-minimizing (LRM) hedging strategy of Schweizer (1991) for a European call and compare it with Merton's (1976) hedging strategy and the Black/Scholes hedging strategy based on the diffusion volatility and on the total volatility, respectively, as discussed by Naik and Lee (1990). Finally, we consider Bates' (1991) pricing model for systematic jump risk and compare a locally variance-minimizing (LVM) strategy and a Delta-hedging strategy.

3.1 Locally Risk-Minimizing Hedging Strategy

In an incomplete market where the actual discounted stock price process follows a martingale, Föllmer and Sondermann (1986) introduce a *mean-self-financing*, *risk-minimizing* hedging strategy: They define a risk process

$$R_t(\phi, \eta) \equiv E((\Gamma_T - \Gamma_t)^2 | \mathcal{F}_t)$$

representing the expected remaining quadratic hedging error. A strategy is called *risk-minimizing* if it minimizes R_t for each $t \leq T$. It exists in their setting but might not exist if the price process follows a continuous time *semi*martingale. Hence Schweizer (1991) extends this concept by introducing a *locally* risk-minimizing strategy. He starts with the following definition.

Definition 2 *A trading strategy $\Delta = (\delta, \epsilon)$ is called a perturbation if δ is bounded and $\delta S_T + \epsilon B_T = 0$.*

That means that a strategy (ϕ, η) generates the same final cash flow as the perturbed strategy $(\phi + \delta, \eta + \epsilon)$. With $\Delta|_{]s,t]} = (\delta \mathbf{1}_{]s,t]}, \epsilon \mathbf{1}_{[s,t[})$ and with a partition τ of $[0, T]$ where $0 = t_0 < t_1 < \ldots < t_N = T$ Schweizer (1991) introduces the risk quotient:

$$r^\tau((\phi, \eta), \Delta) = \sum_{t_i} \frac{R_{t_i}((\phi, \eta) + \Delta|_{]t_i, t_{i+1}]}) - R_{t_i}((\phi, \eta))}{E(\langle M, M \rangle_{t_{i+1}} - \langle M, M \rangle_{t_i} | \mathcal{F}_{t_i})} \mathbf{1}_{]t_i, t_{i+1}]} . \tag{18}$$

The numerator determines the additional risk that arises from the perturbation Δ. Division by the denominator standardizes the risk measure, i. e. the additional risk is allowed to be larger when the additional (conditional) covariance of M is large. Schweizer (1991) defines:

Definition 3 *A trading strategy (ϕ, η) is called locally risk-minimizing if* [8]

$$\liminf r^{\tau_n}((\phi, \eta), \Delta) \geq 0 \qquad P_M - a.e.$$

holds true for every perturbation Δ and every increasing sequence $(\tau_n)_n$ with $|\tau_n| \equiv \max |t_{i+1} - t_i| \to 0$.

Each deviation of the locally risk-minimizing strategy leads in the limit to a rise in the risk. Schweizer (1991, Prop. 2.3) shows that – under certain assumptions on Z which are all satisfied here – a strategy is locally risk-minimizing if the discounted cost process Γ is a martingale that is orthogonal to M, i.e. $\langle \Gamma, M \rangle = 0$. The orthogonality result can also be explained as follows: The space of processes $\{\int \Theta dM | E(\int \Theta^2 d\langle M \rangle) < \infty\}$ consists of all contingent claims that can be hedged by the martingale part of price process. In order to hedge as much as possible of an option we project the martingale part of V on the space $\{\int \Theta dM\}$. The not-hedgeable part is equal to Γ plus some (continuous) process with finite variation. It follows that $\langle \Gamma, M \rangle = 0$. We get

Lemma 2 *The hedging error is orthogonal to the martingale part of the price if the hedging strategy is given by*

$$\begin{aligned} \phi &= \frac{d\langle M, V \rangle}{d\langle M, M \rangle} \\ &= (1-\gamma) F_s + \gamma \frac{E_L(\Delta S \Delta F)}{E_L((\Delta S)^2)} \,. \end{aligned} \tag{19}$$

Proof: see appendix.

The hedge ratio consists of a diffusion component and a jump component. The diffusion part corresponds to the first derivative of the value process with respect to the stock price which is weighted with the percentage of the total variance explained by the diffusion component. The jump part is given by an instantaneous β-factor, i.e. the instantaneous covariation of the jumps in the stock price and in the hedge portfolio value which is divided by the instantaneous variance of the jump in the stock price, $E_L(\Delta S \Delta F)/E_L((\Delta S)^2)$. It is weighted with the percentage of the total variance explained by the jump component.

Furthermore, the cost process Γ follows a martingale under the conditions of the following lemma.

[8] $dP_M \equiv dP$ x $d\langle M, M \rangle$ denotes the Doléans measure of $\langle M, M \rangle$ on the predictable σ-field (see, e.g., Schweizer 1991, Métivier 1982 p. 86).

Lemma 3 *Since ζ is assumed to be a martingale density the cost process Γ is a martingale if*

$$g = -\frac{(\alpha - r)\sigma_D}{\sigma_{tot}^2} \qquad \textit{and} \qquad h = 1 - \frac{(\alpha - r)y}{\sigma_{tot}^2}\,.$$

The corresponding martingale density is given by

$$\begin{aligned}\zeta_t &= \exp\left(\frac{-(\alpha - r)\sigma_D}{\sigma_{tot}^2} W_t - \frac{1}{2}\,\frac{(\alpha - r)^2\sigma_D^2}{\sigma_{tot}^4}\,t + \frac{(\alpha - r)\lambda k}{\sigma_{tot}^2}\,t\right) \\ &\quad \prod_{i=1}^{N_t}\left(1 - \frac{(\alpha - r)L_i}{\sigma_{tot}^2}\right). \end{aligned} \tag{20}$$

PROOF: see appendix.

By inserting g and h into equation (4) and rearranging terms one can see that $d\zeta/\zeta = -a\,dM$ where $a = (\alpha - r)Z/(\sigma_{tot}^2 Z^2) = dA/d\langle M, M\rangle$. This defines the *minimal martingale density* introduced by Schweizer (1992). If it is a strict martingale density then $dP^{g,h} = \zeta dP$ is the minimal equivalent martingale measure defined by Schweizer (1991). Only if ζ is strictly positive the minimal equivalent martingale measure exists.[9] However, observe that ζ is strictly positive only if the factors in the last product are positive. Consequently, if $(1 - ((\alpha - r)L_i)/\sigma_{tot}^2)$ is negative with positive probability, $P^{g,h}$ is not an equivalent probability measure. Since so far we have not used the fact that $\ln(1 + L_i)$ is normally distributed we can formulate the following

Proposition 2 *In a Poisson jump diffusion model with arbitrarily distributed, square-integrable jump sizes, L_i, the minimal equivalent martingale measure exists if and only if*

$$\frac{(\alpha - r)L_i}{\sigma_{tot}^2} < 1 \qquad P - a.s. \tag{21}$$

PROOF: This follows directly from the definition of the minimal martingale density and $\zeta > 0 \quad P - a.s.$ It derives also directly from Theorem 2 of Schweizer (1992).

An application to our setting yields the following corollary:

[9] Föllmer and Schweizer (1991) show for continuous processes that the minimal martingale measure is unique if it exists and that its Radon-Nikodym density is equal to the minimal martingale density. Their proof can be given analogously for the Poisson jump diffusion process.

Corollary 1 *In a Poisson jump diffusion model with lognormally distributed jump sizes the minimal equivalent martingale measure exists if and only if*

$$-1 \leq \frac{(\alpha - r)}{\sigma_{tot}^2} \leq 0\,. \qquad (22)$$

PROOF: Since $(1 + L_i)$ is lognormally distributed the values of L_i lie in the open interval $]-1, \infty[$. Therefore, equation (21) is satisfied if and only if equation (22) is true.

Consequently, in typical situations where the market price for risk is positive, $(\alpha - r)/\sigma_{tot} > 0$, the minimal martingale measure does not exist. Furthermore, we should stress again that we have to be cautious about interpreting the value of the hedge portfolio as a call value in an arbitrage-free market.

The following differential equation describes the value of the hedge portfolio for the locally risk-minimizing strategy. Compared with Merton's differential equation, we get an additional jump term proportional to the excess return.

Proposition 3 *The value of the locally risk-minimizing hedge portfolio satisfies the following deterministic partial differential equation:*

$$\begin{aligned} 0 &= F_t - rF + (r - \lambda k)sF_s + \frac{1}{2}F_{ss}\sigma_D^2 s^2 + \lambda E_L(\Delta F) \\ &\quad + \frac{\lambda(\alpha - r)}{\sigma_{tot}^2}\left(F_s s E_L(L_i^2) - E_L(\Delta F L_i) \right)\,. \end{aligned} \qquad (23)$$

PROOF: This follows directly from Proposition 1 and $h = 1 - (\alpha - r)y/\sigma_{tot}^2$.

The value of the locally risk-minimizing hedge portfolio results from:

Theorem 1 *In the Poisson jump diffusion model with constant parameters and lognormally distributed jump sizes in the stock price the value of the locally risk-minimizing hedge portfolio for a European call is given by*

$$F = \sum_{n=0}^{\infty} \sum_{l=0}^{n} a_{n,l} F^{BS}(S_t, K, r_{n,l}, \sigma_n, \tau)\,. \qquad (24)$$

where

$$
\begin{array}{lcl}
F^{BS} & \equiv & \textit{Black/Scholes value} \\
\tau & \equiv & T-t = \ \textit{time to expiration} \\
r_{n,l} & \equiv & r - \tilde{\lambda}\tilde{k} + \frac{n\alpha_J + (n-l)\sigma_J^2}{\tau} \\
\tilde{\lambda} & \equiv & \lambda\ (\ 1 - \frac{(\alpha - r)k}{\sigma_{tot}^2}\ ) \\
\tilde{k} & \equiv & \frac{1}{\tilde{\lambda}}\ (\ \lambda k - \frac{(\alpha-r)\lambda E(L_i^2)}{\sigma_{tot}^2}\ ) \\
\sigma_n^2 & \equiv & \frac{n\sigma_J^2}{\tau} + \sigma_D^2 \\
a_{n,l} & \equiv & \exp(-\tilde{\lambda}(\tilde{k}+1)\tau)\frac{(\tilde{\lambda}(\tilde{k}+1)\tau)^n}{n!}\binom{n}{l}\left(\frac{a}{\tilde{\lambda}(\tilde{k}+1)}\right)^l\left(\frac{b}{\tilde{\lambda}(\tilde{k}+1)}\right)^{n-l} \\
a & \equiv & \left(\lambda + \frac{\lambda(\alpha-r)}{\sigma_{tot}^2}\right)\cdot(k+1) \\
b & \equiv & \frac{-\lambda(\alpha-r)}{\sigma_{tot}^2}\ (\ E(L_i^2) + 2k + 1\ )\ .
\end{array}
$$

PROOF: Evaluating equation (7) yields formula (24).

Having determined the value of the hedge portfolio we can now derive the hedge ratio:

Theorem 2 *The locally risk-minimizing hedge ratio equals*

$$
\begin{aligned}
\phi &= (1-\gamma)F_s + \gamma\frac{E_L(\Delta S\Delta F)}{E_L((\Delta S)^2)} \\
&= (1-\gamma)\sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{n,l}N(d_1(n,l)) \qquad (25) \\
&\quad +\gamma\frac{\lambda}{\sigma_{jump}^2}\sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{n,l}\ \Big\{ -k\,\frac{F^{BS}(S,K,r_{n,l},\sigma_n,\tau)}{S} \\
&\quad +(k+1)^2 e^{\sigma_J^2}\,\frac{F^{BS}(S,K,r_{n+1,l},\sigma_{n+1},\tau)}{S} \\
&\quad -(k+1)\,\frac{F^{BS}(S,K,r_{n+1,l+1},\sigma_{n+1},\tau)}{S}\ \Big\}
\end{aligned}
$$

where in addition to the notations of Theorem 1 we use

$$
d_1(n,l) \ \equiv\ (\ln(S/K) + (r_{n,l} + 0.5\sigma_n^2)\tau)/(\sigma_n\sqrt{\tau}).
$$

PROOF: The result follows from inserting the value of F, according to equation (24), into equation (19).

The value of the locally risk-minimizing hedge portfolio is a weighted sum of Black/Scholes values. The weighting factors sum up to unity. However, some of the weighting factors are negative if condition (22) is not satisfied. If no jumps occur (i.e. $\lambda = 0$) formula (24) specializes to the formula of Black and Scholes (1973); the same is true for the hedge ratios. If we are in the pure jump model of Cox and Ross (1976) with a constant jump size (i.e. $\sigma_D = \sigma_J = 0$) the locally risk-minimizing portfolio value and hedge ratio coincide with those of Cox and Ross. F equals Merton's (1976) value if the expected return of the stock coincides with the riskless return, i.e. the expected excess return is zero and the discounted price process follows a martingale. In both cases we determine the value of the hedge portfolio by simply taking expectations under the original measure. However, the hedge ratios are different since jump risk is not hedged in Merton's model while it is partly hedged by the locally risk-minimizing strategy.

The increments of the cost process can be decomposed in the following way:

$$\begin{aligned} dC &= \Delta F - \phi \Delta S \\ &\quad + (F_s - \phi)\sigma_D S dW \\ &\quad + (\phi \lambda k S - \lambda E_L(\Delta F))dt\,. \end{aligned} \tag{26}$$

where Δ stands for the jump of the process (e.g. $\Delta F = F(t) - F(t^-)$). The expression in the first line gives the hedging error caused by a jump in the stock price. The expression in the second line stems from a change in the diffusion component of the stock. The last term results from a change in time. Therefore, we have a hedging error even if no jumps occur. Figure 1 shows for representative parameters the hedging error of the locally risk-minimizing hedging strategy due to changes in S (for fixed t).

3.2 Merton's Hedging Strategy

Merton (1976) assumes that jump risk is diversifiable and devises a hedge portfolio where jump risk is not hedged, therefore $\phi = F_s$. He derives the deterministic partial differential equation for F given through equation (12). Solving the equation for lognormally distributed jump sizes in the stock price yields the following formula for a European call and the following hedge ratio, respectively:

$$\begin{aligned} F &= F^{Me}(S, K, \tau, r, \sigma_D, \lambda, \alpha_J, \sigma_J) \\ &= \sum_n e^{-\lambda' \tau} \frac{(\lambda' \tau)^n}{n!} F^{BS}(S, K, \tau, r_n, \sigma_n) \end{aligned} \tag{27}$$

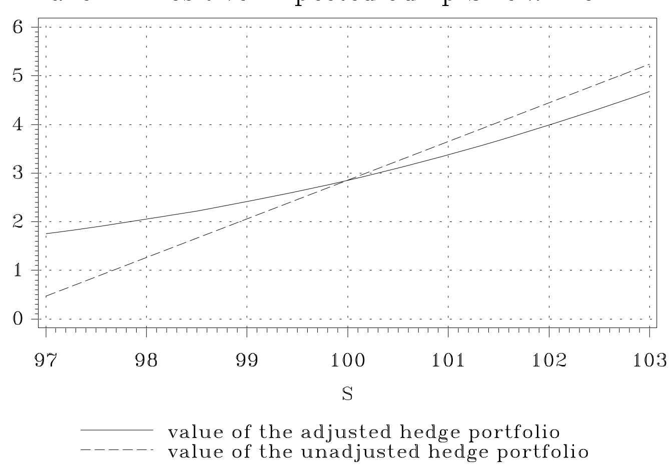
and

$$\begin{aligned} \phi &= F_s^{Me} \\ &= \sum_n e^{-\lambda' \tau} \frac{(\lambda' \tau)^n}{n!} N(d_1(n)) \end{aligned} \tag{28}$$

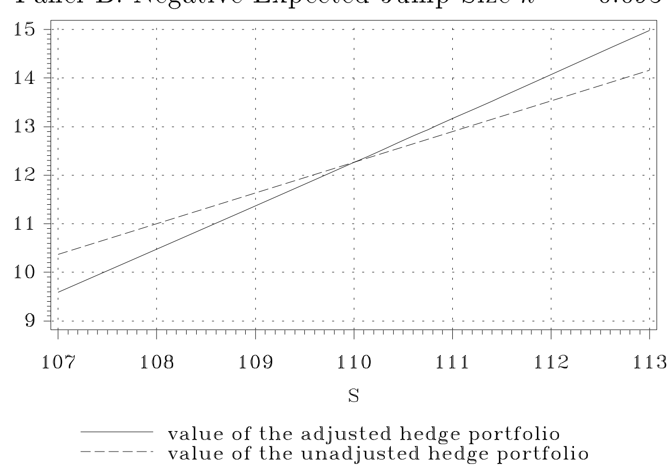
Figure 1: **Locally Risk-Minimizing Hedging Error due to Stock Price Changes**

The solid line depicts the value of the hedge portfolio as a function of the stock price, $F(S)$, for the parameters $r = 0.1$, $K = 100$, $T = 1/12$, $\alpha = 0.2$, $\lambda = 3$, $\sigma_{tot} = 0.3$ and $\gamma = 0.8$. The dashed line gives the value of the *unadjusted* hedge portfolio as a function of S when the hedge ratio ϕ as well as η are determined for $S_- = 100$ (Panel A) and $S_- = 110$ (Panel B), respectively, i.e. $\phi(S_-)S + \eta(S_-)B$. The slope of the dashed line is equal to ϕ. Since the hedge ratio does not correspond to the first derivative (it can be either greater or smaller) the dashed line is not the tangent to the portfolio value function. The hedging error due to a change in S (t fixed) equals the vertical differences between these two lines. It can be negative as well as positive. A hedging error appears if the stock price jumps, but even if the stock price changes continuously, the hedging error due to a change in S is not equal to zero.

Panel A: Positive Expected Jump Size $k = 0.105$



Panel B: Negative Expected Jump Size $k = -0.095$



where

$$\begin{aligned}
F^{BS} &\equiv \text{Black/Scholes value}, \\
\tau &\equiv T - t, \\
\lambda' &\equiv \lambda(k+1), \\
\sigma_n &\equiv \sigma_D^2 + \tfrac{n\sigma_J^2}{\tau}, \\
r_n &\equiv r - \lambda k + \tfrac{n\alpha_J}{\tau} \\
d_1(n) &\equiv (\ln(S/K) + (r_n + 0.5\sigma_n^2)\tau)/(\sigma_n\sqrt{\tau})\,.
\end{aligned}$$

According to Merton (1976) the hedging error equals

$$\begin{aligned}
dC &= \Delta F^{Me} - F_s^{Me}\Delta S \\
&\quad + \left(\lambda k S F_s^{Me} - \lambda E_L(\Delta F^{Me})\right) dt\,. \qquad (29)
\end{aligned}$$

Contrary to the locally risk-minimizing strategy the hedge ratio now consists of a diffusion component but not of a jump component. Hence the hedging error does not contain a diffusion component as can be see from equation (29). It now consists of two terms. The expression in the first line gives the hedging error in case of a jump, which is always positive since F^{Me} is a convex function in S. Figure 2 shows for representative parameters the hedging error of Merton's strategy due to changes in S (for fixed t). The second expression represents the hedging error that arises continuously in time. It is always negative thus compensating for the first term. We see that the total hedging error is not zero even if the stock price does not jump. The value of the hedge portfolio is again a weighted sum of Black/Scholes values. Although this strategy does not hedge jump risk the value of the hedge portfolio takes into account that with each jump the variance of the stock price changes.

In a formal sense, with the locally risk-minimizing hedging strategy we projected the martingale part of V on $\{\int \Theta dM\}$. It followed that $\langle M, \Gamma\rangle = 0$. Here, we only hedge with the diffusion component. Therefore, we project the martingale part of V on the space $\{\int \Theta dM^c | E(\int \Theta^2 d\langle M^c, M^c\rangle) < \infty\}$ consisting of all contingent claims that can be hedged by the diffusion part of the stock price process. It follows that $\langle \Gamma, M^c\rangle = 0$. From the stochastic differential equation of Γ, (16), as well as the orthogonality of M^c, and M^d and t (i.e. $\langle M^c, M^d\rangle = 0$, and $\langle M^c, t\rangle = 0$) we infer that $\langle \Gamma, M^c\rangle = 0$ is equivalent to the hedge ratio being equal to the first derivative of the hedge portfolio value:[10]

$$\phi = \frac{d\langle V^{Me}, M^c\rangle}{d\langle M^c, M^c\rangle} = V_z^{Me} = F_s^{Me} . \quad (30)$$

We know from Proposition 1 that setting $h = 1$ yields Merton's partial differential equation, and from Lemma 1 that we have $g = -(\alpha - r)/\sigma_D$. Inserting h as well as ϕ into equation (17) yields again the hedging error. We also derive that Merton's strategy is mean-self-financing.[11] For the martingale density we infer

$$\zeta_t = \exp\left\{ -\frac{(\alpha - r)}{\sigma_D} W_t - \frac{1}{2}\frac{(\alpha - r)^2}{\sigma_D^2} t \right\} . \quad (31)$$

Since ζ is a *strict* martingale density, $P^{g,h}$ is an equivalent martingale measure.[12]

3.3 Black/Scholes Hedging Strategy

We have seen that Merton's hedging strategy hedges diffusion risk completely but not jump risk. However, it takes into account that the jump component of the stock price process contributes to the variance of the process. We now consider a Black/Scholes hedging strategy which even ignores the variance of the jump component as discussed by Naik and Lee (1990). This would be the case if a Black/Scholes world is assumed and the variance of the process is estimated for a period where no jump occurs. Consequently, the value of the hedge portfolio corresponds to the Black/Scholes

[10] Since we can put the integrand ϕ in front and since $\langle A, \cdot\rangle = 0$ we get

$$\begin{aligned} d\langle \Gamma, M^c\rangle &= d\langle V - \int \phi dZ, M^c\rangle \\ &= d\langle V, M^c\rangle - \phi d\langle Z, M^c\rangle \\ &= d\langle V, M^c\rangle - \phi d\langle M^c, M^c\rangle . \end{aligned}$$

Orthogonality is thus equivalent to $\phi = d\langle M^c, V\rangle / d\langle M^c, M^c\rangle$. We have $d\langle M^c, M^c\rangle = \sigma_D^2 Z^2 dt$ and $d\langle V, M^c\rangle = V_z Z^2 \sigma_D^2 dt$ and thus $\phi = V_z = F_s$.

[11] Insert h and ϕ into equation (41) of the proof of Lemma 3 (see appendix). That yields the martingale condition for Γ

[12] We would obtain the same result by arguing thus: Since the variance of the jump component is perceived but jump risk is not hedged the density process ζ should not eliminate the jump component but leave it unchanged. This can be achieved by setting $h = 1$. From the martingale property of the density, equation (5), we have $g = \frac{-(\alpha - r)}{\sigma_D}$. Finally, requiring a mean-self-financing strategy yields $\phi = V_z$.

value based on the diffusion volatility, σ_D:

$$\begin{aligned} F^{BS} &= F^{BS}(S, K, \tau, r, \sigma_D) \\ &= SN(d_1(\sigma_D)) - Ke^{-r\tau}N(d_2(\sigma_D)) \end{aligned} \tag{32}$$

where

$$\begin{aligned} d_1 &= (\ln(S/K) + (r + 0.5\sigma_D^2)\tau)/(\sigma_D\sqrt{\tau}) \\ d_2 &= d_1 - \sigma_D\sqrt{\tau} \\ \tau &= T - t\,. \end{aligned}$$

The hedge ratio equals the first derivative of the portfolio value with respect to the stock price:

$$\phi = F_s^{BS} = N(d_1)\,. \tag{33}$$

Conditional on no jump this strategy is self-financing. However, contrary to the locally risk-minimizing strategy and to Merton's strategy this strategy is *not* mean-self-financing. If a jump occurs the hedging error is always positive,

$$dC = \Delta F^{BS} - F_s^{BS}\Delta S > 0\,, \tag{34}$$

since the Black/Scholes call value is a convex function in S. This hedge plan clearly 'underhedges' the option. According to Naik and Lee (1990), a perfect hedge requires a *jump financing security* – that is a security that pays exactly dC. Since such a security does not exist the approach of Naik and Lee (1990) is tantamount to totally ignoring the jump component of the stock price process. That means that the hedger acts as if he only observed the diffusion component and its volatility.

In a formal sense this approach is equivalent to choosing $h = 0$ as can be seen from the partial differential equation (13). Condition (5) for martingale densities then yields $g = -(\alpha - r - \lambda k)/\sigma_D$. Hence, the density process equals

$$\begin{aligned} \zeta_t &= \exp\left\{\frac{-(\alpha - r - \lambda k)}{\sigma_D}W_t - \frac{1}{2}\left(\frac{-(\alpha - r - \lambda k)}{\sigma_D}\right)^2 t - N_t + \lambda t\right\} \\ &\quad \cdot \prod_{s \le t}(1 - \Delta N_s)e^{\Delta N_s} \\ &= \exp\left\{\frac{-(\alpha - r - \lambda k)}{\sigma_D}W_t - \frac{1}{2}\left(\frac{-(\alpha - r - \lambda k)}{\sigma_D}\right)^2 t + \lambda t\right\} \cdot \mathbf{1}_{\{N_t = 0\}}\,. \end{aligned} \tag{35}$$

The martingale density is no longer signed[13]. However, it is not strictly positive but becomes zero when a jump occurs. That means that the martingale density assigns zero probability to jump events thus eliminating the explicit influence of the jump component on the hedge portfolio. Nonetheless, the value of the hedge portfolio jumps, when S jumps since the value of the hedge portfolio is a continuous function of S. Figure 3 illustrates for representative parameters the hedging error due to changes in S (t fixed). It shows that this hedging error – and thus the total hedging error since the money market account refinances the portfolio for infinitesimal changes – is positive if S jumps. This also implies an arbitrage opportunity if the call is traded for F^{BS}. We get:

Lemma 4 *In a Poisson jump diffusion model the value of a European call must be larger than the Black/Scholes value based on the diffusion variance.*

PROOF: Buy the call for F^{BS} and sell the hedge portfolio. Then the cash inflow equals the hedging error $dC = \Delta F^{BS} - F_s^{BS} \Delta S$ which is nonnegative with probability 1 and strictly positive with positive probability. Therefore, this strategy forms an arbitrage opportunity.

A Black/Scholes hedging strategy based on the total variance, σ_{tot}^2, corresponds to an investor who ignores jump risk but who uses historical volatility estimators from a period with stock price jumps. Hence the value of the hedge portfolio is

$$\begin{aligned} F^{BS} &= F^{BS}(S, K, \tau, r, \sigma_{tot}) \\ &= SN(d_1(\sigma_{tot})) - Ke^{-r\tau} N(d_2(\sigma_{tot}))\,. \end{aligned}$$

The hedge ratio and the hedging error are given by

$$\begin{aligned} \phi &= F_s^{BS}(\sigma_{tot}) = N(d_1(\sigma_{tot})) \\ dC &= \Delta F - F_s \Delta S - \frac{1}{2} F_{ss} S^2 \sigma_{jump}^2 dt\,, \end{aligned}$$

respectively. This Black/Scholes hedging strategy is no longer self-financing even if no jumps occur until the option's maturity: $dC = -0.5 F_{ss} S^2 \sigma_{jump}^2 dt$. [14]

[13] By 'signed' we mean negative with positive probability.

[14] The Black/Scholes value based on the total variance is not computable by changing the measure. The Girsanov transformation of a Brownian motion always yields a Brownian motion with the same variance. Therefore, no martingale density exists that could transform the jump diffusion process into a geometric Brownian Motion based on the total variance.

3.4 Hedging Strategies in Bates' Model

We now regard Bates' systematic jump risk model (1991) and assume that *all the jump risk* is purely systematic[15] and is therefore priced. Bates computes a value for European options in an equilibrium model where the representative investor has a time separable utility function with constant relative risk aversion R. The resulting option pricing formula for a European call corresponds to Merton's formula with a transformed jump part:

$$F^{Ba}(S, K, \tau, \sigma_D, r, \lambda, \alpha_J, \sigma_J, R) = F^{Me}(S, K, \tau, \sigma_D, r, \lambda^*, \alpha_J^*, \sigma_J)$$

where

$$\begin{aligned} \lambda^* &\equiv \lambda \exp(-R\alpha_J + 0.5R(1+R)\sigma_J^2) \\ \alpha_J^* &\equiv \alpha_J - R\sigma_J^2 \quad . \end{aligned}$$

F^{Ba} can be calculated with the help of the following martingale density

$$d\zeta_t = \zeta_{t^-} \frac{-(\alpha - r) - \lambda^* k^* + \lambda k}{\sigma_D} dW + \int \zeta_{t^-}((y+1)^{-R} - 1)(\nu(dy, dt) - H(dy)dt)$$

with $k^* = \exp(\alpha^*) - 1$. For $R = 0$ density and price collapse to those of Merton.

Assuming Bates' option price to be correct, we propose two hedging strategies where the values of the corresponding hedge portfolios track these prices all the time.[16]

Delta Hedge

A common way to hedge options is by choosing a Delta-hedging strategy. In analogy to the Black/Scholes model or the Merton model for jumps the hedge ratio is chosen equal to the first derivative of the call value with respect to the stock price:

$$\phi = F_s^{Ba} .$$

The hedging error is equal to

$$\begin{aligned} dC &= \Delta F^{Ba} - F_s^{Ba} \Delta S \\ &\quad + (F_s^{Ba} \lambda^* k^* S - E_{L^*}(\Delta F^{Ba}))dt . \end{aligned}$$

[15] I.e. S is perfectly correlated to the wealth process of the economy.

[16] Recall that in general the value of the LRM hedge portfolio is different from Bates' option price.

Locally Variance Minimizing Hedging Strategy (LVM)

As mentioned before, a strategy is *locally risk-minimizing* if the discounted hedging error Γ is a martingale which is orthogonal to the martingale part of the price process.[17] Loosely speaking, we minimize the local variance of the hedging error $d\langle\Gamma,\Gamma\rangle$ under the condition that the *mean* hedging error is zero at all times t. A similar hedging strategy for the Bates model reads as follows: Minimize the *local variance* of the hedging error $d\langle\Gamma,\Gamma\rangle$, under the condition that at all times t the value of the hedge portfolio equals Bates' model price. Again the hedge ratio ϕ can be obtained by projecting the martingale part of Bates' value, V^{Ba}, on $\{\int \Theta dM\}$.[18] This leads to a hedge ratio that has a very similar form to the locally risk-minimizing hedge ratio:[19]

$$
\begin{aligned}
\phi &= \frac{d\langle V^{Ba}, M\rangle}{d\langle M, M\rangle} = (1-\gamma)F_s^{Ba} + \gamma \frac{E_L(\Delta S \Delta F_s^{Ba})}{E_L((\Delta S)^2)} \\
&= (1-\gamma)\sum_n a_n^* N(d_1(n)) \\
&\quad +\gamma \frac{\lambda}{\sigma_{jump}^2} \sum_{n=0}^{\infty} a_n^* \Bigg\{ -k\, \frac{F^{BS}(S,K,r_n^*,\sigma_n,\tau)}{S} \\
&\quad +(k+1)^2 e^{\sigma_J^2}\, \frac{F^{BS}(S,K,r_{n+1}^{*''},\sigma_{n+1},\tau)}{S} \\
&\quad -(k+1)\, \frac{F^{BS}(S,K,r_{n+1}^{*'},\sigma_{n+1},\tau)}{S} \Bigg\} \; .
\end{aligned}
$$

where

$$
\begin{aligned}
a_n^* &\equiv \exp(-\lambda^*(k^*+1))\tfrac{(\lambda^*(k^*+1))^n}{n!} \\
r_n^* &\equiv r - \lambda^* k^* + n\alpha^*/\tau \\
r_n^{*'} &\equiv r_n^* + R\sigma_J/\tau \\
r_n^{*''} &\equiv r_n^* + (R+1)\sigma_J/\tau \; .
\end{aligned}
$$

This yields the following hedging error:

$$
\begin{aligned}
dC &= \Delta F^{Ba} - \phi \Delta S + (F_s^{Ba} - \phi) S \sigma_D dW \\
&\quad + \left\{ (F_s^{Ba} - \phi)((\alpha - r) - \lambda k) S - \lambda^* E^*(\Delta F^{Ba}) + F_s^{Ba} S \lambda^* k^* \right\} dt\, .
\end{aligned}
$$

Figure (4) shows for representative parameters the hedging error of these two hedging strategies due to changes in S (t fixed).

[17] Under regularity assumptions.

[18] However, note that this time Bates' value instead of the locally risk-minimizing portfolio value is projected.

[19] see appendix.

Figure 2: **Merton's Hedging Error due to Stock Price Changes**

The solid line depicts Merton's value as a function of the stock price, $F^{Me}(S)$, for the parameters $r = 0.1$, $K = 100$, $T = 1/12$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $\lambda = 3$, $k = 0.105$. The dashed line represents the value of the *unadjusted* hedge portfolio as a function of S where the hedge ratio ϕ as well as η are determined for $S_- = 100$, i.e. $\phi(S_-)S + \eta(S_-)B$. It corresponds to the tangent of the solid line since the hedge ratio equals the first derivative of the portfolio value with respect to the stock price. The hedging error caused by a change in the stock price (for fixed t) equals the vertical difference of these lines. It is zero when the change in S is only infinitesimal and it is strictly positive when the stock price jumps since the value function is a convex function of S. However, due to the position in the money market account this strategy is mean-self-financing when changes in t are considered.

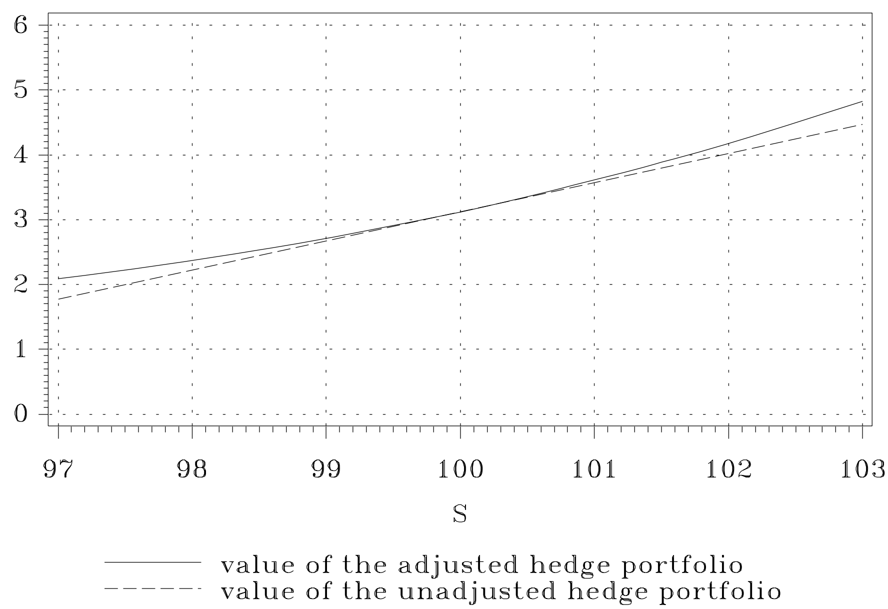
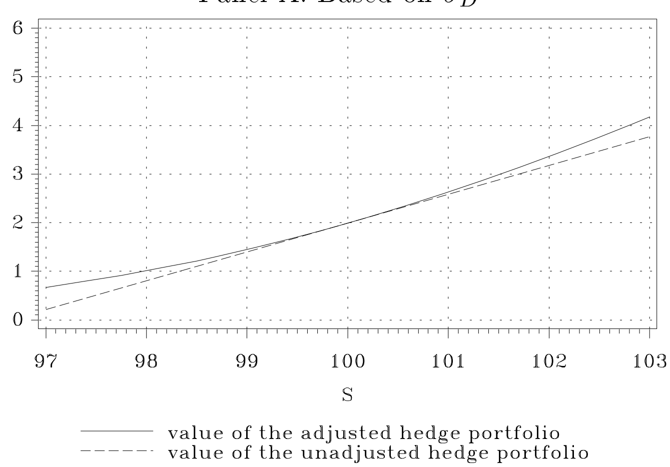


Figure 3: **Black/Scholes Hedging Error due to Stock Price Changes**

The solid lines depict the Black/Scholes values as a function of the stock price, $F^{BS}(S)$, for the parameters $r = 0.1$, $K = 100$, $T = 1/12$ $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $\lambda = 3$ and $k = 0.105$. Panel A is based on σ_D and Panel B on σ_{tot}. The dashed lines represent the value of the *unadjusted* hedge portfolio as a function of S where the hedge ratio ϕ as well as η are determined for $S_- = 100$, i.e. $\phi(S_-)S + \eta(S_-)B$. They correspond with the tangents to the solid lines in S_- since $\phi = F_s^{BS}$. The hedging error due to changes in S (t fixed) equals the vertical differences between these two lines. It is always positive since the value function is a convex function in S. A hedging error appears if the stock price jumps, but if the stock price changes continuously, the hedging error due to changes in S is equal to zero. Whereas in Panel A the total hedging error is also zero for continuous changes in S (the money market account finances the rearrangements of the portfolio) this is not the case in Panel B.

Panel A: Based on σ_D



Panel B: Based on σ_{tot}

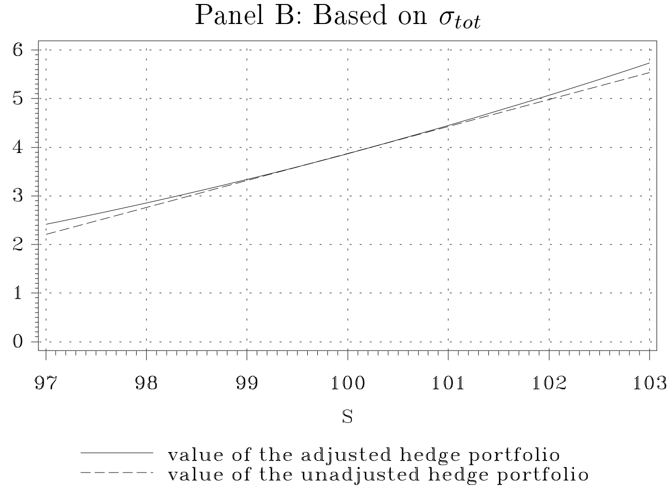
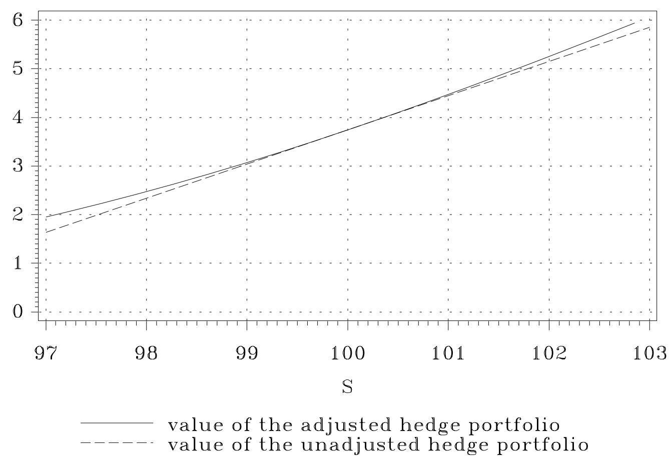


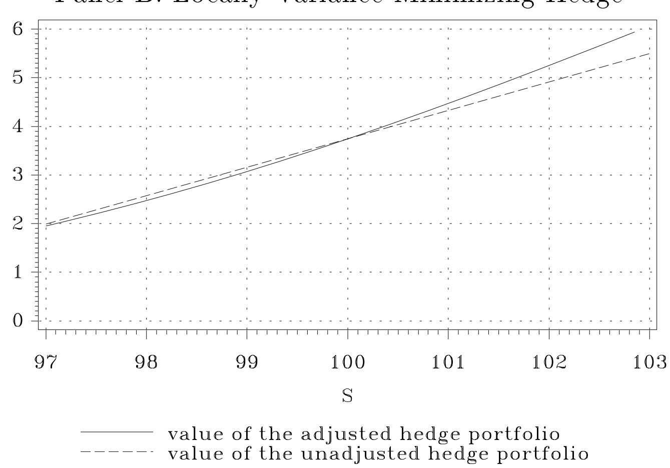
Figure 4: **Bates' Hedging Error due to Stock Price Changes**

The solid lines depict the Bates values as a function of the stock price, $F^{Ba}(S)$, for the parameters $r = 0.1$, $K = 100$, $T = 1/12$ $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $\lambda = 3$ and $k = 0.105$. Panel A shows the hedging error based on the Delta hedge and Panel B based on the locally variance-minimizing hedge. The dashed lines represent the value of the *unadjusted* hedge portfolio as a function of S where the hedge ratio ϕ as well as η are determined for $S_{-} = 100$, i.e. $\phi(S_{-})S + \eta(S_{-})B$. The hedging error due to changes in S (t fixed) equals the vertical differences between these two lines. In Panel A the dashed line corresponds with the tangent to the solid line in S_{-} since the $\phi = F_s^{Ba}$. Consequently the hedging error due to changes in S is always positive as in Merton's or the Black/Scholes case. In Panel B the hedge ratio is not equal to the first derivative. The hedging error due to changes in S can be either positive or negative.

Panel A: Delta Hedge



Panel B: Locally Variance Minimizing Hedge



3.5 Arbitrage Boundaries

As we have noted in section 2.2, the Poisson jump diffusion model contains no arbitrage opportunities. With the help of Merton's model we can give boundaries for the arbitrage-free call value.

Proposition 4 *There are no arbitrage opportunities for a single call if and only if the call value F is smaller than the stock price and larger than the Black/Scholes value based on the diffusion volatility:*

$$F \in \,] \, F^{BS}(\sigma_D), S \, [\, . \tag{36}$$

PROOF: If we choose the function h to be positive and constant and determine g according to equation (5) the corresponding density process ζ is strictly positive and thus determines an equivalent martingale measure $P^{g,h}$ (see equation (6)). The partial differential equation for the value of the hedge portfolio $F = F(g,h)$ coincides with Merton's differential equation but with a transformed jump parameter $\hat{\lambda} = \lambda \cdot h$. We derive $F(g,h) = F^{Me}(\hat{\lambda})$. Therefore, all call values $F \in \left\{ F^{Me}(\hat{\lambda}) : \hat{\lambda} > 0 \right\}$ can be written as expected discounted values with respect to the equivalent martingale measure $P^{g,h}$, thus allowing no arbitrage. Furthermore, it can be shown that (for σ_D, σ_J, k constant) we get $\lim_{\hat{\lambda} \to 0} F^{Me}(\hat{\lambda}) = F^{BS}(\sigma_D)$ and also $\lim_{\hat{\lambda} \to \infty} F^{Me}(\hat{\lambda}) = S$, independent of the sign of the expected percentage jump size k. The latter is remarkable because for $\hat{\lambda} \to \infty$ the stock price converges to zero in measure. It follows that for a single call all values in the open interval $\left\{ F^{Me}(\hat{\lambda}) : \hat{\lambda} > 0 \right\} =]F^{BS}(\sigma_D), S[$ are feasible without creating arbitrage opportunities. On the other hand we can determine simple arbitrage strategies if the call value equals S and $F^{BS}(\sigma_D)$ (as in Lemma 4), respectively. Therefore, S and F^{BS} are strict boundaries. All call values calculated in the following subsections satisfy these boundaries.

4 Numerical Analysis

We are now interested in the sensitivity of the LRM portfolio value and hedge ratio with respect to the model parameters. Therefore, we compute portfolio values and hedge ratios for typical parameter values. We want to focus especially on changes in the money ratio and in the expected percentage jump size. Furthermore, we analyse the differences between the hedge ratios of the LRM and LVM strategies and the alternative strategies. Finally, we perform a Monte Carlo simulation to compare the distribution of total hedging costs.

4.1 LRM Strategy for Representative Parameters

Table 1 shows how the locally risk-minimizing portfolio value depends on the exercise price K, time to expiration T, jump intensity λ, and the expected jump size per unit time λk. We take the total stock variance, σ_{tot}^2, as well as the percentage of the total stock variance explained by the jump, γ, as constant. Accordingly, the variance of the diffusion component σ_D^2 as well as the variance of the jump component σ_{jump}^2 are constant and the variance of the jump size in the return σ_J^2 decreases as k increases. As expected, we find that the value of the hedge portfolio is an increasing function of time to maturity and a decreasing function of the strike price. For given jump intensity λ, with increasing magnitude of the expected jump size per unit time λk, the value of the portfolio decreases, except for out-of-the money calls with a short time to expiration where the converse is true. This observation confirms the result of Trautmann/Beinert (1995) who find a similar relationship in the idiosyncratic jump risk model of Merton (1976) and the systematic jump risk model of Bates (1991). The influence of λk on the hedge portfolio's value decreases with increasing λ. This reflects the fact that for constant total variance, as λ increases, the jump component converges to a second independent diffusion process implying that the whole process is again a geometric Brownian motion. In this case we know that the drift $\alpha = \alpha_D + \lambda k$ has no influence on the option value. Consequently, λk as the drift of the jump component looses its influence for increasing λ.

Table 2 shows the corresponding hedge ratios. All other parameters kept constant the hedge ratio decreases with increasing strike price. The relationship between the hedge ratio and time to maturity is similar to that in the Black/Scholes model (see, e.g., Cox/Rubinstein (1985, p. 223)). The hedge ratio always increases with the drift of the jump component, λk. Our intuitive explanation for this result reads

Table 1: **Values of the Locally Risk-Minimizing Hedge Portfolio**

		$S = 100$, $\alpha = 0.15$, $r = 0.1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$					
		$\lambda = 1$			$\lambda = 10$		
λk	K	$T = 1/12$	$T = 1/2$	$T = 1$	$T = 1/12$	$T = 1/2$	$T = 1$
	90	11.64	18.16	23.94	11.30	17.16	22.66
–0.1	100	2.96	11.08	17.75	3.60	10.91	16.82
	110	0.29	5.63	12.44	0.67	6.41	12.14
	90	11.28	16.89	22.31	11.22	17.01	22.47
0	100	2.72	9.79	15.91	3.53	10.77	16.64
	110	0.45	4.98	10.78	0.74	6.34	11.99
	90	10.96	15.76	20.85	11.14	16.85	22.29
0.1	100	2.56	8.91	14.52	3.48	10.63	16.46
	110	0.63	4.85	9.89	0.81	6.28	11.84

as follows: While the Black/Scholes hedging strategy adjusts the hedge ratio *after* a jump occurred, the locally risk-minimizing strategy obviously *anticipates*, on an average basis, the hedge ratio appropriate after the next jump. Therefore, it is smaller for small λk and larger for large λk. Again, the larger λ the smaller is the influence of λk on the hedge ratio. As argued before, this results from the convergence of the jump component to a geometric Brownian motion.

While the hedging strategies of Merton and Black/Scholes do *not* depend on the expected excess return $(\alpha - r)$ the locally risk-minimizing hedging strategy does. Figure 5 shows, for different k, the dependence of the locally risk-minimizing portfolio value on the excess return. As before, we fix the total stock variance as well as the variance explained by the jump component while the volatility of the jump size in the return σ_J is chosen accordingly. For $k = 0$, as $(\alpha - r)$ increases, the portfolio value varies only slightly. However, for $k = -0.1$, as $(\alpha - r)$ increases so does the portfolio value, whereas for $k = 0.1$, the portfolio value is a decreasing function of $(\alpha - r)$. Figure 5 also depicts the dependence of the hedge ratio with respect to the excess return. The hedge ratio increases with the excess return independent of k.

Table 2: **Values of the Locally Risk-Minimizing Hedge Ratio**

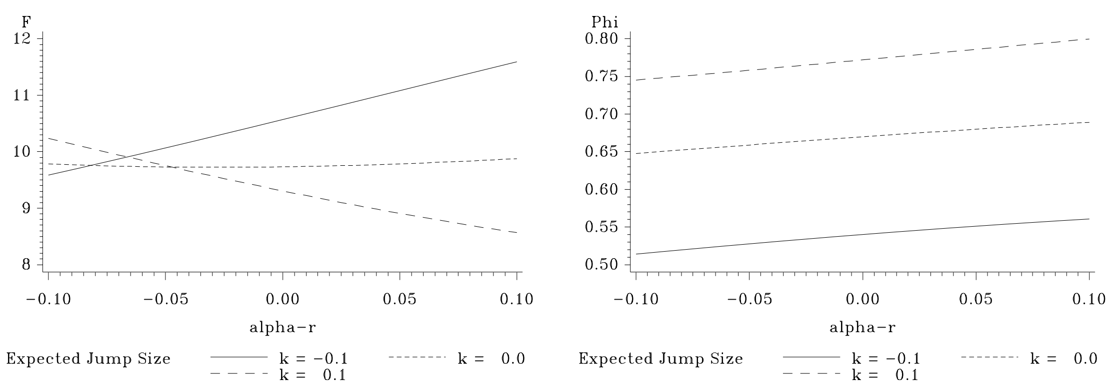
		$S = 100,\ \alpha = 0.15,\ r = 0.1,\ \sigma_{tot} = 0.3,\ \gamma = 0.8$					
		$\lambda = 1$			$\lambda = 10$		
λk	K	$T = 1/12$	$T = 1/2$	$T = 1$	$T = 1/12$	$T = 1/2$	$T = 1$
	90	0.63	0.68	0.73	0.85	0.79	0.79
–0.1	100	0.42	0.55	0.63	0.53	0.63	0.68
	110	0.20	0.40	0.52	0.20	0.45	0.56
	90	0.80	0.80	0.81	0.87	0.80	0.80
0	100	0.61	0.68	0.72	0.57	0.64	0.69
	110	0.39	0.53	0.62	0.23	0.47	0.57
	90	0.93	0.90	0.89	0.90	0.82	0.81
0.1	100	0.78	0.79	0.80	0.61	0.66	0.70
	110	0.54	0.63	0.69	0.26	0.48	0.58

Figure 5: **Excess Return and the Locally Risk-Minimizing Strategy**

$S = 100,\ K = 100,\ r = 0.1,\ T = 1/2,\ \lambda = 1,\ \sigma_{tot} = 0.3,\ \gamma = 0.8$

Hedge Portfolio Value

Hedge Ratio



4.2 Comparison of Hedging Strategies

As mentioned above, for $(\alpha - r) = 0$ the portfolio value of the locally risk-minimizing strategy is equal to Merton's value. Therefore, one can also infer from Figure 5 how the portfolio values of these two strategies differ for different excess returns. For $k = 0$ the portfolio values are very similar, independent of the excess return. On the other hand the portfolio values differ considerably if the absolute values for k and $(\alpha - r)$ are high.

Figure 6 illustrates for different parameters k, the hedge ratios of the alternative strategies as functions of the stock price. If the expected excess return is positive and the expected jump size is significantly different from zero and positive (negative) then the LRM and LVM hedge ratios are substantially larger (smaller) than the delta hedge ratios for out-of-the-money (in-the-money) calls. Moreover, it shows that the LRM and LVM hedge ratios are least sensitive to changes in the stock price. This is a very important property because it means that changes in the underlying stock require smaller adjustments in these risk-minimizing hedge ratios. Therefore, the corresponding transaction costs and the hedging error due to discrete adjustments of the portfolio are smaller than for the remaining strategies. Although not explicitly presented here the lower sensitivity of the locally risk-minimizing hedge ratio with respect to the stock price remains true even if the excess return is zero, $(\alpha - r) = 0$. Recall that in this case Merton's value is equal to the locally risk-minimizing portfolio value while the hedge ratios are different. As shown in Figure 7, for fixed parameters σ^2_{tot} and γ, Merton's value and the portfolio value of the locally risk-minimizing strategy converge if λ goes to infinity. The same is true for the hedge ratios. This again is due to the fact that in this case the price process converges to a diffusion process and thus to a complete model where *all* prices and hedging strategies must be equivalent. The difference in the Merton strategy and the locally risk-minimizing strategy is largest for small λ. If, on the other hand, λ *as well as* σ^2_{jump} go to zero we also have convergence of the portfolio values and the hedge ratios for *all* strategies since in this case the process converges to a Brownian motion with volatility σ^2_D.

4.3 Monte Carlo Simulation of Total Hedging Costs

A Monte Carlo Simulation[20] based on 10,000 sample paths is used to derive the frequency distribution of the total (discounted) hedging costs for all alternative

[20]We used the empirical martingale simulation method of Duan and Simonato (1996).

Figure 6: **The Hedge Ratios as a Function of the Stock Price**

Parameters: $K = 100$, $r = 0.1$, $T = 1/12$, $\alpha = 0.15$, $\lambda = 1$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $R = 1$.

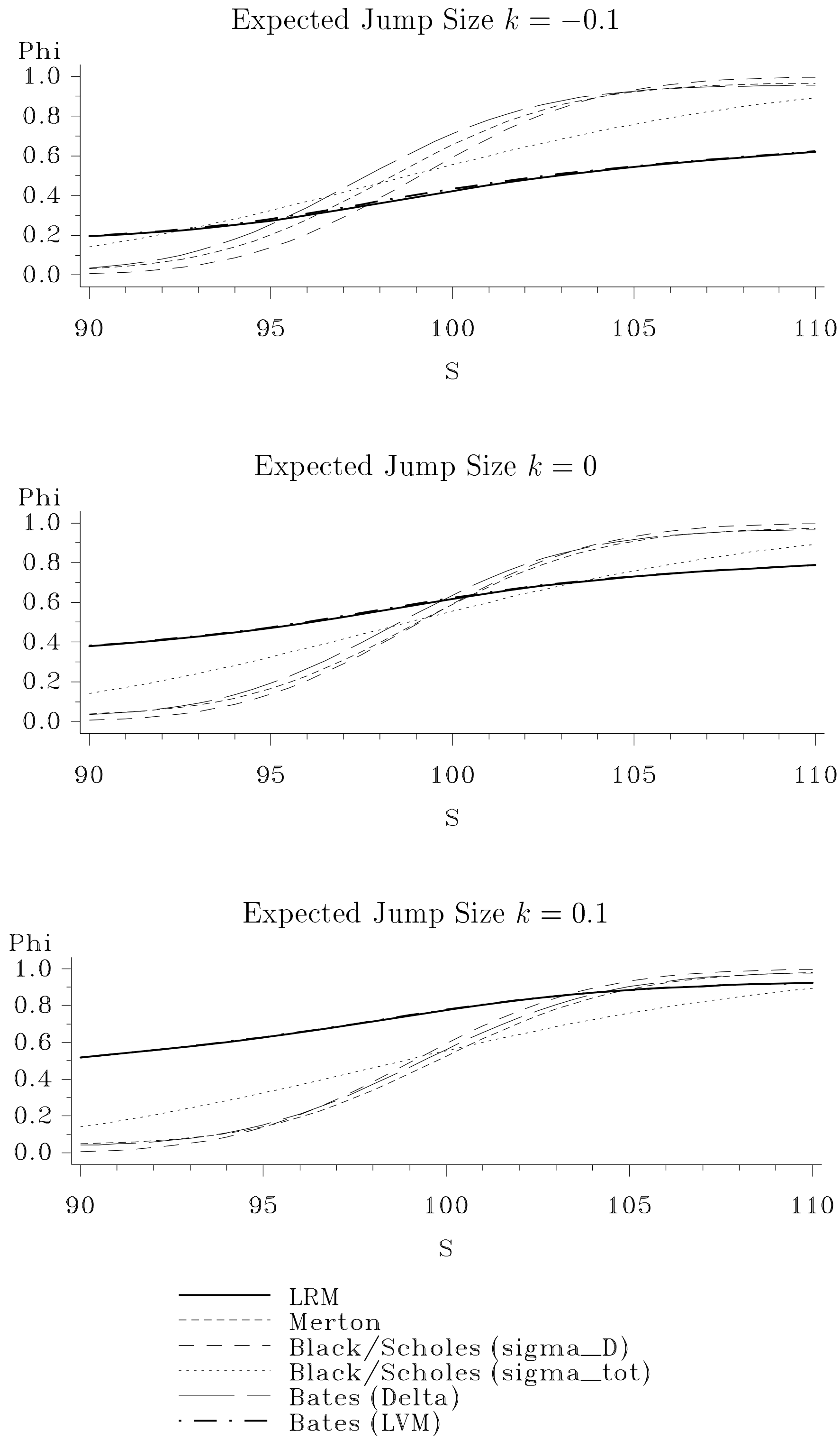
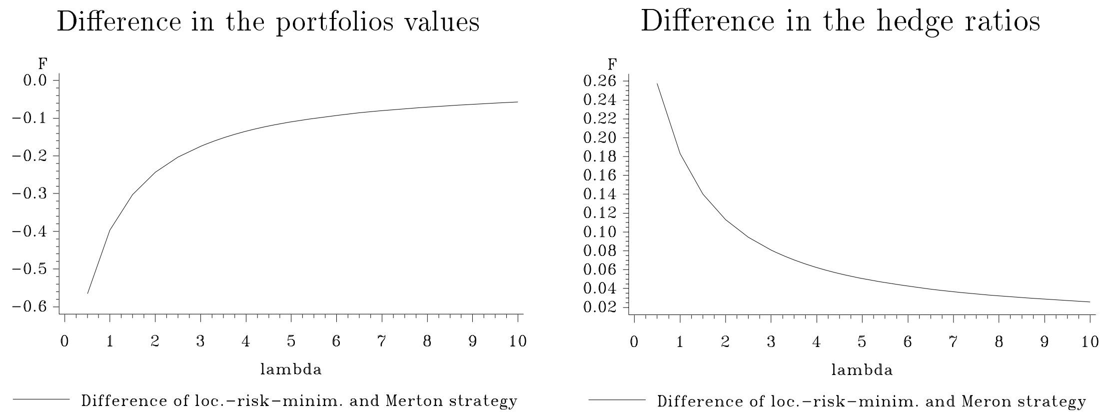


Figure 7: **Difference of LRM hedge and Merton's hedge as function of λ**

Parameters: $K = 100$, $S = 100$, $r = 0.1$, $T = 1/2$, $\alpha = 0.15$, $\lambda k = 0$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$.



hedging strategies. The hedge portfolio was adjusted only once a week; this discrete rebalancing of the hedge portfolio results in an additional error term since in a strict sense the hedge ratios are determined for continuous adjustments. Theoretically, the initial costs and the mean costs of the LRM strategy and Merton's strategy should coincide; the divergence is mainly due to this discrete hedging policy. Table 3 shows the *initial hedging costs* as well as *mean, standard deviation, skewness, kurtosis, minimum, maximum* and nine *quantiles* of the total (discounted) costs for hedging a European call, where the expected jump is of considerable size, $k = -0,2$. It is remarkable that the initial hedging costs, Γ_0, differ by more than 9.50 dollars between Bates model and the Black/Scholes model based on σ_D, whereas the difference in mean hedging costs amounts to only about 1 dollar. Standard deviation and skewness are smallest for the LRM and LVM strategies, but kurtosis is highest for the LRM hedge. The quantiles for the LRM strategy and the LVM strategy lie close together. It is striking that the worst case behaviour is best for the LRM strategy and LVM strategy: the 99%-, 95%- and 90%-quantiles of the total hedging costs are substantially lower. E.g., the total hedging costs are above 34.31 (34.20) dollars only in 1% of the cases when applying the LRM (LVM) strategy. However, using a delta hedging strategy costs more than 43.28, 43.43, 44.41 and 48.31 dollars, respectively, with 1% probability. In 5% of the cases a delta hedging strategy requires more than 33.83, 34.66, 35.27, and 37.79 dollars, respectively, whereas 27.39 (27.52) dollars are sufficient with a LRM (LVM) approach. Figure 8 visualizes the empirical frequency distribution of the total hedging costs. It can be seen, that the mode of the total hedging costs is lowest for the Black/Scholes strategy based on σ_D.[21] The alternative delta-hedging strategies have higher modes of total hedging costs. The modes are highest for the LRM and LVM strategies.

For a less extreme jump size, k, the differences in the shapes of the frequency distributions are similar though not as pronounced.

Table 3: **Simulated Distribution of Total Hedging Costs**

$S = 100,\ K = 100,\ \lambda = 1,\ k = -0.2,\ \sigma_{tot} = 0.3,\ \gamma = 0.8,\ T = 1,\ \alpha = 0.15,\ r = 0.1,\ R = 1$

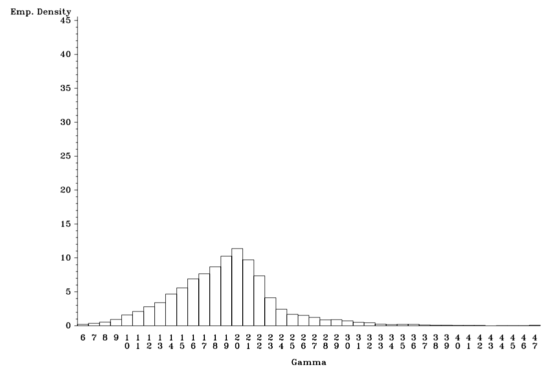
	Hedging Strategies					
Total Hedging Costs	LRM	Bates (LVM)	Bates (Delta)	Merton	B/S (σ_{tot})	B/S (σ_D)
(Initial Costs)	(19.00)	(20.74))	(20.74)	(18.07)	(16.73)	(11.20)
Mean	18.91	18.86	18.01	17.96	18.08	17.80
Stand. Dev.	4.92	4.95	7.89	8.18	7.41	9.51
Skewness	0.82	0.81	1.55	1.57	1.73	1.51
Kurtosis	3.08	2.80	1.95	1.99	2.91	1.71
Maximum	55.84	55.46	57.94	59.82	60.73	65.56
99% Quantile	34.31	34.20	43.43	44.41	43.28	48.31
95% Quantile	27.39	27.52	34.66	35.27	33.83	37.79
90% Quantile	24.16	24.40	30.62	30.65	29.36	32.47
75% Quantile	21.28	21.25	21.18	21.85	21.05	23.11
50% Quantile	18.96	18.87	14.79	14.49	14.69	11.76
25% Quantile	15.92	15.82	12.97	12.69	13.57	11.27
10% Quantile	12.95	12.89	11.42	11.48	12.41	11.10
5% Quantile	11.19	11.17	10.60	10.81	11.60	11.00
1% Quantile	8.33	8.33	9.42	9.34	10.02	10.71
Minimum	4.75	4.78	8.24	8.80	7.71	9.52

[21] If no jump occurs, the costs should coincide with the initial costs since only in case of a jump a hedging error occurs (when ignoring the hedging error due to the discrete rebalancing).

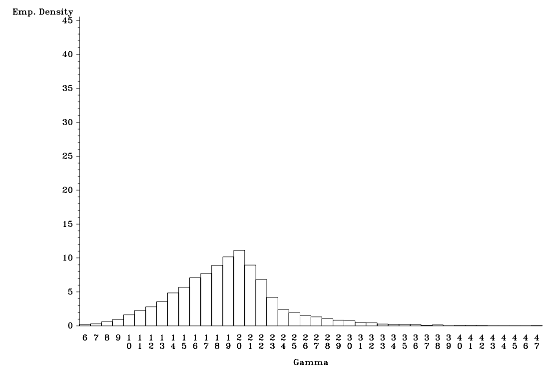
Figure 8: **Distribution of Total Hedging Costs**

$S = 100$, $K = 100$, $\lambda = 1$, $k = -0.2$, $\sigma_{tot} = 0.3$, $\gamma = 0.8$, $T = 1$, $\alpha = 0.15$, $r = 0.1$, $R = 1$. The heights of the columns give the frequency (in %) that the duplicating costs are between $\Gamma - 0.5$ and $\Gamma + 0.5$. At 47 the frequency that the duplication costs are above 46.5 is depicted.

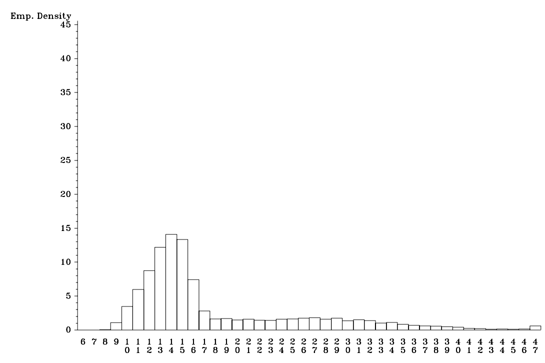
Panel A: Locally Risk-minimizing strategy



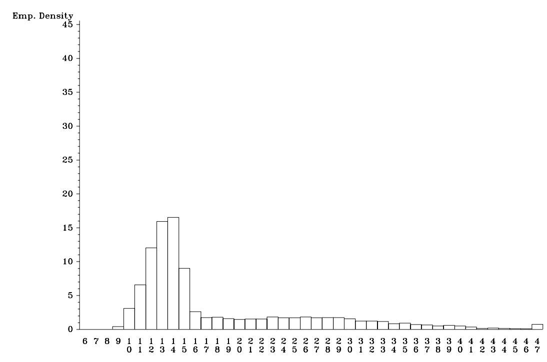
Panel B: Bates' strategy (LVM)



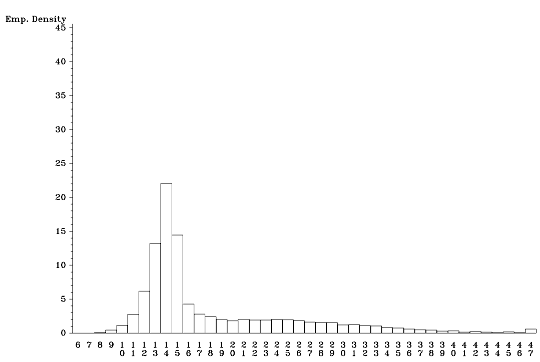
Panel C: Bates' strategy (Delta)



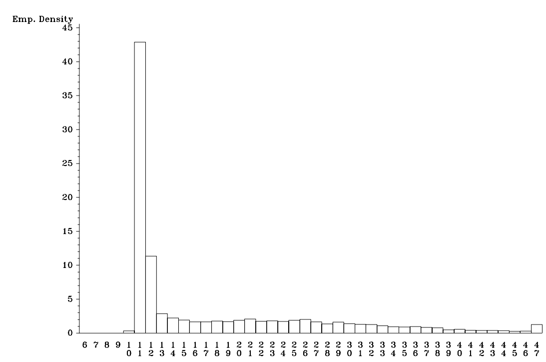
Panel D: Merton's strategy



Panel E: Black/Scholes strategy (σ_{tot})



Panel F: Black/Scholes strategy (σ_D)



5 Conclusion

This paper derives explicitly two formulae for hedging options in a jump diffusion framework. The first formula is an application of Schweizer's (1991) LRM approach while hedging according to the second formula minimizes the local variance of the hedging error in Bates' (1991) equilibrium option pricing model. Both hedge formulae consist of a diffusion component and a jump component and thus hedge diffusion as well as jump risk partly. The value of the LRM hedge portfolio is a weighted sum of Black/Scholes values. Some of the weighting factors are negative if, for example, the market price of risk is positive. Negative weighting factors result if the martingale density is negative with positive probability. In this case the corresponding martingale measure is not an equivalent martingale measure in contrast to the assumptions of Colwell and Elliott (1993). However, as long as the call value is above the Black/Scholes value based on the diffusion variance and below the stock price itself there is no arbitrage opportunity.

We compared the LRM strategy and LVM strategy with delta hedging strategies in the models of Merton (1976), Black/Scholes (1973), and Bates. Delta hedging strategies only consist of a diffusion component and thus hedge diffusion risk but do not hedge jump risk. By numerical analysis we found that the LRM and LVM hedge ratios are less sensitive to stock price changes than the alternative delta hedging strategies. Moreover, if the expected excess return is positive and the expected jump size is significantly different from zero and positive (negative) then the LRM and LVM hedge ratios are substantially larger (smaller) than delta hedge ratios for out-of-the-money (in-the-money) calls. We also showed that the worst case behaviour of the LRM and LVM strategies is superior: The 99%-, 95%-, and 90% quantiles of the total hedging costs are substantially lower.

6 Appendix

In the following all integrand processes $Z, \zeta, V, V_t \ldots$ should be regarded at time t^- and at $Z(t^-)$. But for the sake of simplicity we omit the argument.

Proof of Lemma 1

With the definition of the quadratic covariation, $[\cdot,\cdot]$ (see e.g. Protter (1990, p. 58) or Jacod/Shiryaev (1987, p. 52)) and since $[Z,\zeta] - \langle Z,\zeta\rangle$ is a local martingale (see e.g. Protter (1990, p. 98)) we get for the product of ζ and Z

$$\begin{aligned}
d\zeta Z &= Zd\zeta + \zeta dZ + d[Z,\zeta] \\
&= Zd\zeta + \zeta dZ + d[Z,\zeta] - d\langle Z,\zeta\rangle + d\langle Z,\zeta\rangle \\
&= Z\zeta \left\{ gdW + \int (h-1)(d\nu - H(dy)dt) \right\} \\
&\quad + Z\zeta \left\{ (\alpha - r)dt + \sigma_D dW + \int y(d\nu - H(dy)dt) \right\} \\
&\quad + d[Z,\zeta] - d\langle Z,\zeta\rangle + Z\zeta \left\{ \sigma_D g dt + \int (h-1) y H(dy) dt \right\} \\
&= \text{local martingale} + Z\zeta \left\{ (\alpha - r) + \sigma_D g + \int (h-1) y H(dy) \right\} dt\,.
\end{aligned}$$

Therefore ζZ is a local martingale and ζ a martingale density if and only if the last term is a local martingale. But since this is a continuous martingale with finite variation it follows that this martingale must be constant (see, e.g., Protter (1990, p. 64) and thus:

$$\left\{ (\alpha - r) + \sigma_D g + \int (h-1) y H(dy) \right\} \mathbf{1}_{\{\zeta \neq 0\}} = 0\ . \tag{37}$$

For the Black/Scholes hedge with $h = 0$ it follows that $(\alpha - r) + \sigma_D g - \lambda k = 0$ and thus $g = -(\alpha - r - \lambda k)/\sigma_D$. For Merton's hedge we get $h = 1$. Consequently $(\alpha - r) + \sigma_D g = 0$ and thus $g = -(\alpha - r)/\sigma_D$.

Proof of Proposition 1

With Ito's formula as well as $d\langle Z^c, Z^c\rangle = Z^2\sigma_D^2 dt$ and $\int yZ\nu(dy,dt) = \Delta Z = Z(t) - Z(t^-)$ we derive for V

$$
\begin{aligned}
dV &= V_t dt + V_z dZ + \frac{1}{2} V_{zz} Z^2 \sigma_D^2 dt + \Delta V - V_z \Delta Z \\
&= V_t dt + V_z \left\{ (\alpha - r) Z dt + \sigma_D Z dW + \int yZ(\nu(dy,dt) - H(dy)dt) \right\} \\
&\quad + \frac{1}{2} V_{zz} Z^2 \sigma_D^2 dt + \Delta V - V_z \Delta Z \\
&= V_z \sigma_D Z dW + \Delta V \\
&\quad + \left\{ V_t + V_z(\alpha - r) Z + \frac{1}{2} V_{zz} Z^2 \sigma_D^2 - V_z \int ZyH(dy) \right\} dt\,.
\end{aligned}
\tag{38}
$$

According to the definition of the quadratic covariation, since $[V,\zeta] - \langle V,\zeta\rangle$ is a local martingale and since $\int \Big(V(Z(1+y)) - V(Z)\Big)\nu(dy,dt) = \Delta V = V(t) - V(t^-)$ the product $V\zeta$ equals

$$
\begin{aligned}
dV\zeta &= Vd\zeta + \zeta dV + d[V,\zeta] \\
&= Vd\zeta + \zeta dV + d[V,\zeta] - d\langle V,\zeta\rangle + d\langle V,\zeta\rangle \\
&= Vd\zeta + \zeta V_z \sigma_D Z dW + \zeta \Delta V \\
&\quad + \zeta \left\{ V_t + V_z(\alpha - r) Z + \frac{1}{2} V_{zz} Z^2 \sigma_D^2 - V_z \int ZyH(dy) \right\} dt \\
&\quad + d[V,\zeta] - d\langle V,\zeta\rangle \\
&\quad + \left\{ V_z \sigma_D Z \zeta g dt + \int \zeta \Big(V(Z(1+y)) - V(Z)\Big)(h-1)H(dy) \right\} dt \\
&= \underbrace{Vd\zeta + \zeta V_z \sigma_D Z dW}_{\text{local martingale}} \\
&\quad + \underbrace{\int \zeta \Big(V(Z(1+y)) - V(Z)\Big)(\nu - H(dy)dt) + d[V,\zeta] - d\langle V,\zeta\rangle}_{\text{local martingale}} \\
&\quad + \zeta \left\{ V_t + V_z(\alpha - r) Z + \frac{1}{2} V_{zz} Z^2 \sigma_D^2 + V_z \sigma_D Z g - V_z \lambda Z k \right. \\
&\quad \left. + \int \Big(V(Z(1+y)) - V(Z)\Big) hH(dy) \right\} dt\,.
\end{aligned}
$$

Since $(V\zeta)_t = E(c(B_T Z_T)\zeta_T / B_T | \mathcal{F}_t)$ is a local martingale so must be

$$
\begin{aligned}
&\int_0^r \zeta \left\{ V_t + V_z(\alpha - r) Z + \frac{1}{2} V_{zz} Z^2 \sigma_D^2 + V_z \sigma_D Z g - V_z \lambda Z k \right. \\
&\quad \left. + \int \Big(V(Z(1+y)) - V(Z)\Big) hH(dy) \right\} dt\,.
\end{aligned}
\tag{39}
$$

This is a continuous local martingale with finite variation. Therefore it is zero. Consequently, so is the term in $\{\}$ if $\zeta \neq 0$. Thus V satisfies the following differential equation:

$$V_t + V_z(\alpha - r)z + \frac{1}{2}V_{zz}z^2\sigma_D^2 + V_z\sigma_D zg - V_z\lambda zk \\ + \int \Big(V(z(1+y)) - V(z)\Big)hH(dy) = 0 \tag{40}$$

Since ζ is a martingale density condition (37) must be satisfied. Inserting this into equation (40) we get equation (9). On the other hand if we insert equation (9) into the stochastic differential equation for V, according to (38), we get equation (10). With $V = e^{-rt}F$ equation (11) follows. ∎

Proof of Lemma 2

We required Γ to be orthogonal to the martingale part of the stock price process M. Since we can put the integrand ϕ in front and since $\langle A, M\rangle = 0$ we get

$$\begin{aligned} d\langle \Gamma, M\rangle &= d\langle V - \int \phi dZ, M\rangle \\ &= d\langle V, M\rangle - \phi d\langle Z, M\rangle \\ &= d\langle V, M\rangle - \phi d\langle M, M\rangle \,. \end{aligned}$$

Orthogonality is thus equivalent to

$$\phi = \frac{d\langle M, V\rangle}{d\langle M, M\rangle}\,.$$

V is a special semimartingale and can thus be written as

$$dV = \underbrace{(\ldots)dt}_{dA_V} + \underbrace{V_z\sigma_D ZdW}_{dV^c} + \underbrace{\int \Big(V(Z(1+y)) - V(Z)\Big)(\nu(dy,dt) - H(dy)dt)}_{dV^d}$$

where A_V is a predictable process with finite variation, V^c is a continuous martingale and V^d is a purely discontinuous martingale. We have $\langle A_V, M\rangle = 0$. Moreover the instantaneous variances can be depicted as $d\langle M^c, M^c\rangle = \sigma_D^2 Z^2 dt$ as well as $d\langle M^d, M^d\rangle = \sigma_{jump}^2 Z^2 dt$ and $d\langle M, M\rangle = \sigma_{tot}^2 Z^2 dt$. We can therefore write

$$
\begin{aligned}
\phi &= \frac{d\langle M^c + M^d, A_V + V^c + V^d\rangle}{d\langle M, M\rangle} \\
&= \frac{d\langle M^c, V^c\rangle + d\langle M^d, V^d\rangle}{d\langle M, M\rangle} \\
&= \frac{d\langle M^c, M^c\rangle}{d\langle M, M\rangle} \cdot \frac{d\langle M^c, V^c\rangle}{d\langle M^c, M^c\rangle} + \frac{d\langle M^d, M^d\rangle}{d\langle M, M\rangle} \cdot \frac{d\langle M^d, V^d\rangle}{d\langle M^d, M^d\rangle} \\
&= (1-\gamma)V_z + \gamma \frac{E_L(\Delta Z \Delta V)}{E_L((\Delta Z)^2)} \\
&= (1-\gamma)F_s + \gamma \frac{E_L(\Delta S \Delta F)}{E_L((\Delta S)^2)} .
\end{aligned}
$$

Proof of and Lemma 3

From equation (16) it follows that Γ is a (local) martingale if and only if

$$
\begin{aligned}
&\int \Big\{ -V_z Z(\alpha - r) - V_z Z \int (h-1) y H(dy) \\
&- \int \Big(V(Z(1+y)) - V(Z) \Big)(1-h) H(dy) + \phi(\alpha - r) Z \Big\} \, dt
\end{aligned}
$$

is a local martingale (since all other terms on the right are local martingale). But this is a continuous process with finite variation and thus it is equivalent to

$$
\begin{aligned}
0 &= \{ -V_z z(\alpha - r) - V_z z \int (h-1) y H(dy) \\
&\quad - \int \Big(V(z(1+y)) - V(z) \Big)(1-h) H(dy) + \phi(\alpha - r) z \} . \qquad (41)
\end{aligned}
$$

In addition ζ should be a martingale density. Thus we have equation (37). Together with equation (19) we find that the following functions g and h satisfy conditions (41) and (37) for all options $c(S_T)$:

$$
g = -\frac{(\alpha - r)\sigma_D}{\sigma_{tot}^2} \qquad h = 1 - \frac{(\alpha - r) y}{\sigma_{tot}^2} .
$$

Derivation of the locally variance minimizing strategy in Bates' model

For each hedge ratio ϕ the (discounted) hedging error is

$$\Gamma_t = V_t^{Ba} - V_0^{Ba} - \int_0^t \phi dZ \, .$$

We want to find a strategy such that the local variance $d\langle\Gamma,\Gamma\rangle_t$ is minimized. Denote by $M(V^{Ba})$ the martingale part of V^{Ba} under the original measure P and by $A(V^{Ba})$ the drift term. The martingale part of Γ is denoted by L, it reads

$$\begin{aligned} L_t &\equiv \Gamma_t - A(V^{Ba})_t + \int_0^t \phi dA \\ &= M(V^{Ba})_t - \int_0^t \phi dM \, . \end{aligned}$$

Since $A(V^{Ba})$ and $\int \phi dA$ are continuous and of finite variation

$$\langle L, L\rangle = \langle \Gamma, \Gamma\rangle \, .$$

Therefore $\min d\langle\Gamma,\Gamma\rangle$ is equivalent to $\min d\langle L, L\rangle$. This is achieved by projecting $M(V^{Ba})$ on $\{\int \Theta dM\}$ or equivalently by requiring L to be orthogonal to M. In analogy to the derivation of the locally risk minimizing strategy we obtain

$$\begin{aligned} \phi &= \frac{d\langle M(V^{Ba}), M\rangle}{d\langle M, M\rangle} \\ &= \frac{d\langle V^{Ba}, Z\rangle}{d\langle Z, Z\rangle} \\ &= (1-\gamma)V_z^{Ba} + \gamma \frac{E_L(\Delta V^{Ba} \Delta Z)}{E_L((\Delta Z)^2)} \\ &= (1-\gamma)F_s^{Ba} + \gamma \frac{E_L(\Delta F^{Ba} \Delta S)}{E_L((\Delta S)^2)} \end{aligned}$$

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