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A Reduced-Form Model for Warrant Valuation

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Abstract

This paper studies warrant valuation using a reduced-form model. Analogous to the credit risk literature, structural models require complete information about the asset value process and the firm's liabilities. In contrast, reduced-form models require only information about the firm's stock price process. We introduce a reduced-form model where the warrant holder is a price taker, and we relate our model to structural models appearing in the literature.

Keywords: warrants, structural models, reduced-form models, incomplete information

JEL Classifications: G13, G32

1. Introduction

Historically, the valuation of warrants has been an important topic in both corporate finance and investment theory (see Ross, Westerfield, and Jaffe, 1993). Recently, with the repurchase of hundreds of millions of dollars of bank warrants

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This paper uses many of the insights of a Working paper by the same authors entitled "Warrant Valuation in Complete Markets," October 1991.

from the U.S. Treasury under the Troubled Asset Relief Program (TARP) Capital Purchase Program,[1] warrant valuation has taken on renewed public interest (see Congressional Oversight Panel (COP), 2009; Wilson, 2009). In light of this renewed interest, the purpose of this paper is to introduce a new class of models for valuing warrants called "reduced form."

Analogous to the credit risk literature, warrant valuation models can be classified as either "structural" or "reduced form" based on the information sets used in constructing the model (see Jarrow and Protter, 2004). Structural models are those that assume the market has complete information with respect to the firm's asset value process—traded and observable—and that the market knows the details of the firm's entire dynamic liability structure. In contrast, reduced-form models assume that the market only has information with respect to the firm's stock price process and the outstanding warrants. As is well known in the credit risk literature, this difference in information structure can completely change the evolution of the observed stock price process. This change in the evolution of the stock price process in the reduced information setting is not currently recognized in the existing literature on warrant valuation.

The introduction of the reduced-form model and analyzing its implications are the contributions of our paper. We argue that given the current structure of corporate accounting and market transparency, reduced-form models are more relevant to valuing warrants in actual markets, for example, for use in the valuation of the TARP bank warrants discussed above. In addition, we show that many of the conclusions regarding the behavior of the stock price process in the existing structural approach to warrant valuation need not apply in this reduced information setting. In particular, contrary to repeated statements in the existing literature, the Black-Scholes formula may apply to the valuation of European warrants in actual markets.

2. The structural approach

This section briefly reviews the structural approach to warrant valuation, the purpose of which is to emphasize some of its characteristics for comparison with the reduced-form model introduced below. The structural approach to warrant valuation is the predominant modeling methodology appearing in the existing literature. The structural approach assumes that complete information is available to the market with respect to the firm's asset value process and the firm's entire dynamic liability structure. In a structural model, warrants are valued as derivatives on the firm's asset value process.

[1] For more details, see the U.S. Treasury website, www.financialstability.gov/latest/tg_06262009.html.

2.1. Literature review

In the structural approach to warrant valuation, there are two issues to study in regard to warrant exercise when exercise is unrestricted (sequential exercise and partial exercise), and the solutions differ with respect to whether the warrant holders are large traders or price takers.

Sequential exercise is the issue of whether American warrants should be exercised only at maturity or sequentially in time, even in the absence of regular dividends. If the warrant holders are large traders, Emanuel (1983), Cox and Rubinstein (1985), Constantinides (1984), Ingersoll (1987), and Koziol (2006) demonstrate that sequential exercise can increase the value of warrants. Spatt and Sterbenz (1988) study oligopoly warrant holders and show that there are reinvestment policies of the firm for which sequential exercise is not beneficial. More recently, Linder and Trautmann (2009) provide sufficient conditions for sequential exercise not to be optimal. If warrant holders are price takers, Constantinides (1984) and Koziol (2006) show that there exist Nash equilibrium where price takers will exercise warrants as a block, although the price will be less than or equal to that of a large trader. In these models, the results depend critically on both the use of the cash proceeds by the firm and the specific liability structure assumed.

Partial exercise is the issue of whether European warrants are exercised differently than otherwise equivalent call options, given the impact of warrant share price dilution and the infusion of cash into the firm. Again, the results depend on the firm's liability structure and the use of the cash proceeds. Galai and Schneller (1978) and Cox and Rubinstein (1985), assuming a firm with only equity and warrants, show that the optimal exercise strategy is the same as an otherwise equivalent call option. However, the price of the warrant is less than that of an otherwise equivalent call written on the stock of an otherwise identical firm without warrants by a dilution factor. In contrast, Crouhy and Galai (1994) and Koziol (2006), studying a firm with equity, debt, and warrants, show that the optimal exercise strategy results in warrant exercise occurring at levels above the strike price when exercise is "block"-restricted. Koziol (2006) shows in addition for the unrestricted case that the optimal warrant exercise strategy is also optimal for call options and that warrant and call values coincide. Hanke and Potzelberger (2002) show how the existence of warrants affects the valuation of options written on the firm's stock.

All of these structural models of warrant valuation, for simplicity, assume that interest rates are constant. This is an unrealistic assumption for the valuation of long dated warrants. We relax this assumption in our review of the structural approach below.

2.2. Assumptions

To illustrate the structural model, we use a model similar to that in Crouhy and Galai (1994), except that we allow for stochastic interest rates. We assume that all traders are price takers.

We assume a continuous trading model with trading horizon $[0, \tau]$ for $\tau < \infty$. There is a probability space (Ω, G, Q) on which exists a d-dimensional Brownian motion $\{Z_1(t), \ldots, Z_d(t) : t \in [0, \tau]\}$ initialized at zero with the augmented filtration $\{G_t : t \in [0, \tau]\}$. The information set $\{G_t : t \in [0, \tau]\}$ represents the complete information available to the firm's management.

We assume that the default free spot rate of interest $r(t)$ is stochastic and $\{G_t : t \in [0, \tau]\}$ adapted. We let

$$B(t) = e^{\int_0^t r(s)\,ds}$$

denote the value of the money market account at time t initialized at $B(0) = 1$.

We also assume that the assets of the firm trade (and are therefore observable) with asset value $A(t)$ at time t.

Assumption 1: *Firm asset dynamics*

$$dA(t) = \delta A(t)\,dt + \xi A(t)\,dZ_1(t), \tag{1}$$

where $A(0) > 0$, $\xi > 0$, δ *are constants.*

We assume that markets are arbitrage free and complete. Of course, this assumption is consistent with an model of Heath, Jarrow, and Morton (1992) for the evolution for the term structure of interest rates.

Assumption 2: *Arbitrage free and complete market. There exists a unique probability measure* $\widetilde{Q}$ *equivalent to* Q *such that* $\frac{A(t)}{B(t)}$ *is a* $\widetilde{Q}-$ *martingale with respect to* $\{G_t : t \in [0, \tau]\}$.

The next topic to be addressed is the firm's capital structure, and how warrants impact its composition. We start with a simple firm, and discuss complications to its capital structure after the analysis is complete. Initially, we consider a firm with time t firm value $V(t)$, which has risky debt, a warrant, and common equity outstanding. For this firm, we distinguish between firm value and the underlying assets of the firm. The assets underlying the firm, $A(t)$, are of constant returns to scale technology. The firm consists of investment in these assets, and cash inflows just expand the firm's assets proportionally.

The relevant details of the capital structure are that at time T for $0 < T < \tau$ the debt of the firm matures. The debt is a zero-coupon issue with face value F. We denote its time t value as $D(t)$. Also outstanding are n European warrants with exercise price K and maturity date t^* where $0 \le t^* \le T \le \tau$. We denote the warrant's time t value as $W(t)$. There are N shares of common equity outstanding at time $t < t^*$, each with a per share value equal to $S(t)$. Total equity value for $t < t^*$ is given by $NS(t)$. No dividends are paid to these equity shares.

Hence, firm value, denoted $V(t)$, is equal to:

$$V(t) = \begin{cases} NS(t) + D(t) + nW(t) & \text{if } 0 \leq t < t^* \\ NS(t) + D(t) & \text{if } t^* \leq t < T \text{ and warrants not exercised} \\ [N+n]S(t) + D(t) & \text{if } t^* \leq t < T \text{ and warrants exercised} \end{cases}. \tag{2}$$

At time t^*, the warrant may be exercised and n new shares issued. This explains the value for the firm after time t^* in Equation (2).

Define $\overline{A}$ to be the level of the firm's assets above which price taking warrant holders decide to exercise the warrant at time t^*, that is, exercise occurs if and only if $\overline{A} \leq A(t^*)$. The determination of $\overline{A}$ is postponed until later in this section.

Given $\overline{A}$, an accounting identity gives that the firm value is also equal to the value of the underlying assets, that is,

$$V(t) = \begin{cases} A(t) & \text{if } 0 \leq t < t^* \\ A(t) & \text{if } t^* \leq t < T \text{ and } A(t^*) < \overline{A} \\ \left[\dfrac{A(t^*) + nK}{A(t^*)}\right] A(t) & \text{if } t^* \leq t < T \text{ and } \overline{A} \leq A(t^*) \end{cases}. \tag{3}$$

Equation (3) makes explicit the changes that occur in the firm's investment policies if the warrant is exercised at time t^* (when $\overline{A} \leq A(t^*)$) and an influx of nK dollars enters the firm. The firm invests this cash into the constant returns to scale assets $A(t^*)$. The amount invested goes from $A(t^*)$ to $[A(t^*) + nK]$ at time t^*, a discrete jump.

2.3. Valuations

Given this information, we can now value the firm's liabilities and equity. First, we determine the stochastic process followed by the firm's equity. At the maturity of the firm's debt, the equity value, per share, is

$$S(T) = \begin{cases} \dfrac{\max\{A(T) - F, 0\}}{N} & \text{if } A(t^*) < \overline{A} \\ \dfrac{\max\left\{\left[\dfrac{A(t^*) + nK}{A(t^*)}\right] A(T) - F, 0\right\}}{N+n} & \text{if } \overline{A} \leq A(t^*) \end{cases}. \tag{4}$$

Using martingale pricing, we have that

$$S(t) = \begin{cases} \widetilde{E}\left(\dfrac{\max\{A(T) - F, 0\}}{N \cdot B(T)} \middle| G_t \right) B(t) & \text{if } A(t^*) < \overline{A} \\ \widetilde{E}\left(\dfrac{\max\left\{ \left[\dfrac{A(t^*) + nK}{A(t^*)} \right] A(T) - F, 0 \right\}}{[N + n]\, B(T)} \middle| G_t \right) B(t) & \text{if } \overline{A} \leq A(t^*) \end{cases} \tag{5}$$

for $t^* \leq t \leq T$ and

$$S(t) = \widetilde{E}\left(\begin{array}{c} \dfrac{\max\{A(T) - F, 0\}}{N \cdot B(T)} 1_{\{A(t^*) < \overline{A}\}} + \\ \dfrac{\max\left\{ \left[\dfrac{A(t^*) + nK}{A(t^*)} \right] A(T) - F, 0 \right\}}{[N + n]\, B(T)} 1_{\{A(t^*) \geq \overline{A}\}} \end{array} \middle| G_t \right) B(t) \tag{6}$$

for $0 \leq t < t^*$ where $\widetilde{E}(\cdot)$ is expectation under the martingale measure $\widetilde{Q}$.

The value of the common stock, per share, is seen to be the expected discounted value of its cash flow at time T. An implication of this statement is that $\frac{S(t)}{B(t)}$, and therefore, $S(t)$ is sample path continuous in t for all $t \in [0, T]$. This follows by a standard theorem from stochastic calculus that on a Brownian filtration all martingales are continuous almost everywhere (see Karatzas and Shreve, 1988, p. 182). In particular, the stock price is continuous in t at the expiration date of the warrant. From Equation (5), we also see that for fixed t^*, $S(t^*)$ is discontinuous in $A(t^*)$ as $A(t^*) \uparrow \overline{A}$. However, this is not a discontinuity in t.

Given price taking traders, following an optimal exercise strategy, the value of the warrant at time t^* is

$$\begin{aligned} W(t^*) &= \max\{S(t^*) - K, 0\} \\ &= \begin{cases} \widetilde{E}\left(\dfrac{\max\left\{ \left[\dfrac{A(t^*) + nK}{A(t^*)} \right] A(T) - F, 0 \right\}}{[N + n]\, B(T)} \middle| G_{t^*} \right) B(t^*) - K & \text{if } \overline{A} \leq A(t^*) \\ 0 & \text{if } A(t^*) < \overline{A}, \end{cases} \end{aligned} \tag{7}$$

where $\overline{A}$ is the solution to

$$\widetilde{E}\left(\dfrac{\max\left\{ \left[\dfrac{A(t^*) + nK}{A(t^*)} \right] A(T) - F, 0 \right\}}{[N + n]\, B(T)} \middle| G_{t^*} \right) B(t^*) - K = 0. \tag{8}$$

The warrant's value at expiration is seen to be zero if it expires out of the money, and otherwise $S(t^*) - K$. The decision to exercise is uniquely determined by the assets of the firm at time t^*. If $\overline{A} \leq A(t^*)$ then exercise is optimal. The boundary value $\overline{A}$ is determined in Equation (8) as that quantity such that $S(t^*) = K$. It is the value of $A(t^*)$ such that the warrant holders are indifferent between exercising or not.

Using martingale pricing, we get that for $0 \leq t < t^*$,

$$W(t^*) \\ = \widetilde{E}\left(\left[\widetilde{E}\left(\frac{\max\left\{\left[\dfrac{A(t^*)+nK}{A(t^*)}\right]A(T)-F,0\right\}}{[N+n]\,B(T)}\middle| G_{t^*}\right)-\frac{K}{B(t^*)}\right]1_{\{\overline{A}\leq A(t^*)\}}\middle| G_t\right)B(t). \tag{9}$$

An implication of Equation (9) is that $\frac{W(t)}{B(t)}$ is a $\widetilde{Q}-$ martingale and hence $W(t)$ is sample path continuous in t.

Finally, by Equation (2), we can examine the stochastic behavior of the total value of the equity and the risk debt. With respect to the total value of the equity, since $S(t)$ is continuous at t^*, there will be a discontinuous jump in the total value of the equity from $NS(t^*)$ to $(N + n)S(t^*)$ at time t^* if the warrants are exercised. We will return to this observation later.

The value of risky debt is obtained by substituting Equations (6) and (9) into Equation (2). We are interested in studying the behavior of $D(t)$ as $t \to t^*$. Prior to time t^*, by Equation (6) we have that

$$D(t) = A(t) - NS(t) - nW(t) \quad \text{for } t < t^*. \tag{10}$$

At time t^*,

$$D(t^*) = \begin{cases} A(t^*) - NS(t^*) & \text{if } A(t^*) < \overline{A} \\ A(t^*) + nK - [N+n]S(t^*) & \text{if } \overline{A} \leq A(t^*) \end{cases}. \tag{11}$$

Rearranging terms gives

$$D(t^*) = \begin{cases} A(t^*) - NS(t^*) & \text{if } A(t^*) < \overline{A} \\ A(t^*) - NS(t^*) - n[S(t^*) - k] & \text{if } \overline{A} \leq A(t^*) \end{cases} \tag{12}$$

or

$$D(t^*) = A(t^*) - NS(t^*) - nW(t^*). \tag{13}$$

Thus, there is no jump in the value of the risky debt at time t^* as $A(t)$, $S(t)$, and $W(t)$ are continuous in t. This implies that the bondholders rationally anticipate whether the warrants will be optimally exercised at time t^*, and there is no jump in their value at time t^*. The shift in wealth from the stockholders to the bondholders due to the cash inflow of nK dollars occurs continuously in time as t approaches t^*. Consequently, the change in the value of the firm at time t^* under warrant exercise nK is solely reflected as a discrete change in the total value of the equity outstanding $nS(t^*)$ and the loss due to the value of the warrants $nW(t^*)$ leaving the balance sheet.

It can be shown (see Schulz and Trautmann, 1994; Koziol, 2006) that due to the cash infusion at the exercise date, the risk of the stock can change, resulting in a shift in the stock's volatilities. This shift, however, depends on the market having the complete information set $\{G_t : t \in [0, \tau]\}$ as detailed above.

It is straightforward to generalize the preceding analysis to include additional financial liabilities on the firm's balance sheet. Needed are a specification of the boundary conditions as in Equations (3), (6), and (9), and then, an application of the risk neutral valuation procedure yields each financial liability's value. Changes in the investment policies of the firm can be incorporated by adjustments to Equation (3).

2.4. Limitations

The advantage of the structural approach to warrant valuation is that the value of the firm's equity and warrants are endogenous to the model. Hence, analogous to credit risk models, the structural approach is conceptually useful for understanding the valuation of warrants in complete information markets. But, the structural approach is less useful for practical applications.

This is due to the current structure of corporate accounting and market transparency. First, the firm's asset value process is not tradeable nor observable. This observation implies that the martingale pricing methodology no longer applies. Second, we need to know the firm's entire liability structure over the life of the warrant. For realistic corporations, their liabilities are dynamic, not static, and quite complex. Some of the liabilities are off-balance sheet as well. Third, a firm's future investment projects are unknown (certainly not constant returns to scale technology), and their dividend policy (and share issuance/repurchase policies) stochastic.

The dynamic nature of the firm's balance sheet makes the determination of the warrant's value difficult. Furthermore, since price taking (and nonprice taking) warrant holders need to rationally anticipate the actions of the other warrant holders in (a Nash) equilibrium, incomplete information will change the resulting valuations and strategies. Consequently, an alternative method is needed for applications. This alternative method, the reduced-form approach, is provided in the next section.

3. The reduced-form approach

The reduced-form approach to valuation uses only information that is available to the market, that is the stock price that trades and is observable, and the terms of the warrant contract. The warrants are priced as derivatives on the stock price process. Although not necessary for the reduced-form approach, we assume that the warrant holders are price takers.[2] The paper by Schulz and Trautmann (1994) can be viewed as the first application of reduced-form models to warrant valuation.

3.1. Assumptions

The setup is similar to the structural approach. We assume a continuous trading model with trading horizon $[0, \tau]$ for $\tau < \infty$. There is a probability space (Ω, F, Q) on which exists a m-dimensional Brownian motion $\{Z_1(t), \ldots, Z_m(t) : t \in [0, \tau]\}$ initialized at zero with the augmented filtration $\{F_t : t \in [0, \tau]\}$ where $m \leq d$. We assume that $F_t \subset G_t$ for all t. This later assumption captures the notion that the market has less information in $\{F_t\}$ than the firm's management has in $\{G_t\}$.

We assume that the default free spot rate of interest $r(t)$ is stochastic and $\{F_t : t \in [0, \tau]\}$ adapted. We let

$$B(t) = e^{\int_0^t r(s)\,ds}$$

denote the value of the money market account at time t initialized at $B(0) = 1$. Also traded are a common stock with time t price $S(t)$ and a European warrant on the stock with expiration date $t^* < \tau$ and exercise price K, with time t price $W(t)$. We assume that both $S(t)$ and $W(t)$ are $\{F_t : t \in [0, \tau]\}$ adapted.

Reduced-form models do not assume that the assets of the firm trade. Nor, do they need to assume that the firm's investment policies or dynamic liability structure are known to the market. This incomplete information is captured, implicitly, in the following assumption.

Assumption 3: *Stock price dynamics*

$$dS(t) = \mu(t)S(t)\,dt + \sum_{i=1}^{m} \eta_i(t)S(t)\,dZ_i(t) \tag{14}$$

for all $0 \leq t \leq \tau$ *where* $S(0) > 0$ *is a constant,* $\mu(t)$ *is the stock return's instantaneous drift, a random process, and* $\eta_i(t)$ *for* $i = 1, \ldots, m$ *are the stock's instantaneous volatilities, also random processes. Both the stock's drifts and volatilities need to satisfy suitable measurability and integrability conditions so that Equation (14) is well defined.*[3]

First, this assumption implies that the stock price process is continuous in time. In particular, it is continuous at the warrant's expiration date t^*. As shown in the previous section, this is consistent with warrant share dilution and cash infusion into the firm at warrant expiration, even under the complete information set $\{G_t : t \in [0, \tau]\}$. The stock price process reflects the fact that the warrant exists and are traded. As such,

[2] The relaxation of this assumption can proceed along the lines of Jarrow (1994).

[3] See Karatzas and Shreve (1988) for the relevant conditions.

it reflects both the anticipated actions with respect to the warrants' exercise and the resulting shifts in the underlying firm's capital structure and investment decisions (under the Brownian filtration).

The stock process has random drifts and volatilities. This aspect of assumption 3 is also consistent with the structural approach to warrant valuation under complete information. From the structural approach to warrant valuation, we would not expect the volatilities to be constant across time. This is especially true at the warrant's expiration date, where the volatilities can jump due to the cash infusion.

However, the fact that the stock's volatilities are stochastic and nonconstant in a structural model does not imply that they need to be stochastic nor nonconstant in a consistent reduced-form model. It depends on the characteristics of the reduced information set $\{F_t : t \in [0, \tau]\}$. Indeed, it is possible that the evolution of the stock price $S(t)$ under $\{G_t : t \in [0, \tau]\}$ follows Equation (6) with nonconstant volatility, but for $S(t)$ to have a constant volatility under $\{F_t : t \in [0, \tau]\}$.[4] This observation is contrary to a common belief in the existing literature (see Crouhy and Galai, 1994; Bensoussan, Crouhy, and Galai, 1995a, 1995b; Hanke and Potzelberger, 2002; Koziol, 2006). In particular, Equation (14) could be consistent with geometric Brownian motion.

Last, Equation (14) allows for a correlation between the stock's return and interest rate changes as reflected in the fact that the same m Brownian motions influence both the stock price process and the evolution of the term structure of interest rates.

As before, we need to assume that markets are arbitrage free and complete.

Assumption 4: *Arbitrage free and complete markets. There exists a unique probability measure $\widetilde{Q}$ equivalent to Q such that $\frac{S(t)}{B(t)}$ is a $\widetilde{Q}-$ martingale with respect to* $\{F_t : t \in [0, \tau]\}$.

3.2. Valuation

Given assumptions 3 and 4, we can value the warrants as otherwise equivalent call options on the stock. The European warrant's payout at its expiration date t^* is given by the standard boundary condition:

$$W(t^*) = \max\{S(t^*) - K, 0\}. \tag{15}$$

The stock price at time t^*, continuously anticipates the exercise of the warrant. If the warrant is in the money at time t^*, then $S(t)$ reflects this as $\lim_{t \uparrow t^*} S(t) = S(t^*)$. Similarly, if the warrant is out of the money at time t^*, then $S(t)$ continuously anticipates this. This continuity of the stock price process is due to the fact that

[4] For example, in the credit risk literature (see Jarrow and Protter, 2004), in a structural model the firm's debt can follow a nonconstant volatility continuous sample path process but in the implied reduced-form model, it can follow a constant volatility discontinuous sample path process.

the resolution of uncertainty in our economy is characterized by the paths of a m-dimensional Brownian motion, a continuous process. Thus, to decide on exercise of the warrant, the warrant holder needs only examine the current stock price $S(t^*)$ and exercise the warrant if it exceeds the exercise price K.

Consequently, the warrant's time t value is given by

$$W(t) = \widetilde{E}\left(\frac{\max\{S(t^*) - K, 0\}}{B(t^*)} \middle| F_t \right) B(t). \tag{16}$$

It is the expected discounted value of the warrant's payoff.

For emphasis, we provide an immediate consequence of this valuation formula. Of course, this lemma requires for its validity that the warrant exists and is traded so that the stock price process accurately reflects the influence of the warrant on the firm's value. In accordance with Koziol's (2006) results for the structural approach we get the following result:

Lemma 1 *(Otherwise equivalent call options): Under perfect competition among warrant holders (unrestricted exercise) an ordinary American call option on the stock with exercise price K and maturity date t^* will have the same price as an otherwise identical American warrant on the stock.*

Hedge ratios are determined from Equation (16) in the standard way. For example, given a closed form solution, the hedge ratio $\frac{\partial W(t)}{\partial S(t)}$ can be calculated by ordinary differentiation. Otherwise, a numerical procedure is necessitated.

The valuation formula in Equation (16) is under stochastic interest rates and it is for completely arbitrary capital structures and investment policies in the underlying firm. The capital structures and investment policies are implicitly incorporated in the "equilibrium" and exogenously given stock price process. Consequently, the choice of the particular stock price process utilized is crucial in the reduced-form approach.

Some theoretical arguments as to which stochastic processes are preferable can be obtained from the structural approach to warrant valuation. However, as noted above, one must examine the behavior of the stock price process under the reduced information set $\{F_t : t \in [0, \tau]\}$ and not the complete information set $\{G_t : t \in [0, \tau]\}$. Examples of such an analysis await subsequent research.

From a practical perspective, the ultimate determination of the stochastic process is the data itself. The reduced-form approach lends itself well to such an empirically based specification. This is in contrast to the structural approach to warrant valuation that requires knowledge of the firm's asset value process—an unobservable quantity. For example, if a lognormal distribution for stock returns fits the data and forward rate processes have deterministic volatilities, then a Black-Scholes type valuation formula will result (see Amin and Jarrow, 1992). In fact, using the approximation arguments contained in Jarrow and Rudd (1982), Black-Scholes values with implicit volatilities may still provide reasonable approximations even when the underlying processes are not lognormal. This is consistent with the evidence in Schulz and Trautmann (1994).

Of course, in the standard way, the above analysis can be generalized to apply to American type warrants with early exercise privileges (see Amin and Jarrow, 1992). It can also be extended to include jumps in both the stock price process and the evolution of the term structures (see, Jarrow and Madan, 1995).

4. Conclusion

This paper introduces a new approach to warrant valuation based on a reduced-form model. The reduced-form model only depends on the information available to market participants. As such, it can be applied in practice to the valuation of warrants, for example, it has been used to value bank TARP warrants issued to the U.S. Treasury during the current credit crisis. The structural approach, due to its onerous information requirements, does not provide a usable alternative. Furthermore, many of the implications on the evolution of the stock price and its volatility of the structural approach to warrant valuation, obtained under complete information, do not apply in a reduced information based setting.

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