Gaussian Multi-factor Interest Rate Models: Theory, Estimation, and Implications for Option Pricing

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Abstract

Gaussian interest rate models are attractive because of their analytical tractability. Examples are the one-factor models of Ho/Lee and Vasicek, and two-factor models resulting from combinations of both. We show by factor analysis that the variation in spot rates of the German Government bond market in the period 1980-93 can essentially be explained by two factors. Consequently, we restrict ourselves to one-factor and two-factor models which are estimated by nonlinear regression. We compare these models with respect to their ability to explain observed changes in the term structure of interest rates.

On the basis of theoretical insights and our empirical findings we conclude that the two one-factor models considered should be discarded, because of the models' inability to explain a twist of the term structure where short term and long term rates move in opposite directions. Although the combination of the Ho/Lee and Vasicek one-factor model as proposed by Heath/Jarrow/Morton is able to explain the term structure twist, we recommend a two-factor Vasicek model (which contains the other models as special cases) for pricing interest rate options. The latter model is the only one which captures a u-shaped volatility function induced by the term structure twist. This situation occured in two of the seven 2-year subperiods analyzed. A comparison of model prices for calls on zero bonds which indicates that neglecting the volatility smile might cause a substantial valuation error completes the paper.

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1 Introduction

The valuation of interest rate contingent claims has attracted a growing interest in recent years and various models have been proposed. There are essentially two approaches to the modeling of the term structure. The general equilibrium approach used for example by COX/INGERSOLL/ROSS (1985) and LONGSTAFF/SCHWARTZ (1992) starts from a description of the economy and derives the term structure of interest rates endogenously. In contrast the arbitrage approach starts from assumptions about the stochastic evolution of one or more interest rates and derives prices of contingent claims by imposing the no arbitrage condition. One and two factor models of only the short rate were first suggested by VASICEK (1977) and BRENNAN/SCHWARTZ (1979). These models are generally not consistent with the initial term structure. HULL/WHITE (1993) provide a general procedure for short rate models which allows to match an arbitrary initial yield curve. The alternative approach to model the complete term structure was pioneered by HO/LEE (1986) and generalized by HEATH/JARROW/MORTON (HJM) (1992). HJM allow for a fixed but unspecified number of stochastic factors. The application of such a model requires the determination of the number of factors driving the term structure movement and the specification of appropriate volatility coefficients.

We restrict our analysis to Gaussian models with deterministic volatility coefficients, because of their analytical and numerical tractability. Moreover we believe that the assumption of normally distributed interest rates is quite reasonable, if the probability of negative interest rates is small. We will discuss this issue based on our parameter estimates. From our point of view the often suggested alternative of lognormally distributed interest rates is not appropriate since we do not expect interest rates to have a drift and variance rate proportional to the level of interest rates.

In this paper we show by principal component analysis of the spot rate movement that in the German bond market two factors are relevant, and that these factors can be identified as a shift and a twist of the term structure. Furthermore, we develop a specific two-factor model within the class of HJM-models which can be estimated by nonlinear regression. Our model includes as special cases the continuous time version of the Ho/Lee model, the extended Vasicek model, and a model proposed by HJM (1992) as an example for a simple two-factor model. We estimate these models for seven two-year periods and identify the types of term structure movement, that can be explained by these models.

The Ho/Lee model is appropriate only when the predominant movement of the term structure is a parallel shift, resulting in interest rate volatilities which are the same for all maturities. In contrast the Vasicek model is based on the often observed phenomenon of lower volatilities for long term spot rates compared to short term spot rates. A phenomenon that can be explained by the considered two-factor models, but not by the one-factor models, is a twist of the term structure with short and long rates moving in opposite directions. Moreover we find that in subperiods which are dominated by a twist of the term structure, the spot rate variance shows a smile pattern. This smile effect is captured only by our model which is essentially a two-factor Vasicek model. Option price simulations show that neglecting the smile effect might cause substantial price differences.

Whereas we find that the more elaborate models are to be prefered, AMIN/MORTON (1994) obtain a slightly different result. They compare six one-factor models of the HJM type using implied volatility estimates. In their analysis one parameter models yield more stable parameter values over time and they are able to earn larger and more consistent abnormal returns. Amin and Morton conclude that among the one parameter models the Ho/Lee model seems to be the most preferred. FLESAKER (1993) in contrast rejects the Ho/Lee model for Eurodollar futures options traded at the Chicago Mercantile Exchange.

The paper is organized as follows. The basic model and the properties of the four Gaussian models under consideration are analyzed in section 2. In section 3 we describe the estimation procedure and discuss necessary assumptions. The results of the principal component analysis and the nonlinear regressions are presented in section 4. Implications for option pricing are discussed in section 5. Section 6 concludes the paper.

2 Gaussian Models in the HJM Framework

The HJM approach to pricing interest rate contingent claims starts with a family of forward rate processes and initializes it to an arbitrary but fixed initial forward rate curve. This approach was pioneered by HO/LEE (1986) for a simple discrete one-factor model and extended to a continuous time multi-factor setting by HEATH/JARROW/MORTON (1992). They consider a continuous trading economy with a trading interval $[0, \bar{T}]$. The uncertainty in the economy is characterized by the probability space $(\Omega, \mathcal{F}, Q)$ and the filtration $F = \{\mathcal{F}_t : t \in [0, \bar{T}]\}$, satisfying the usual conditions. Furthermore they assume a continuum of discount bonds $P(t, T)$ with maturities $T \in [0, \bar{T}]$. Let the bond price at time t be given by $P(t, T)$, where T is the maturity of the zero bond, then the *forward rate* at time t for instantaneous and riskless borrowing at date T is

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T} = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}.$$

Equivalently one can define the *bond price* in terms of the forward rates by

$$P(t, T) = \exp\left(-\int_t^T f(t, s) ds\right). \tag{1}$$

With the assumption of a continuum of discount bonds there is a riskless investment opportunity at any time t yielding the *riskless short rate*

$$r(t) = f(t, t).$$

A roll-over-position at the short rate $r(t)$ is termed *money market account*. The value of the money market account initialized at time 0 with one dollar investment is

$$B(t) = \exp\left(\int_0^t r(s)ds\right).$$

Bond values expressed in units of the money market account

$$Z(t,T) = \frac{P(t,T)}{B(t)}$$

are called *relative bond prices* and can be shown to form a martingale with respect to some risk neutral measure $\tilde{Q}$. Analogously one can show that *forward prices*

$$F(t,t^*,T) = \frac{P(t,T)}{P(t,t^*)}$$

are a martingale with respect to the forward risk adjusted measure Q^*. $F(t,t^*,T)$ is the price of a zero bond with maturity T and delivery date t^* at time t.

2.1 Arbitrage-Free Term Structure Dynamics

Let the stochastic evolution of forward rates follow a diffusion process of the form

$$f(t,T) - f(0,T) = \int_0^t \mu(v,T)dv + \sum_{k=1}^{K} \int_0^t \sigma_k(v,T)dW_k(v), \tag{2}$$

where $\mu(v,T)$ is the drift of the forward rate with maturity T while $\sigma_k(v,T)$ for $k = 1,\ldots,K$ are its volatility coefficients,[1] and $W_k(v)$ are independent (standard) Brownian motions. Since bond prices depend on forward rates, the drift and volatility of the bond price process

$$\begin{aligned} P(t,T) - P(0,T) &= \int_0^t \mu^p(v,T)P(v,T)dv \qquad (3) \\ &\quad + \sum_{k=1}^{K} \int_0^t \sigma_k^p(v,T)P(v,T)dW_k(v). \end{aligned}$$

must be connected to the drift and volatility coefficients of the forward rate process. HJM (1992) show using Ito's lemma and a generalized version of Fubini's theorem that

$$\sigma_k^p(t,T) = -\int_t^T \sigma_k(t,y)dy \quad \text{for} \quad k = 1,\ldots,K, \tag{4}$$

$$\mu^p(t,T) = r(t) - \int_t^T \mu(t,y)dy + \frac{1}{2}\sum_{k=1}^{K} \sigma_k^p(t,T)^2. \tag{5}$$

[1]The drift and volatility coefficients in the general HJM model may also depend on the path of the Brownian motions. Since this paper concentrates on Gaussian models with deterministic drift and diffusion coefficients, we omit the dependency of ω to keep the notation clear and simple.

Applying Ito's Lemma to the definition of the forward price yields the forward price process

$$\begin{aligned} F(t,t^*,T) - F(0,t^*,T) &= \int_0^t \mu(v,t^*,T)F(v,t^*,T)dv \\ &\quad + \sum_{k=1}^{K} \int_0^t \sigma_k(v,t^*,T)F(v,t^*,T)dW(v) \end{aligned} \tag{6}$$

with

$$\begin{aligned} \sigma_k(t,t^*,T) &= \sigma_k^p(t,T) - \sigma_k^p(t,t^*) \\ \mu(t,t^*,T) &= (\mu^p(t,T) - \mu^p(t,t^*)) - \sum_{k=1}^{K} \sigma_k^p(t,t^*)[\sigma_k^p(t,T) - \sigma_k^p(t,t^*)]. \end{aligned}$$

We refer to the volatility coefficients $\sigma_k(t,T), \sigma_k^p(t,T), \sigma_k(t,t^*,T)$ collectively as the volatility structure. We assume that there exist K zero bonds with maturity $0 < S_1 < \cdots < S_K \leq \bar{T}$ such that the diffusion matrix $(\sigma_k(t,S_k))_{K\times K}$ is nonsingular. This ensures the uniqueness of market prices of risk and the uniqueness of the equivalent martingale measure if the model is arbitrage-free.

Theorem 2.1 (No Arbitrage Conditions)
The following conditions are equivalent. A K-factor model satisfying these conditions is called locally arbitrage-free.

(A-1) *There exists a unique equivalent martingale measure $\tilde{Q}$ such that relative bond prices $Z(t,T)$ are martingales with respect to this measure.*

(A-2) *There exist market prices of risk $\lambda_k(t), k = 1,\ldots,K$, with*

$$\mu^p(t,T) - r(t) = \sum_{k=1}^{K} \sigma_k^p(t,T)\lambda_k(t), \quad \textit{for all} \quad T \in [t,\bar{T}]. \tag{7}$$

(A-3) *The forward rate drift is uniquely determined by the volatility structure and the market prices of risk:*

$$\mu(t,T) = \sum_{k=1}^{K} \sigma_k(t,T)\left[\lambda_k(t) + \int_t^T \sigma_k(t,y)dy\right] \tag{8}$$

(A-4) *There exists a unique equivalent martingale measure Q^* such that forward rates $f(t,t^*)$ are martingales with respect to this measure.*

Proof: In appendix A.

Condition (A-2) is the well-known no-arbitrage condition requiring that the excess-return of every bond regardless of maturity equals the sum of K risk premiums. Condition (A-3) relates the forward rate drift to the volatility structure and the market prices of risk. The arbitrage-free term structure dynamics are therefore completely specified by the volatility structure and market prices of risk. The martingale measure in condition (A-1) is often called risk neutral measure, since the expected instantaneous return of all bonds with respect to this measure equals the riskless short rate. The martingale measure in condition (A-4) is the forward-risk-adjusted measure introduced by JAMSHIDIAN (1991) which facilitates the calculation of closed form solutions for European style interest rate contingent claims.

Using condition (A-2) one can easily show that forward prices in a locally arbitrage-free model satisfy

$$\begin{aligned} F(t,t^*,T) - F(0,t^*,T) &= \sum_{k=1}^{K} \int_0^t [\lambda_k(v) - \sigma_k^p(v,t^*)]\sigma_k(v,t^*,T)F(v,t^*,T)dv \\ &\quad + \sum_{k=1}^{K} \int_0^t \sigma_k(v,t^*,T)F(v,t^*,T)dW_k(v) \end{aligned}$$

and are martingales with respect to Q^*

$$F(t,t^*,T) - F(0,t^*,T) = \sum_{k=1}^{K} \int_0^t \sigma_k(v,t^*,T)F(v,t^*,T)dW_k^*(v).$$

Applying Ito's Lemma to the logarithm of the forward price and using the fact that $F(t^*,t^*,T) = P(t^*,T)$ yields for the stochastic bond price in t^*, given the information $\mathcal{F}_t$,

$$\begin{aligned} P(t^*,T)\,|\,\mathcal{F}_t &= \frac{P(t,T)}{P(t,t^*)} \exp\Bigg\{ \sum_{k=1}^{K} \int_t^{t^*} [\lambda_k(v) - \sigma_k^p(v,t^*)]\sigma_k(v,t^*,T)dv \\ &\quad - \frac{1}{2}\sum_{k=1}^{K} \int_t^{t^*} \sigma_k(v,t^*,T)^2 dv + \sum_{k=1}^{K} \int_t^{t^*} \sigma_k(v,t^*,T)dW_k(v) \Bigg\}. \end{aligned}$$

Thus the bond price at any time t^* given the information $\mathcal{F}_t$ depends on the forward price of the bond at time t, the volatility structure and the market prices of risk. For the purpose of contingent claim valuation we are only interested in the stochastic bond price in terms of $\tilde{W}(t)$ and $W^*(t)$, which are Brownian motions with respect to the risk-neutral measure $\tilde{Q}$ and the forward-risk-adjusted measure Q^*.

$$\begin{aligned} P(t^*,T)\,|\,\mathcal{F}_t &= \frac{P(t,T)}{P(t,t^*)} \exp\Bigg\{ -\sum_{k=1}^{K} \int_t^{t^*} \sigma_k^p(v,t^*)\sigma_k(v,t^*,T)dv \\ &\quad - \frac{1}{2}\sum_{k=1}^{K} \int_t^{t^*} \sigma_k(v,t^*,T)^2 dv + \sum_{k=1}^{K} \int_t^{t^*} \sigma_k(v,t^*,T)d\tilde{W}_k(v) \Bigg\} \end{aligned} \tag{9}$$

$$P(t^*,T)\,|\,\mathcal{F}_t = \frac{P(t,T)}{P(t,t^*)}\exp\left\{-\frac{1}{2}\sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)^2 dv + \sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)dW_k^*(v)\right\} \tag{10}$$

$\tilde{Q}$ and Q^* are identified by Girsanov's theorem. The determiniation of $\tilde{Q}$ and Q^* requires the knowledge of the market prices of risk, but the stochastics of the bond price are completely specified by (9) and (10), which are independent of the market prices of risk. This allows for a preference-free valuation of contingent claims.

2.2 Gaussian Models

A *Gaussian model* is one in which the volatility structure $\sigma_k(v,T)$ is deterministic. Hence forward rates follow a Gaussian process and are normally distributed in a Gaussian model.[2] The bond price in a Gaussian model is lognormally distributed with respect to $\tilde{Q}$ and Q^*[3]

$$\begin{aligned}
\mathrm{E}_{\tilde{Q}}(P(t^*,T)|\mathcal{F}_t) &= \frac{P(t,T)}{P(t,t^*)}\exp\left\{-\sum_{k=1}^{K}\int_t^{t^*}\sigma_k^p(v,t^*)\sigma_k(v,t^*,T)dv\right\} \\
\mathrm{E}_{Q^*}(P(t^*,T)|\mathcal{F}_t) &= \frac{P(t,T)}{P(t,t^*)} \\
\mathrm{Var}_{\tilde{Q}}(\ln P(t^*,T)|\mathcal{F}_t) &= \mathrm{Var}_{Q^*}(\ln P(t^*,T)|\mathcal{F}_t) \\
&= \sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)^2 dv.
\end{aligned}$$

Gaussian models are often criticized because of the lognormal bond price distribution, since it implies normally distributed spot rates

$$R(t,T) = -\frac{\ln P(t,T)}{T-t},$$

which may become negative with positive probability. An important issue for the application of Gaussian models is therefore the assessment of the probability of negative spot rates based on estimated parameters. In the subsequent sections we will analyze the properties of four Gaussian models, two one-factor and two two-factor models.

[2] Jamshidian (1991) further restricts Gaussian models to term structure models with an especially simple volatility structure, such that the short rate also follows a Gaussian process.

[3] The bond price distributions with respect to the risk neutral and forward risk adjusted measure differ only in the expected bond price. The distribution with respect to the original measure Q can not be specified without further assumptions on the market prices of risk.

2.3 Continuous Time Version of the Ho/Lee-Model

The continuous time limit of the HO/LEE-Model (1986) was independently derived by JAMSHIDIAN (1991) and HEATH/JARROW/MORTON (1992). In the HJM-framework the model is fully specified by the volatility coefficient of the forward rate process:

$$\sigma(v, T) = \sigma$$

The bond price $P(t^*, T)$ in terms of $\tilde{W}(t)$ is[4]

$$P(t^*, T) \,|\, \mathcal{F}_0 = \frac{P(0,T)}{P(0,t^*)} \exp\left\{ -\frac{\sigma^2}{2} T t^* (T - t^*) - \sigma (T - t^*) \tilde{W}(t^*) \right\},$$

which implies for the spot rate with time to maturity $T - t^*$ at time t^*

$$R(t^*, T) \,|\, \mathcal{F}_0 = -\frac{\ln P(t^*, T)}{T - t^*} = -\frac{\ln \frac{P(0,T)}{P(0,t^*)}}{T - t^*} + \frac{\sigma^2}{2} T t^* + \sigma \tilde{W}(t^*). \tag{11}$$

Future spot rates are therefore normally distributed with

$$\mu^R(t^*, T) = -\frac{\ln \frac{P(0,T)}{P(0,t^*)}}{T - t^*} + \frac{\sigma^2}{2} T t^*$$

and

$$\sigma^R(t^*, T)^2 = \sigma^2 t^*$$

given the Information $\mathcal{F}_0$.

The simple volatility structure allows only for a small set of future yield curves, since the stochastic factor $\tilde{W}(t)$ has the same effect on all spot rates independent of maturity. Hence, at any point of time all possible yield curves are parallel. This should not be confused with a parallel shift of the term structure in time, which leads to arbitrage opportunities.[5] Furthermore, one can easily see that starting with a normal or flat term structure inverse yield curves are not possible. Figure 1 illustrates these limitations of the Ho/Lee model.

2.4 Term Structure Consistent Vasicek Model

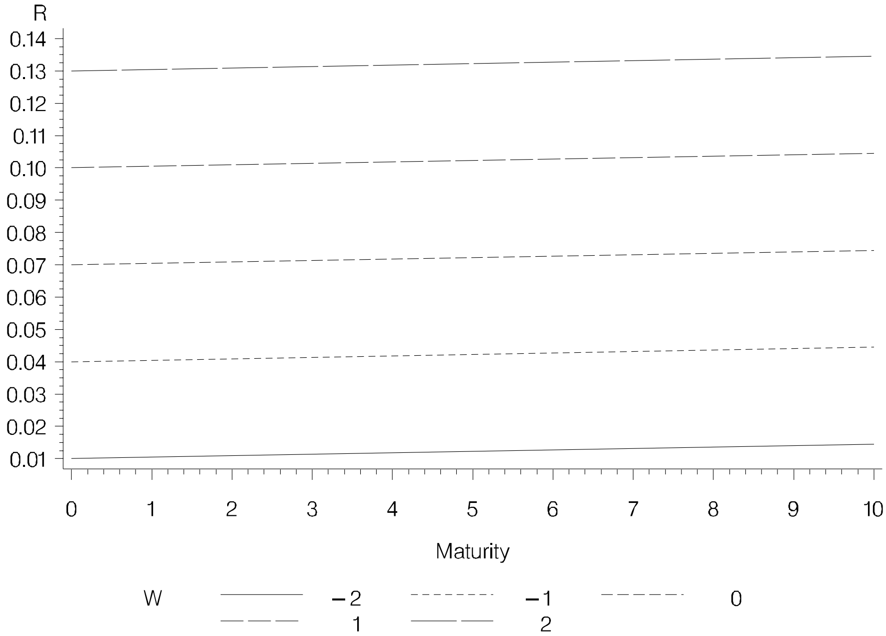
The VASICEK model (1977) is based on an Ornstein-Uhlenbeck process for the short rate. The short rate dynamics imply an endogenous term structure which is not necessarily consistent with the observable term structure. HULL/WHITE (1990) have shown that consistency can be achieved by introducing a time-dependent drift. The functional form of the drift in terms of the initial term structure and volatility

[4] In the discussion of the four special Gaussian models we restrict ourselves to the bond prices and spot rates given the information at time 0.

[5] This was first noted by BOYLE (1978). In the given model parallel shifts are excluded by the second drift term, which depends on the maturity.

Figure 1: **Yield Curves in the Ho/Lee-Model**

This figure shows 5 possible yield curves after one year when the initial term structure is flat at 7%. The 5 realizations of the Brownian motion are given by $\tilde{W}(t) = -2, -1, 0, 1, 2$. We choose a large volatility coefficient of $\sigma = 0.3$ to avoid the impression of a parallel shift in time.



structure has been derived by JAMSHIDIAN (1991) for the general Gaussian one-factor model. In the HJM-framework the term structure consistent Vasicek model is specified by

$$\sigma(v, T) = \sigma e^{-\kappa(T-v)},$$

where κ is a real valued constant. The bond price $P(t^*, T)$ in terms of $\tilde{W}(t)$ is

$$P(t^*, T) = \frac{P(0,T)}{P(0,t^*)} \exp\left\{-M(t^*, T) - \frac{\sigma}{\kappa}(1 - e^{-\kappa(T-t^*)})\tilde{x}(t^*)\right\},$$

with

$$M(t^*, T) = \frac{\sigma^2}{4\kappa^3}\left[(1 - e^{-\kappa(T-t^*)})^2(1 - e^{-2\kappa t^*}) + 2(1 - e^{-\kappa(T-t^*)})(1 - e^{-\kappa t^*})^2\right].$$

$\tilde{x}(t^*)$ is the unique solution to the Ornstein-Uhlenbeck process[6]

$$\tilde{x}(t^*) = \int_0^{t^*} e^{-\kappa(t^*-v)} d\tilde{W}(v).$$

The spot rate in the extended Vasicek model satisfies

$$R(t^*,T) = -\frac{\ln P(t^*,T)}{T-t^*} = -\frac{\ln \frac{P(0,T)}{P(0,t^*)}}{T-t^*} + \frac{M(t^*,T)}{T-t^*} + \sigma\frac{(1-e^{-\kappa(T-t^*)})}{\kappa(T-t^*)}\tilde{x}(t^*) \quad (12)$$

and is normally distributed with[7]

$$\begin{aligned}\mu^R(t^*,T) &= -\frac{\ln \frac{P(0,T)}{P(0,t^*)}}{T-t^*} + \frac{M(t^*,T)}{T-t^*} \\ \sigma^R(t^*,T)^2 &= \frac{\sigma^2}{2\kappa^3}\left(\frac{(1-e^{-\kappa(T-t^*)})}{(T-t^*)}\right)^2\left(1-e^{-2\kappa t^*}\right).\end{aligned}$$

given the information $\mathcal{F}_0$.

Contrary to the Ho/Lee model, the spot rate volatility is a decreasing function of time to maturity which reduces the probability of negative spot rates and is consistent with the often encountered phenomena of lower volatilities at the long end of the term structure. Figure 2 illustrates this difference and shows that the extended Vasicek model allows for normal, flat and inverse yield curves, even when the initial yield curve is flat.

2.5 The Heath/Jarrow/Morton Model

Heath/Jarrow/Morton (1992) propose a Gaussian two-factor model, which is essentially a combination of the Ho/Lee and Vasicek model.[8] The volatility structure in the HJM model is given by

$$\begin{aligned}\sigma_1(v,T) &= \sigma_1 \\ \sigma_2(v,T) &= \sigma_2 e^{-\kappa(T-v)}.\end{aligned}$$

The bond price $P(t^*,T)$ in terms of $\tilde{W}(t)$ is

$$\begin{aligned}P(t^*,T) &= \frac{P(0,T)}{P(0,t^*)}\exp\{-M_1(t^*,T) - M_2(t^*,T) \\ &\quad -\sigma_1(T-t^*)\tilde{W}_1(t^*) - \frac{\sigma_2}{\kappa_2}(1-e^{-\kappa(T-t^*)})\tilde{x}(t^*)\Big\},\end{aligned}$$

[6]The corresponding differential equation is

$$d\tilde{x}(t) = -\kappa\tilde{x}(t)dt + dW(t)$$

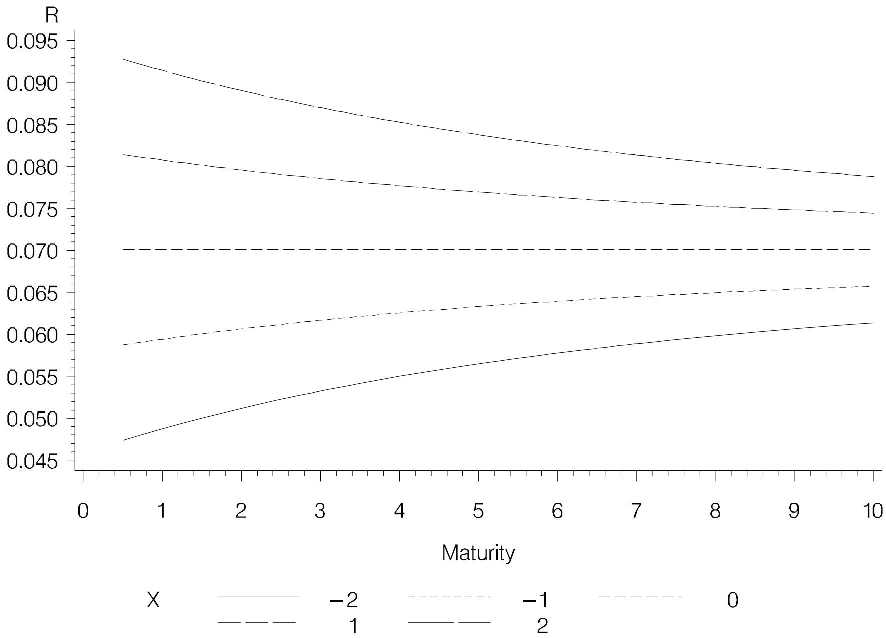
with initial value x(0) = 0.

[7]The risk neutral distribution follows from the properties of the Ornstein-Uhlenbeck process, cf Karatzas/Shreve (1988, p 358).

[8]We will term this model in the following HJM model even though the HJM approach is much more general and includes not only Gaussian models.

Figure 2: **Yield Curves in the Vasicek-Model**

This figure shows 5 possible yield curves after one year when the initial term structure is flat at 7%. The 5 realizations of the Ornstein-Uhlenbeck process are given by $\tilde{x}(t) = -2, -1, 0, 1, 2$. The volatility and reversion parameter are $\sigma = 0.0121, \kappa = 0.2564$.



with

$$M_1(t^*, T) = \frac{\sigma_1^2}{2} T t^* (T - t^*)$$
$$M_2(t^*, T) = \frac{\sigma_2^2}{4\kappa^3} \left[(1 - e^{-\kappa(T-t^*)})^2 (1 - e^{-2\kappa t^*}) + 2(1 - e^{-\kappa(T-t^*)})(1 - e^{-\kappa t^*})^2 \right].$$

$\tilde{x}(t)$ is the unique solution to the Ornstein-Uhlenbeck process like in the Vasicek model. The spot rate in the HJM model satisfies

$$R(t^*, T) = -\frac{\ln \frac{P(0,T)}{P(0,t^*)}}{T - t^*} + \frac{M_1(t^*, T)}{T - t^*} + \frac{M_2(t^*, T)}{T - t^*} \qquad (13)$$
$$\sigma_1 \tilde{W}_1(t^*) + \sigma_2 \frac{(1 - e^{-\kappa(T-t^*)})}{\kappa(T - t^*)} \tilde{x}(t^*).$$

and is normally distributed with

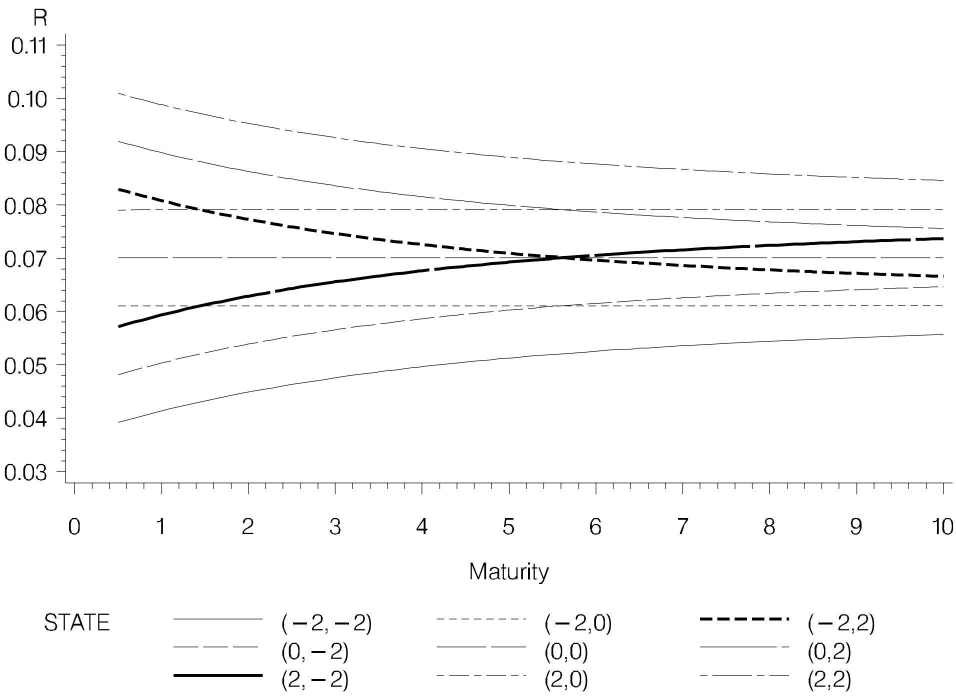
$$\mu^R(t^*, T) = -\frac{\ln \frac{P(0,T)}{P(0,t^*)}}{T - t^*} + \frac{M_1(t^*, T)}{T - t^*} + \frac{M_2(t^*, T)}{T - t^*}$$

$$\sigma^R(t^*,T)^2 = \sigma_1^2 t^* + \frac{\sigma_2^2}{2\kappa^3}\left(\frac{1-e^{-\kappa(T-t^*)}}{T-t^*}\right)^2\left(1-e^{-2\kappa t^*}\right)$$

given the information $\mathcal{F}_0$. From (13) it is obvious that the two-factor HJM model allows for a much larger variety of term structure movements than the one-factor models from Ho/Lee and Vasicek. More importantly, the HJM model allows for twists of the yield curve whereas the Vasicek model only allows for nonparallel shifts. Figure 3 illustrates two possible twists and seven nonparallel shifts of the term structure.

Figure 3: **Yield Curves in the HJM model**

This figure shows 9 possible yield curves after one year when the initial term structure is flat at 7%. The 9 realizations of $(\tilde{W}_1(t), \tilde{x}(t))$ are given by $(-2,-2), (-2,0), \ldots, (2,0), (2,2)$. The volatility and reversion coefficients are $\sigma_1 = 0.0045, \sigma_2 = 0.0122, \kappa = 0.4416$.



2.6 A Two-factor Vasicek Model

A twist of the term structure often leads to high spot rate volatilities for long and short maturities but rather low spot rate volatilities for medium maturities. This effect (henceforth called *smile-effect*) cannot be explained by the HJM model because the spot rate variance $\sigma^R(t^*,\tau)^2$ is a monotone decreasing function of time to maturity $\tau = T - t^*$. Therefore we propose the Gaussian two-factor model given by the volatility structure

$$\begin{aligned}\sigma_1(v,T) &= \sigma_1 e^{\kappa_1(T-v)}\\ \sigma_2(v,T) &= \sigma_2 e^{-\kappa_2(T-v)}\end{aligned}$$

with $\kappa_1 \geq 0$, $\kappa_2 \geq 0$. This model can be viewed as a two-factor Vasicek model (2FV model) and is able to explain the smile effect. The bond price $P(t^*,T)$ in this model is

$$\begin{aligned}P(t^*,T) &= \frac{P(0,T)}{P(0,t^*)}\exp\left\{-M_1(t^*,T) - M_2(t^*,T) + \frac{\sigma_1}{\kappa_1}(1-e^{\kappa_1(T-t^*)})\tilde{x}_1(t^*)\right.\\ &\qquad \left.-\frac{\sigma_2}{\kappa_2}(1-e^{-\kappa_2(T-t^*)})\tilde{x}_2(t^*)\right\},\end{aligned}$$

with

$$\begin{aligned}M_1(t^*,T) &= -\frac{\sigma_1^2}{4\kappa_1^3}\left[(1-e^{\kappa_1(T-t^*)})^2(1-e^{2\kappa_1 t^*}) + 2(1-e^{\kappa_1(T-t^*)})(1-e^{\kappa_1 t^*})^2\right]\\ M_2(t^*,T) &= \frac{\sigma_2^2}{4\kappa_2^3}\left[(1-e^{-\kappa_2(T-t^*)})^2(1-e^{-2\kappa_2 t^*}) + 2(1-e^{-\kappa_2(T-t^*)})(1-e^{-\kappa_2 t^*})^2\right]\end{aligned}$$

and

$$\begin{aligned}\tilde{x}_1(t^*) &= \int_0^{t^*} e^{\kappa_1(t^*-v)}d\tilde{W}_1(v)\\ \tilde{x}_2(t^*) &= \int_0^{t^*} e^{-\kappa_2(t^*-v)}d\tilde{W}_2(v).\end{aligned}$$

The spot rate in the 2FV-model

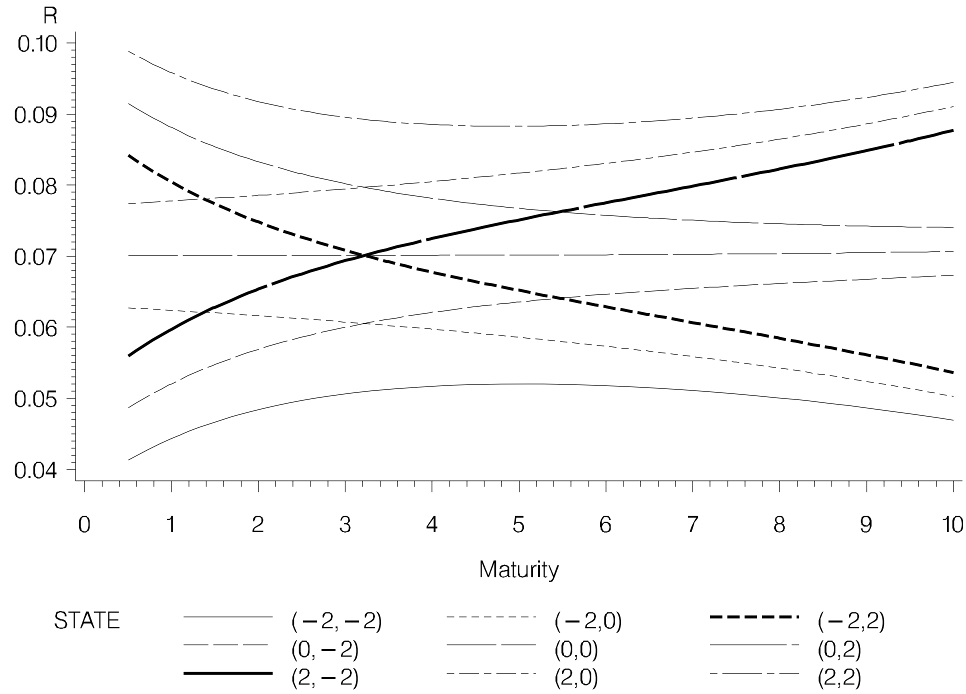
$$\begin{aligned}R(t^*,T) &= -\frac{\ln\frac{P(0,T)}{P(0,t^*)}}{T-t^*} + \frac{M_1(t^*,T)}{T-t^*} + \frac{M_2(t^*,T)}{T-t^*}\\ &\quad +\sigma_1\frac{(e^{\kappa_1(T-t^*)}-1)}{\kappa_1(T-t^*)}\tilde{x}_1(t^*) + \sigma_2\frac{(1-e^{-\kappa_2(T-t^*)})}{\kappa_2(T-t^*)}\tilde{x}_2(t^*)\end{aligned} \tag{14}$$

is normally distributed with

$$\begin{aligned}\mu^R(t^*,T) &= -\frac{\ln\frac{P(0,T)}{P(0,t^*)}}{T-t^*} + \frac{M_1(t^*,T)}{T-t^*} + \frac{M_2(t^*,T)}{T-t^*}\\ \sigma^R(t^*,T)^2 &= \frac{\sigma_1^2}{2\kappa_1^3}\left(\frac{e^{\kappa_1(T-t^*)}-1}{T-t^*}\right)^2\left(e^{2\kappa_1 t^*}-1\right)\\ &\quad +\frac{\sigma_2^2}{2\kappa_2^3}\left(\frac{1-e^{-\kappa_2(T-t^*)}}{T-t^*}\right)^2\left(1-e^{-2\kappa_2 t^*}\right).\end{aligned}$$

Figure 4: **Yield Curves in the 2FV-Model**

This figure shows 9 possible yield curves after one year when the initial term structure is flat at 7%. The 9 realizations of $(\tilde{x}_1(t), \tilde{x}_2(t))$ are given by $(-2,-2), (-2,0), \ldots, (2,0), (2,2)$. The volatility and reversion parameter are $\sigma_1 = 0.0035, \kappa_1 = 0.1859, \sigma_2 = 0.0129, \kappa_2 = 0.7662$ and yield a strong smile effect.



The term structure movement is illustrated in figure 4. Possible yield curves after one year are similar to the ones in the HJM model but with a much higher variation of spot rates with 10 years time to maturity. The drawback of the increasing volatility at the long end of the term structure is the higher probability of negative spot rates compared to the HJM model. We will discuss this issue and the importance of the smile effect in section 4.

3 Volatility Estimation

We now use principal component analysis and nonlinear regression to estimate the volatility structure. The principal component analysis[9] is mainly used to identify the number of independent factors. It is applicable to models where the volatility coefficients are left unspecified, apart from regularity conditions. Such models can be viewed as term structure models with *exogenous* volatility structures. For the estimation of models with endogenous volatility structures such as the four Gaussian models presented above, we propose a nonlinear regression approach. [10]

3.1 Assumptions

HJM model the stochastic evolution of the complete term structure. Alternatively this can be done by modelling the term structure of forward rates, the term structure of spot rates or the discount function. All three could theoretically be used for the purpose of volatility estimation but the results can be quite different. The problem results from the fact that in most bond markets there is no 'complete' set of zero bonds. Generally the term structure has to be estimated from a set of coupon bonds. The discount function or spot rates can be estimated with sufficient accuracy but estimates of forward rates corresponding to the slope of the discount function are fairly inaccurate. Therefore one should be cautious when estimating the volatility structure directly from

$$df(t,T) = \mu(t,T)dt + \sum_{k=1}^{K} \sigma_k(t,T)dW_k(t).$$

Rather one should use log-changes of bond prices

$$d\ln P(t,T) = \left[\mu^p(t,T) - \frac{1}{2}\sum_{k=1}^{K} \sigma_k^p(t,T)^2\right] dt + \sum_{k=1}^{K} \sigma_k^p(t,T)dW_k(t). \tag{15}$$

[9] The estimation of the volatility structure by means of a principal component analysis was first proposed by Heath/Jarrow/Morton (1990). Similar empirical tests were carried out by Kahn (1991) and Beckers (1993). In contrast Steeley (1990,1992) and Bühler/Schulze (1993) use the principal component analysis with the primary objective to determine the number of factors driving the term structure movement and to identify these factors.

[10] Recall that in both modell classes the term structure is exogenously given. Fitting the historical term structures is therefore irrelevant in our context. In contrast, Brown/Dybvig (1986) estimate the Cox/Ingersoll/Ross model by a cross-sectional analysis of bond prices, thus calibrating the model to fit the term structure at one point of time. Gibbons/Ramaswamy (1993) and Haverkamp (1993) estimate the Cox/Ingersoll/Ross and Longstaff/Schwartz model, respectively, by the generalized method of moments using first and second order moments. While first and second order moments are of interest for the estimation of endogenous term structure models such as the Cox/Ingersoll/Ross and Longstaff/Schwartz model, only second order moments should be used for the estimation of exogenous term structure models.

or absolute changes of spot rates[11]

$$dR(t,T) = -\frac{1}{T-t}\left[\mu^p(t,T) - \frac{1}{2}\sum_{k=1}^{K}\sigma_k^p(t,T)^2 - R(t,T)\right]dt - \sum_{k=1}^{K}\frac{\sigma_k^p(t,T)}{T-t}dW_k(t).$$

For reasons which will become clear later we prefer to use absolute spot rate changes.

The term structure movement with respect to the risk neutral and forward adjusted measure is completely specified by the volatility structure but the movement with respect to the original probability measure depends on the market prices of risk. Since the risk neutral and forward risk adjusted distributions are not observable, assumptions regarding the market prices of risk are necessary to estimate the volatility structure. To facilitate estimation we assume that all market prices of risk are constant over time. Time invariance is also assumed for the volatility coefficients. They are assumed to depend only on the time to maturity $\tau = T - t$ but not on the time t.[12] With these assumptions the only time dependent component of the spot rate drift

$$\begin{aligned} & -\frac{1}{T-t}\left[\mu^p(t,T) - \frac{1}{2}\sum_{k=1}^{K}\sigma_k^p(t,T)^2 - R(t,T)\right] \\ = \quad & -\frac{1}{\tau}\left[r(t) - R(t,T) + \sum_{k=1}^{K}(\sigma_k^p(\tau)\lambda_k - \frac{1}{2}\sigma_k^p(\tau)^2)\right] \end{aligned}$$

is the difference between the instantaneous short rate and the spot rate of the corresponding maturity divided by this maturity, $(r(t) - R(t,T))/\tau$. Since the instantaneous short rate is not directly observable we have to approximate it by the spot rate with the shortest maturity in our sample. This approximation bears little risk, because the difference $(r(t) - R(t,T))/\tau$ tends to be very small. In contrast, when estimating the volatility structure using log-changes of bond prices, these have to be adjusted for the *absolute* value of the instantaneous short rate. Therefore we prefer to use spot rate changes.

Since the term structure of spot rates can only be estimated for discrete points of time we further assume that the time-continuous stochastic process is a reasonable approximation of the discrete absolute spot rate changes

$$\Delta R(t,\tau) = R(t+\Delta t, t+\tau-\Delta t) - R(t, t+\tau),$$

which are now stated in terms of the time to maturity τ instead of the maturity date T. Spot rate changes are observed for the time to maturities $\tau_1, \ldots, \tau_M$ at the times $t_1, \ldots, t_N$. The discrete approximation of the spot rate process is

$$\Delta R(t_n,\tau_m) = -\frac{1}{\tau}\left[r(t_n) - R(t_n,\tau_m) + \mu^*(\tau_m)\right]\Delta t + \sum_{k=1}^{K}\sigma_k^R(\tau_m)\Delta W_k(t_n), \qquad (16)$$

[11] The stochastic differential equation for the spot rates can easily be obtained by application of Ito's lemma to equation (15).

[12] They have to depend on the time to maturity to model the pull-to-par-effect of the bond price movement.

with

$$\mu^*(\tau) \equiv \sum_{k=1}^{K} \sigma_k^p(\tau)\lambda_k - \frac{1}{2}\sigma_k^p(\tau)^2$$
$$\sigma_k^R(\tau_m) \equiv -\frac{\sigma_k^p(\tau_m)}{\tau}$$

3.2 Principal Component Analysis

Traditionally factor analysis is used to explore the possible underlying structure in a set of interrelated variables, with the aim to reduce the number of original variables to a much smaller set of hypothetical factors. Applied to the analysis of term structure movements, we are interested in (1) the number of factors driving the evolution of spot rates with different maturities (our variables), and (2) the economic interpretation of the hypothetical factors. Apart from this traditional use of the principal component analysis, it can be applied to the estimation of the volatility structure without restricting it by a parametric form. We will demonstrate this based on absolute spot rate changes, but in principal the same analysis could be done using absolute forward rate changes or log-changes of bond prices.[13] The absolute spot rate changes have to be corrected for the difference of the instantaneous short rate and the spot rate of the corresponding maturity divided by the maturity. Subtracting in addition the time independent drift $\mu^*(\tau)$, we get corrected spot rate changes

$$\Delta R^*(t_n, \tau_m) = \Delta R(t_n, \tau_m) + \frac{1}{\tau}\left[r(t_n) - R(t_n, \tau_m) + \mu^*(\tau_m)\right]\Delta t,$$

which according to (16) must satisfy[14]

$$\begin{pmatrix} \Delta R^*(t_1,\tau_1) & \cdots & \Delta R^*(t_N,\tau_1) \\ \Delta R^*(t_1,\tau_2) & \cdots & \Delta R^*(t_N,\tau_2) \\ \vdots & & \vdots \\ \Delta R^*(t_1,\tau_M) & \cdots & \Delta R^*(t_N,\tau_M) \end{pmatrix}$$
$$= \begin{pmatrix} \sigma_1^R(\tau_1) & \cdots & \sigma_M^R(\tau_1) \\ \sigma_1^R(\tau_2) & \cdots & \sigma_M^R(\tau_2) \\ \vdots & & \vdots \\ \sigma_1^R(\tau_M) & \cdots & \sigma_M^R(\tau_M) \end{pmatrix} \begin{pmatrix} \Delta W_1(t_1) & \cdots & \Delta W_1(t_N) \\ \Delta W_2(t_1) & \cdots & \Delta W_2(t_N) \\ \vdots & & \vdots \\ \Delta W_M(t_1) & \cdots & \Delta W_M(t_N) \end{pmatrix}.$$

Given this multivariate linear model, a principal component analysis can be applied to estimate the volatility matrix $(\sigma_m(\tau_m))_{M,M}$ by the matrix of the factor loadings,

[13] If forward rates are assumed to behave proportional to the level of interest rates as in Heath/Jarrow/Morton (1990), relative instead of absolute forward rate changes have to be used.

[14] Since we use a principal component analysis we assume at the beginning $K = M$ factors, and extract only those factors with a high explanatory power.

the so called factor pattern.[15]

The importance of the principal components can be assessed by the percent of variation explained by the factors. The percent of variation of the first to last factor is given by the ordered Eigenvalues of the covariance matrix of the standardized spot rate changes divided by the number of variables. To determine the number of factors driving the spot rate changes we rely on the simple eigenvalue criterion and the Scree test which already give unambiguous results.[16]

A first interpretation of the extracted factors will be done by examining the shape of the corresponding volatility functions. To identify observable variables which are able to explain the extracted factors we estimate factor scores and test regressions of the form

$$F_{i,n} = \alpha_n + \beta_n x_n + \epsilon_n,$$

where $F_{i,n}$ denotes the factor score of the i-th factor with respect to the n-th observation and x_n is an observable variable like a spot rate for a certain maturity or a spread between a long and a short rate. The spot rate with a short time to maturity as one factor would be consistent with traditional one-factor models like VASICEK (1977) and COX, INGERSOLL, ROSS (1985) of the term structure which are based on the short rate development. A spot rate with a long time to maturity as a second factor is consistent with the BRENNAN/SCHWARTZ (1979) model, whereas a spread as a second factor is consistent with SCHAEFER/SCHWARTZ (1984) who use a long rate and a spread between a long and a short rate because these are generally uncorrelated.[17]

3.3 Nonlinear Regression

In this section we develop a nonlinear regression approach to the estimation of the four Gaussian interest rate models presented in section 2. The description concentrates on the two-factor Vasicek model (2FV-model) since this model includes the others as special cases. The volatility structure in the 2FV-model may be stated in terms of the forward rates, bond prices or spot rates.

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[15]It should be noted that the results of the principal component analysis refer to standardized spot rate changes. The factor loading therefore have to be multiplied by the standard deviation of spot rate changes with the corresponding maturity. To facilitate an interpretation of the estimated volatility matrix we multiply the factor loadings furthermore by $\sqrt{\Delta t}$ to obtain annualized volatilities.

[16]Statistical tests of the number of factors are neglected since they generally lead to a too large number of factors (cf. GORSUCH (1983, p. 164)) and the results based on the eigenvalue criterion and scree tests are already unambiguous.

[17]We have not considered any non-interest rate variables which have been incorporated by RICHARD (1978) and COX, INGERSOLL, ROSS (1985) since the identification of the factors is not crucial to the main purpose of our paper.

	factor 1	factor 2
forward rate volatility	$\sigma_1(t,T) = \sigma_1 e^{\kappa_1(T-t)}$	$\sigma_2(t,T) = \sigma_2 e^{-\kappa_2(T-t)}$
bond price volatility	$\sigma_1^p(t,T) = \frac{\sigma_1}{\kappa_1}(1 - e^{\kappa_1(T-t)})$	$\sigma_2^p(t,T) = \frac{\sigma_2}{\kappa_2}(e^{-\kappa_2(T-t)} - 1)$
spot rate volatility	$\sigma_1^R(t,T) = \frac{\sigma_1}{(T-t)\kappa_1}(e^{\kappa_1(T-t)} - 1)$	$\sigma_2^R(t,T) = \frac{\sigma_2}{(T-t)\kappa_2}(1 - e^{-\kappa_2(T-t)})$

For $\sigma_2 = \kappa_1 = 0$ the 2FV-model simplifies to the Ho/Lee-model. The Vasicek-model results for $\sigma_1 = 0$ and the HJM-model is included for $\kappa_1 = 0$.[18] The volatility coefficients of these models satisfy the stationarity assumption of section 3.1. With the assumption of constant market prices of risk, the drift $\mu^*(t,T)$ depends only on the time to maturity $\tau = T-t$, but not on time t. The variance of spot rate changes is therefore independent of $\mu^*(t,T)$.

Absolute spot rate changes in the 2FV-model are given by

$$\begin{aligned}\Delta R(t,\tau) &= -\frac{1}{\tau}[r(t) - R(t,\tau) + \mu^*(\tau)]\Delta t \\ &\quad + \frac{\sigma_1}{\tau\kappa_1}(e^{-\kappa_1\tau} - 1)\Delta W_1(t) - \frac{\sigma_2}{\tau\kappa_2}(e^{-\kappa_2\tau} - 1)\Delta W_2(t)\end{aligned}$$

Adjusted for the difference of the instantaneous short rate and spot rate

$$\Delta R(t,\tau) + \frac{1}{\tau}[r(t) - R(t,\tau)]\Delta t \sim \mathrm{N}\left(-\frac{1}{\tau}\mu^*(\tau)\Delta t, v^R(\tau)\Delta t\right)$$

they are normally distributed with mean $-\frac{1}{\tau}\mu^*(\tau)\Delta t$ and variance

$$v^R(\tau) = v^R(\tau;\sigma_1,\sigma_2,\kappa_1,\kappa_2) = \left[\frac{\sigma_1}{\tau\kappa_1}(e^{\kappa_1\tau} - 1)\right]^2 + \left[\frac{\sigma_2}{\tau\kappa_2}(e^{-\kappa_2\tau} - 1)\right]^2.$$

Based on a time series of N spot rate changes for M maturities $\tau_1,\dots,\tau_M$ the volatility parameters $\sigma_1,\sigma_2,\kappa_1,\kappa_2$ can be estimated by the nonlinear regression

$$\frac{s^R(\tau_m)^2}{\Delta t} = \left[\frac{\sigma_1}{\tau_m\kappa_1}(e^{\kappa_1\tau_m} - 1)\right]^2 + \left[\frac{\sigma_2}{\tau_m\kappa_2}(e^{-\kappa_2\tau_m} - 1)\right]^2 + \epsilon_m, \quad m = 1\dots,M \quad (17)$$

where $s^R(\tau_m)^2$ denotes the sample variance of the spot rate changes with maturity τ_m. We computed the least squares estimators by the Gauss-Newton method due to Hartley (1961).[19] It should be noted that applying the method of least squares to the variances in this way is equivalent to using the (generalized) method of moments based only on second moments with an identity matrix as weighting matrix.

[18] The spot rate and bond price volatility structure have to be derived for $\kappa_1 = 0$ starting with the forward rate process or may be obtained as a limiting case with $\kappa_1 \to 0$, since $\kappa_1 = 0$ is not defined for the above expressions of the bond price and spot rate volatility.

[19] See for example Seber/Wild (1989, p. 623)

4 Volatility Estimates for the German Bond Market

4.1 Data

The empirical results reported here are based on weekly data of the term structure of spot rates from January 1980 to December 1993. The term structure of spot rates are estimated from weekly prices of German T-Bonds (Bundesanleihen) and T-Notes (Bundesobligationen) using the polynomial method proposed by Chambers/Carleton/Waldman (1984). We use a polynomial of order three which results in a relatively smooth yield curve while incurring a potentially large mean absolute deviation of theoretical bond prices from market prices. A spline method or polynomial of higher order allows for a smaller mean absolute deviation but leads to strongly oscillating yield curves. Since we are interested in the movement of the whole bond market and not the volatility caused by some outliers, we prefer a smoothed yield curve for the purpose of volatility estimation.[20]

We employ a data set containing the spot rates of maturities $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, \ldots, 9\frac{1}{4}, 9\frac{1}{2}$ years.[21] The total sample period is divided into seven two year-subperiods each with 104 observed yield curves. The mean spot rate (standard deviation) in the total sample period is 6.8% (0.21%) for the shortest maturity and 7.4% (0.11%) for the longest maturity. This suggest a mean reversion in the short rate process inducing a spot rate volatility decreasing with the time to maturity. In the long run this may be a good description of the term structure movement, but if we consider the two-year subperiods other effects show up. Figure 5 illustrates the term structure movement from 1/88 to 12/89 which is dominated by a *reversion* of the short rate. The volatility of short rates is much larger than the volatility of long rates. But the following two-year period 1/90–12/91 (figure 6) shows in the first year an almost parallel shift and in the second year a *twist* of the term structure. A twist with long and short rates moving into opposite directions causes high volatilities of long and short rates but lower volatilities for medium maturities. A twist in the term structure is therefore associated with a smile-effect in the volatility structure of spot rates. A subperiod with a modest reversion effect is illustrated in figure 7.

4.2 Principal Component Analysis

The main results of the principal component analysis are summarized in table 1. These indicate that the joint movement of spot rates with different maturities can be explained by two independent stochastic factors. The percentage of the total variance explained by the first factor varies between 72.68% and 83.45%. Adding

[20] In contrast, when valuing interest rate contingent claims, we prefer an estimation method which guarantees the lowest mean absolute since any estimation error of the term structure will bias the results of the contingent claim valuation.

[21] Shorter maturities than half a year were not considered since the estimation procedure tends to yield unstable results in this region.

Figure 5: **Yield Curves 1/88 – 12/89**

This figure shows the term structure movement from 1/88 to 12/89. While the first panel shows all weekly estimated yield curves, the second panel pictures only the yield at the beginning, after one year, and at the end of the two-year period, to illustrate the dominating effect. From 1/88 to 12/89 a strong *mean reversion* is observable.

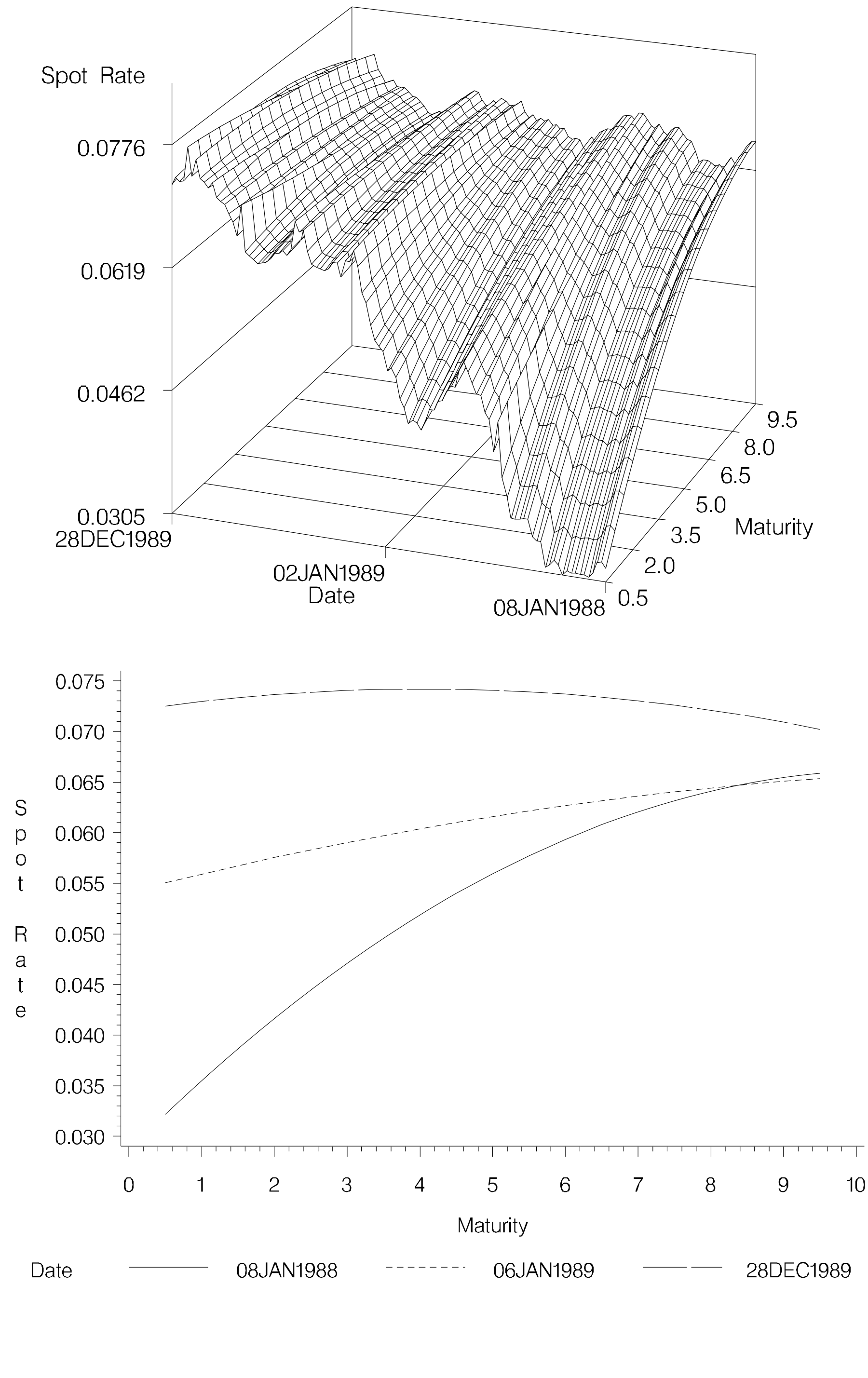


Figure 6: **Yield Curves 1/90 – 12/91**

This figure shows the term structure movement from 1/88 to 12/89. While the first panel shows all weekly estimated yield curves, the second panel pictures only the yield at the beginning, after one year, and at the end of the two-year period, to illustrate the dominating effect. From 1/91 to 12/91 a *twist* is observable.

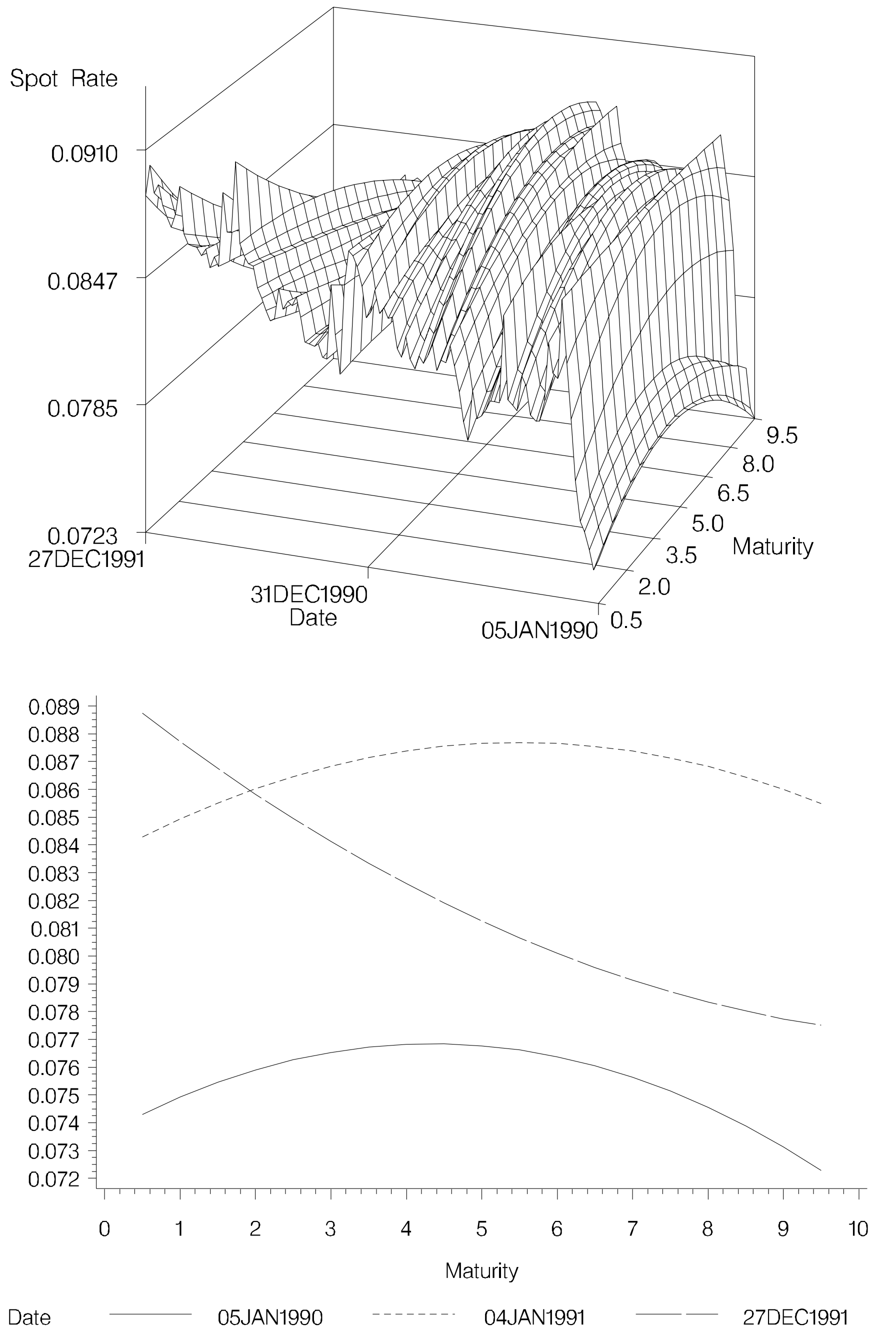
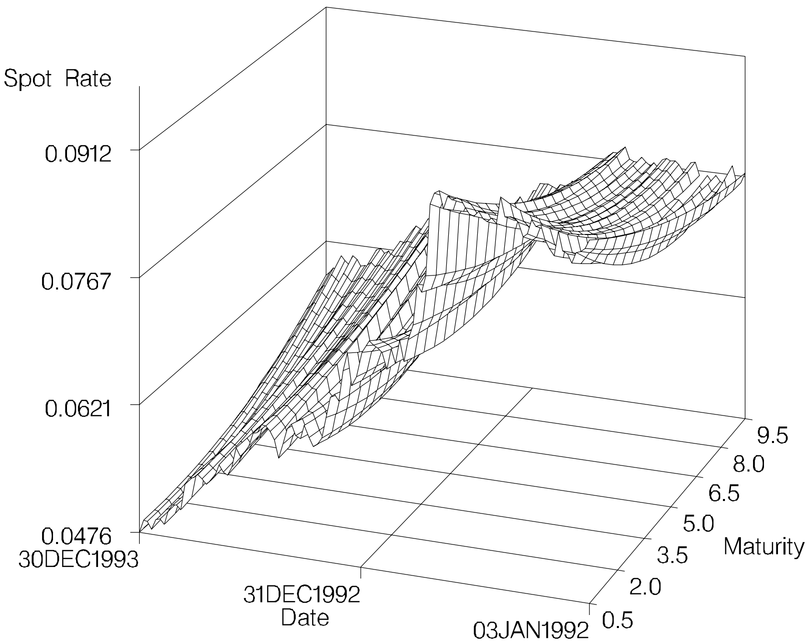
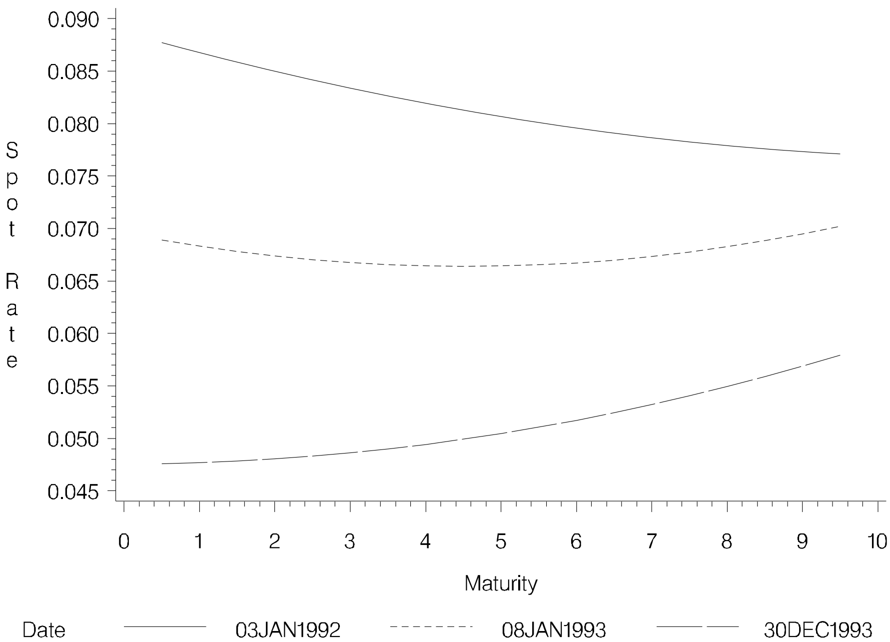


Figure 7: **Yield Curves 1/92 – 12/92**

This figure shows the term structure movement from 1/92 to 12/92. While the first panel shows all weekly estimated yield curves, the second panel pictures only the yield at the beginning, after one year, and at the end of the two-year period, to illustrate the dominating effect. From 1/92 to 12/93 a weak *mean reversion* is observable.





the second factor which explains between 11.75% and 23.22% , approximately 95% of the total variance is explained in all seven subperiods. The eigenvalues of the third factor are in all cases smaller or only slightly larger than one and the percentage of variation explained by this factor is only between 2.3% bis 5.5%.

Table 1: **Explained Variance of the Spot Rates**

Period		factor 1	factor 2	factor 3
1/80–12/81	variance explained	0.8178	0.1447	0.0375
	cumulative	0.8178	0.9625	1.0000
1/82–12/83	variance explained	0.8345	0.1422	0.0234
	cumulative	0.8345	0.9766	1.0000
1/84–12/85	variance explained	0.7268	0.2322	0.0410
	cumulative	0.7268	0.9590	1.0000
1/86–12/87	variance explained	0.8056	0.1396	0.0548
	cumulative	0.8056	0.9452	1.0000
1/88–12/89	variance explained	0.8106	0.1593	0.0301
	cumulative	0.8106	0.9699	1.0000
1/90–12/91	variance explained	0.7999	0.1622	0.0379
	cumulative	0.7999	0.9621	1.0000
1/92–12/93	variance explained	0.8285	0.1175	0.0540
	cumulative	0.8285	0.9460	1.0000

The unrotated factor pattern of the first two factors is plotted in figure 8 for the period 1/88–12/89. The other six subperiods show a similar result. The volatility function of the first factor is slightly decreasing with time to maturity but positive over the whole maturity range. Factor one can therefore be regarded as a nonparallel shift. Factor two can be viewed as a twist factor since the corresponding volatility function is positive for maturities smaller than 4.5 years and negative for larger maturities. A negative volatility in this case means that long rates decrease if short rates increase and vice versa. We will discuss the importance of the twist in the different subperiods in greater detail in section 4.3.[22]

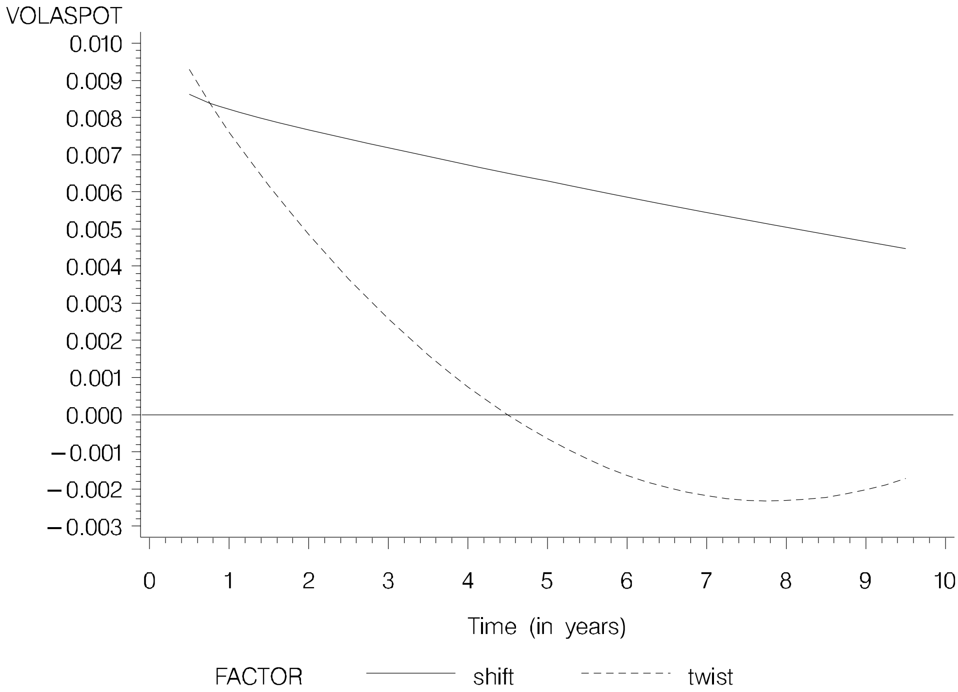
Table 9 (in the appendix) reports the results of the linear regressions of factors on observable variables for the subperiod 1/90–12/91. We omit the results for the six other subperiods since the results are very similar. The shift is best explained by the spot rate change with four years to maturity and the twist is best explained by the spread between the spot rate with 9 and 1/2 year to maturity.[23]

[22]We omit any factor rotation since we use the principal component analysis in this paper mainly to explore the number of independent factors driving the term structure movement.

[23]Bühler/Schulze (1993) obtain similar results for a principal component analysis of monthly yield curves from 1968 to 1988.

Figure 8: **Spot Rate Volatility Structure 1/88 – 12/89**

This figure shows the volatility functions of the shift and twist factor estimated by the factor pattern of the first two principal components for the subperiod 1/88 – 12/89.



4.3 Nonlinear Regression

The parameter estimates of the four Gaussian models for the seven subperiods are presented in table 2 to 5. The asymptotic 95%-confidence intervals of the estimated parameters are given in parentheses. The R^2 statistic is omitted for the Ho/Lee-model since the restriction of the spot rate volatility to a constant does not allow a fitting of a non-constant volatility curve. The nonlinear regression reduces in this case to a regression with only an intercept. The method of least squares applied to such a degenerate case gives an estimator of σ equal to the square root of the average spot rate variance across all maturities. The R^2 is always zero since no variation is explained. Hence, based on an ex-post-analysis of the volatility structure the Ho/Lee-model must be rejected.[24]

The results in table 3 reveal that the *one-factor model of Vasicek* leads to a vital improvement compared to the Ho/Lee model. The R^2 statistic shows that except for the two twist periods 1/86–12/87 and 1/90–12/91 almost 90% percent of the

[24] But this does not necessarily imply that the Ho/Lee-model is also in an ex-ante analysis of option prices the worst of the tested models.

Table 2: **Parameter Estimates for the Ho/Lee-Model**

Period	σ_1
1/80–12/81	0.0142 (0.0130;0.0155)
1/82–12/83	0.0095 (0.0086;0.0104)
1/84–12/85	0.0083 (0.0072;0.0095)
1/86–12/87	0.0079 (0.0075;0.0084)
1/88–12/89	0.0075 (0.0067;0.0083)
1/90–12/91	0.0080 (0.0076;0.0085)
1/92–12/93	0.0075 (0.0069;0.0082)

spot rate variation is explained by the model. All parameter estimates except for the twist periods are highly significant and vary only slightly over time. The Vasicek model cannot fit the observed volatility smile in the two twist periods illustrated in figure 9. The subperiod 1/88–12/89 is one the five subperiods where a good fit of the Vasicek model is possible while the subperiod 1/90–12/91 illuminates the limitations of the Vasicek model.

The *two-factor model of HJM* comprises the two preceding models. The first factor is the parallel shift factor of Ho/Lee and the second factor which is analogous to the Vasicek model can be viewed as a pseudo twist affecting the short rates stronger than the long rates. But the variance function

$$v(\tau) = \sigma_1^2 + \left[\frac{\sigma_2}{\tau\kappa}(e^{-\kappa\tau} - 1)\right]^2$$

is monotonically decreasing

$$\frac{\partial v(\tau)}{\partial \tau} = \underbrace{\left[\frac{\sigma_2}{\tau\kappa}(e^{-\kappa\tau} - 1)\right]}_{<0} \underbrace{\left(\frac{e^{-\kappa\tau} - 1}{\kappa\tau} - e^{-\kappa\tau}\right)(-2)\frac{\sigma_2}{\tau}}_{>0} < 0$$

so that the smile effect cannot be modeled. The fit in the two subperiods exhibiting a strong smile effect is thus very poor although better than for the Vasicek model. This is due to the fact that the variance is monotonically decreasing as in the Vasicek model, but because of the shift factor not necessarily converging to zero. Table 4 and figure 10 summarize the corresponding results for the HJM model.

Table 3: **Parameter Estimates for the Vasicek-Model**

Period	σ	κ	R^2
1/80–12/81	0.0228 (0.0216;0.0239)	0.2583 (0.2181;0.2985)	0.8815
1/82–12/83	0.0162 (0.0155;0.0169)	0.3089 (0.2670;0.3478)	0.9277
1/84–12/85	0.0179 (0.0169;0.0188)	0.5482 (0.4735;0.6229)	0.9403
1/86–12/87	0.0084 (0.0075;0.0093)	0.0243 (-.0184;0.0670)	0.0262
1/88–12/89	0.0131 (0.0125;0.0137)	0.3281 (0.2873;0.3690)	0.9320
1/90–12/91	0.0081 (0.0071;0.0091)	0.0044 (-.0378;0.0466)	0.0009
1/92–12/93	0.0121 (0.0118;0.0124)	0.2564 (0.2361;0.2767)	0.9695

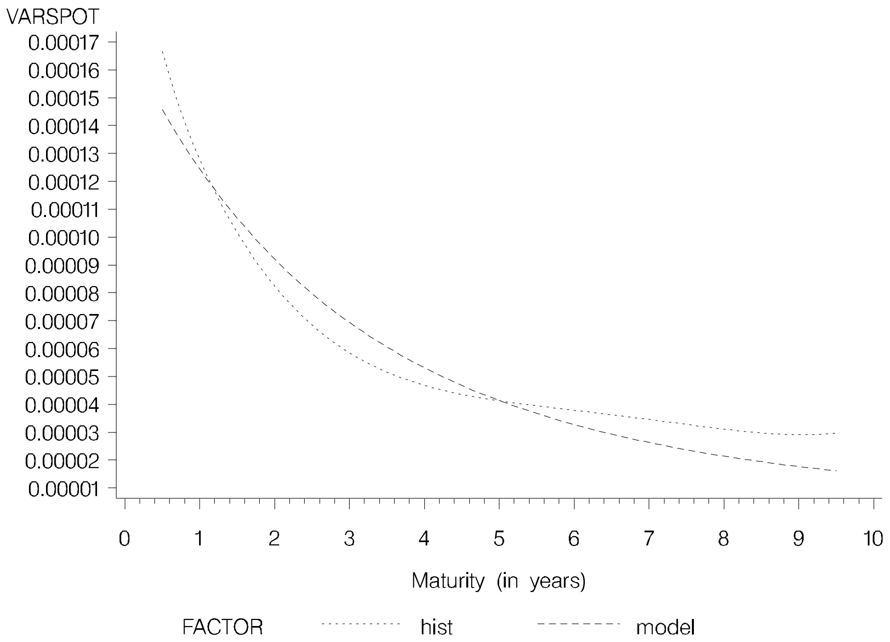
Table 4: **Parameter Estimates for the HJM-Model**

Period	σ_1	σ_2	κ	R^2
1/80–12/81	0.0107 (0.0105;0.0109)	0.0255 (0.0250;0.0260)	0.8012 (0.7515;0.8509)	0.9962
1/82–12/83	0.0065 (0.0064;0.0066)	0.0174 (0.0172;0.0175)	0.6833 (0.6612;0.7054)	0.9990
1/84–12/85	0.0050 (0.0047;0.0054)	0.0193 (0.0186;0.0199)	0.9009 (0.812;40.9894)	0.9905
1/86–12/87	0.0074 (0.0070;0.0079)	0.0145 (0.0055;0.0235)	2.2911 (-.2212;4.8035)	0.5417
1/88–12/89	0.0051 (0.0050;0.0051)	0.0141 (0.0140;0.0142)	0.7149 (0.6946;0.7352)	0.9992
1/90–12/91	0.0076 (0.0071;0.0080)	0.0161 (0.0026;0.0297)	2.7859 (-.9698;6.5416)	0.4884
1/92–12/93	0.0045 (0.0043;0.0048)	0.0122 (0.0120;0.0124)	0.4416 (0.4022;0.4810)	0.9938

Figure 9: **Explained Variance in the Vasicek model**

This figure shows the spot rate variance explained by the Vasicek model compared to the historically spot rate variance for a subperiod with only a reversion effect 1/88 – 12/89 and a subperiod with a strong smile effect 1/90 – 12/91.

1/88 – 12/89



1/90 – 12/91

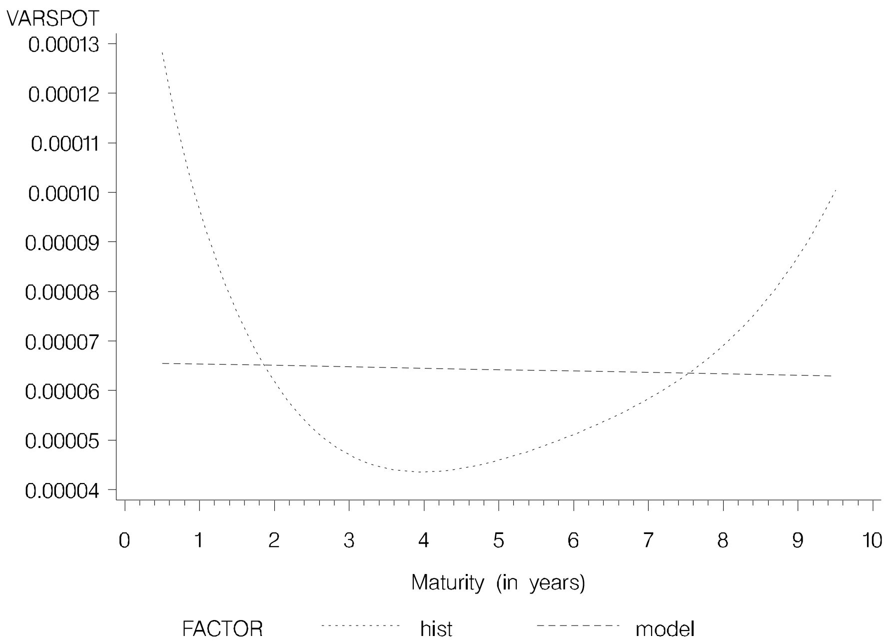
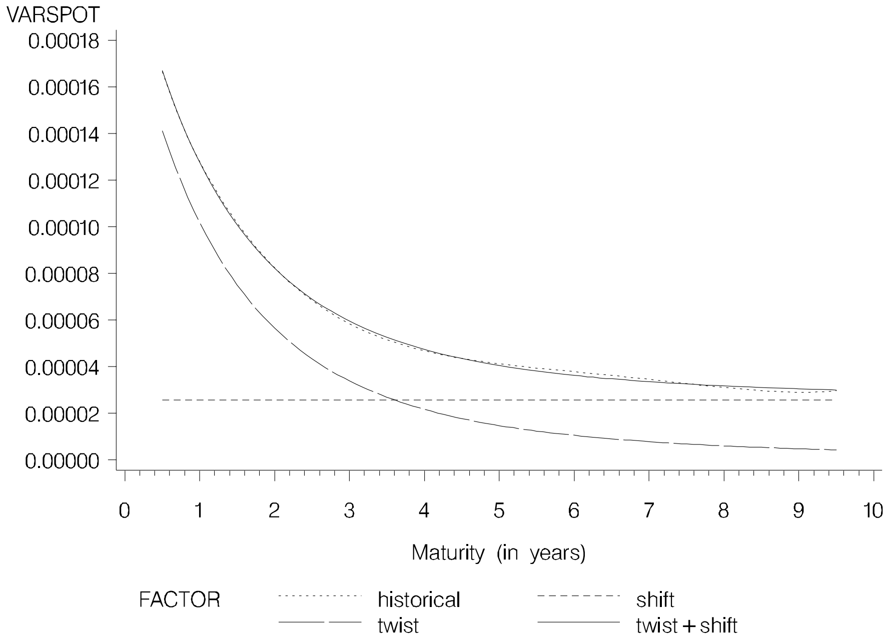


Figure 10: **Explained Variance in the HJM model**

This figure shows the spot rate variance explained by the HJM model compared to the historically spot rate variance for a subperiod with only a reversion effect 1/88 – 12/89 and a subperiod with a strong smile effect 1/90 – 12/91.

1/88 – 12/89



1/90 – 12/91

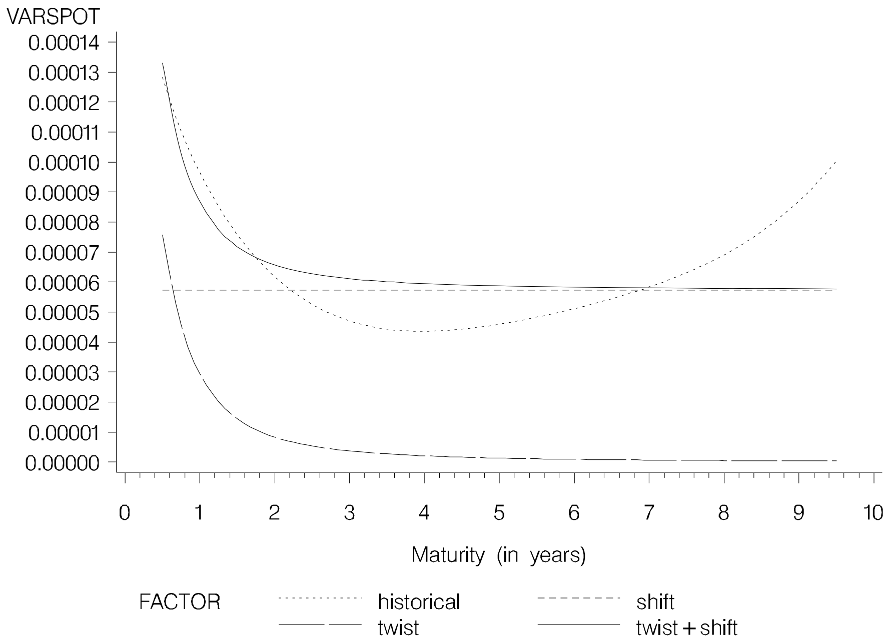


Table 5: **Parameter Estimates for 2FV-Model**

Period	σ_1	κ_1	σ_2	κ_2	R^2
1/80–12/81	0.0107 (0.0095;0.0119)	0.0000 (-.0247; 0.0247)	0.0255 (0.0250; 0.0260)	0.8012 (0.7048; 0.8976)	0.9962
1/82–12/83	0.0050 (0.0048; 0.0052)	0.0546 (0.0461; 0.0631)	0.0176 (0.0175; 0.0176)	0.5968 (0.5837; 0.6099)	0.9999
1/84–12/85	0.0019 (0.0016; 0.0022)	0.2088 (0.1834; 0.2342)	0.0191 (0.0190; 0.0193)	0.7225 (0.7006; 0.7444)	0.9991
1/86–12/87	0.0029 (0.0027; 0.0032)	0.2129 (0.1975; 0.2282)	0.0122 (0.0120; 0.0125)	0.5832 (0.5291; 0.6372)	0.9881
1/88–12/89	0.0051 (0.0047; 0.0055)	0.0000 (-.0181; 0.0181)	0.0141 (0.0140; 0.0143)	0.7152 (0.6734; 0.7570)	0.9992
1/90–12/91	0.0035 (0.0034; 0.0036)	0.1859 (0.1805; 0.1913)	0.0129 (0.0127; 0.0130)	0.7662 (0.7339; 0.7986)	0.9977
1/92–12/93	0.0033 (0.0018; 0.0054)	0.0546 (-.0539; 0.1631)	0.0124 (0.0120; 0.0128)	0.3949 (0.3177; 0.4721)	0.9939

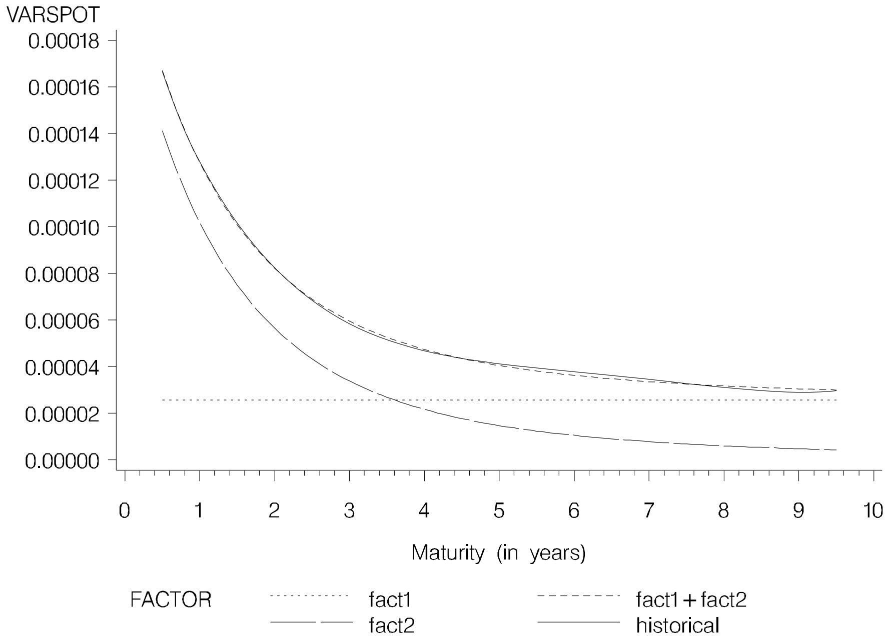
Comparing the R^2 statistics of the HJM model in table 4 and our *2FV model* in table 5 reveals that the advantage of the 2FV model is negligible in five of seven subperiods. In these periods κ_1 is not significantly (statistically or economically) different from zero (except for the subperiod 1/84–12/85). The absence of a smile effect is obvious, when κ_1 is not significantly different from zero, since the 2FV models reduces to the HJM model in this case. But even with a significantly positive κ_1 the smile effect may be negligible. The average slope and convexity of the estimated variance function serves as a measure of the smile effect. When a smile effect is present the average slope should be close to zero while the average convexity should be different from zero. Table 6 shows that this is the case only in the subperiods 1/86–12/87 and 1/90–12/91 but not in the subperiod 1/84–12/85. In the two periods with a significant smile effect the 2FV model outperforms the HJM model clearly. The fit of the estimated variance function is illustrated for two subperiods 1/88-12/89 and 1/90–12/91 in figure 11. The smile effect is largely determined by σ_1 and κ_1. The reversion of the short rate to a long run mean is as in the Vasicek model determined by κ_2. A comparison of the estimation results for subperiods 1/88–12/89 and 1/92–12/93 confirms the strong ($\kappa_1 = 0.7152$) and weak ($\kappa_1 = 0.3949$) mean reversion hypothesized in section 4.1 based on the term structure plots.

We now to assess the probability of negative spot rates in the four Gaussian models based on historical parameter estimates. The probability is plotted in figure 14 in the appendix for spot rate maturities of one to ten years based on the parameter estimates of the subperiod 1/92–12/93. Since we observe the smile effect only in

Figure 11: **Explained Variance in the 2FV-Model**

This figure shows the spot rate variance explained by the two-factor Vasicek model (2FV) compared to the historically spot rate variance for a subperiod with only a reversion effect 1/88 – 12/89 and a subperiod with a strong smile effect 1/90 – 12/91.

1/88 – 12/89



1/90 – 12/91

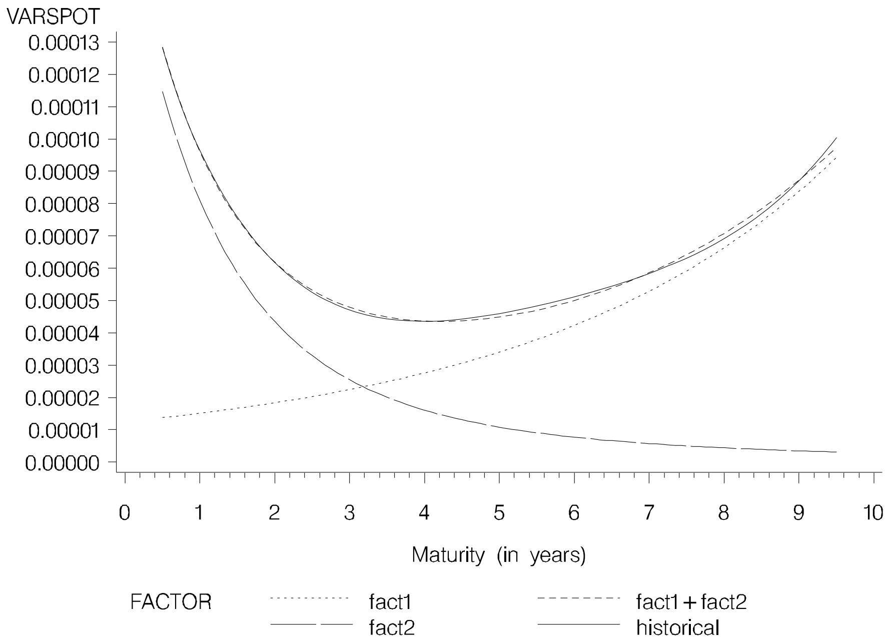


Table 6: **Magnitude of the Smile Effect**

This table summarizes the average slope and convexity of the estimated variance functions as a measure of the smile effect. Both are for expositional purpose multiplied by 100^2. As a measure for the average slope we use the difference of the variance function for the longest and shortest maturity which is equivalent to the integral over the first derivative of the variance function. Analogously we measure the average convexity by the difference of the first derivative of the variance function for the longest and shortest maturity.

Period	average slope	average convexity
1/80–12/81	–2.79041	11.4013
1/82–12/83	–2.03442	9.1701
1/84–12/85	–2.17394	10.3570
1/86–12/87	–0.27706	4.7119
1/88–12/89	–1.37048	5.6380
1/90–12/91	–0.31057	4.7777
1/92–12/93	–1.09019	4.9452

the subperiod 1/90–12/91, we use the parameter estimates of this subperiod to illustrate the drawback of the 2VF model. Apart from the volatility parameters, the probability of negative spot rates is largely influenced by the initial term structure of forward rate. The larger the initial forward rates the lower the probability. We assume an unfavorable scenario of a flat term structure with 4% for all maturities. The volatility of the uncertain future spot rate and hence the probability of negative spot rates increases with time. We consider a time horizon of ten years.

In all four models the probability of negative spot rates after three years is smaller than 0.5% and thus not critical for the valuation of most interest rate futures and options. If bonds with embedded options are to be valued, the behavior of the spot rate dynamics after three years requires a closer look. In the Ho/Lee model with a spot rate variance proportional to time, the probability of negative spot rates after ten years is approximately 5%. The variance of the spot rate with maturity τ in the Vasicek model converges to the constant

$$\frac{\sigma^2}{2\kappa^3}\left(\frac{1-e^{-\kappa\tau}}{\tau}\right)^2.$$

with time t to infinity Accordingly the probability of negative spot rates converges to a constant which is higher for small times to maturity. Based on our parameter estimates the probability is even for spot rates with one year to maturity negligible small (0.309%). The HJM model comprising the two factors of the Ho/Lee and Vasicek model yields probabilities which do not converge to a constant level but are even for a ten year horizon smaller than 1.3%. The examination of the probability of negative spot rates shows a drawback of the 2FV model since the spot rate variance tends to explode with time for long maturities leading to about 10% probability of negative spot rates after five years. These results imply that the problem of negative spot rates in Gaussian models is negligible in the Ho/Lee, Vasicek, HJM and 2FV model when valuing interest rate contingent claims with less than three years to maturity. For longer maturities only the Vasicek and HJM model are appropriate.

5 Implications for Option Pricing

Gaussian interest rate models are attractive because of their analytical tractability. Closed form solutions for futures and options on zero bonds are readily available. It is well known that futures prices are martingales with respect to the risk neutral measure. Since the futures price equals the spot price at maturity, todays futures price is simply the expected future spot price, which is in a Gaussian model

$$H(t,t^*,T) = \mathrm{E}_{\tilde{Q}}[P(t^*,T)|\mathcal{F}_t] = \frac{P(t,T)}{P(t,t^*)} \exp\left\{ \sum_{k=1}^{K} - \int_t^{t^*} \sigma^p(v,t^*)\sigma(v,t^*,T)dv \right\}.$$

Futures on coupon bonds can be valued as portfolios of futures on zero bonds. The lognormal bond price distribution allows further for a Black/Scholes type valuation of options on zero bonds.

Theorem 5.1 (Calls on Zero Bonds)
In a locally arbitrage-free Gaussian K-factor model the price of a call on a zero bond $P(t,T)$ with strike price K and maturity t^* is:

$$C(t,K,t^*,T) = P(t,T)N(d_1) - KP(t,t^*)N(d_2)$$

with

$$\begin{aligned} d_1 &\equiv \frac{1}{\nu(t,t^*,T)} \ln\left(\frac{P(t,T)}{P(t,t^*)K}\right) + \frac{1}{2}\nu(t,t^*,T) \\ d_2 &\equiv d_1 - \nu(t,t^*,T) \\ \nu(t,t^*,T)^2 &\equiv \sum_{k=1}^{K} \int_t^{t^*} \sigma_k(v,t^*,T)^2 dv = \sum_{k=1}^{K} \int_t^{t^*} [\sigma_k^p(v,T) - \sigma_k^p(v,t^*)]^2 dv \end{aligned}$$

Proof: The call price may alternatively be derived with respect to the risk neutral or forward risk adjusted measure. But deriving it with respect to the risk neutral measure requires to compute an expectation of a product of stochastic variables. The forward risk adjusted measure in contrast effectively decouples that product.[25]

[25] cf. Jamshidian (1991)

The forward price of the call is a martingale with respect to Q^*:

$$\begin{aligned} C(t,t^*,T) &= E_{Q^*}[C(t^*,t^*,T)|\mathcal{F}_t] \\ \Leftrightarrow \frac{C(t,T)}{P(t,t^*)} &= E_{Q^*}[C(t^*,T)|\mathcal{F}_t] \end{aligned}$$

The forward price given the information $\mathcal{F}_t$ may thus be calculated by

$$\begin{aligned} C(t,T) &= E_{Q^*}[C(t^*,T)|\mathcal{F}_t]P(t,t^*) \\ &= E_{Q^*}[(P(t^*,T)-K)^+|\mathcal{F}_t]P(t,t^*) \\ &= E_{Q^*}[P(t^*,T)\,1_{\{P(t^*,T)>K\}}|\mathcal{F}_t]P(t,t^*) \\ &\quad -KP(t,t^*)\mathrm{Prob}_{Q^*}(P(t^*,T)>K|\mathcal{F}_t) \\ &= P(t,T)N(d_1)-KP(t,t^*)N(d_2) \end{aligned}$$

The last equation follows by substituting $P(t^*,T)$ with equation (10) and a standard calculation of the expectation. ■

This option pricing formula resembles the Black/Scholes formula in an obvious way. The price of a zero bond $P(t,t^*)$ with maturity t^* replaces $e^{-r(t^*-t)}$ in the traditional Black/Scholes model. Moreover, the variance $\sigma^2(t^*-t)$ is replaced by $\nu(t,t^*,T)^2$, which is given in our Gaussian models by:[26]

$$\begin{aligned} \nu_{HL}(t,t^*,T)^2 &= \sigma^2(T-t^*)^2(t^*-t) \\ \nu_{Vas}(t,t^*,T)^2 &= \frac{\sigma^2}{2\kappa^3}\left(1-e^{-\kappa(T-t^*)}\right)^2\left(1-e^{-2\kappa(t^*-t)}\right) \\ \nu_{HJM}(t,t^*,T)^2 &= \sigma_1^2(T-t^*)^2(t^*-t)+\frac{\sigma_2^2}{2\kappa^3}\left(1-e^{-\kappa(T-t^*)}\right)^2\left(1-e^{-2\kappa(t^*-t)}\right) \\ \nu_{2FV}(t,t^*,T)^2 &= -\frac{\sigma_1^2}{2\kappa_1^3}\left(1-e^{\kappa_1(T-t^*)}\right)^2\left(1-e^{2\kappa_1(t^*-t)}\right) \\ &\quad +\frac{\sigma_2^2}{2\kappa_2^3}\left(1-e^{-\kappa_2(T-t^*)}\right)^2\left(1-e^{-2\kappa_2(t^*-t)}\right) \end{aligned}$$

Despite the similarity of the option pricing formulas in the Black/Scholes model and the Gaussian term structure models there is a crucial difference. Even the BLACK (1976) modification which is based on forward prices to eliminate the changing variance of the bond prices as time elapses is not able to price options on bonds with different maturities in a consistent manner. The volatility has to be estimated for every option separately. In contrast, the four estimated Gaussian models can price all interest rate contingent claims with one set of volatility parameters. For example all four models incorporate the pull-to-par-effect because $\nu(t,t^*,T)$ converges to zero

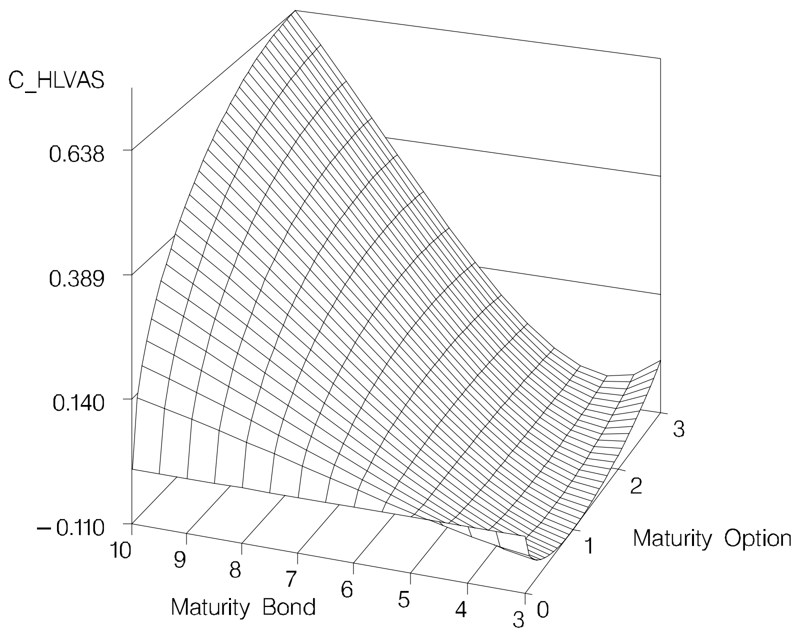
[26]It should be noted that the option pricing formula for the Vasicek model is equivalent to the one derived by JAMSHIDIAN (1989) for the original Vasicek model, which is not necessarily consistent with the initial term structure. This is not surprising, because the adjustment to the initial term structure can be achieved by correcting the drift rate, which does not enter the option pricing formula.

with $t^* \rightarrow T$. In the following we thus analyze the ability of the four models to price different contingent claims. We use at-the-money calls on zero bonds with maturities ranging from 3 to 10 years. The option maturity varies between 0 and 3 years.

First we compare the one-factor models of Ho/Lee and Vasicek. The key difference between the two models is the spot rate variance decreasing with time to maturity in the Vasicek model and being constant in the Ho/Lee model. We take as volatility parameters for the Vasicek model the parameter estimates for the subperiod 1/92–12/93 and calculate the implied volatility for the Ho/Lee model which gives the same option value for the two-year option on the five-year bond. The call price difference between the Ho/Lee and Vasicek model for the previously described set of options is shown in figure 12. Naturally the difference is zero if the option

Figure 12: **Ho/Lee-Call Price versus Vasicek-Call Price**

This figure shows call price differences $C_{HL} - C_{Vas}$ for at-the-money options with a maturity of 0–3 years. The maturity of the underlying bonds with a face value of 100 varies between 3 and 10 years. The initial term structure is flat at 7%. The volatility parameters are $\sigma_{HL} = 0.0067, \sigma_{Vas} = 0.0121, \kappa = 0.2564$.



expires immediately and for the three-year option on a three year bond because prices of at-the-money calls have to be zero in these cases. The price difference is also zero or close to zero for all options on the five-year bond which was used for the calculation of the implied volatility in the Ho/Lee model. But options on bonds with maturities smaller (larger) than five years are undervalued (overvalued) and the absolute (relative) price difference for the two-year option ranges from -0.08072 (-23.35%) to 0.5848 (48.59%). We interpret this large price differentials as a valuation error of the Ho/Lee model because the observed reversion in 1/92–12/93 is not captured by the Ho/Lee model. Prices for a two-year at-the-money call according

Table 7: **Call Prices BS, Ho/Lee, Vasicek, HJM**

This table presents call prices of at-the-money options with a maturity of 2 years for the Black/Scholes, Ho/Lee, Vasicek and HJM model. The volatility parameters are $\sigma_{BS} = 0.0200, \sigma_{HL} = 0.0067, \sigma_{Vas} = 0.0121, \kappa_{Vas} = 0.2564, \sigma_{1,HJM} = 0.0027, \sigma_{2,HJM} = 0.0161, \kappa_{HJM} = 0.4416$. The maturity of the underlying bonds varies between 3 and 10 years and the initial term structure is flat at 7%.

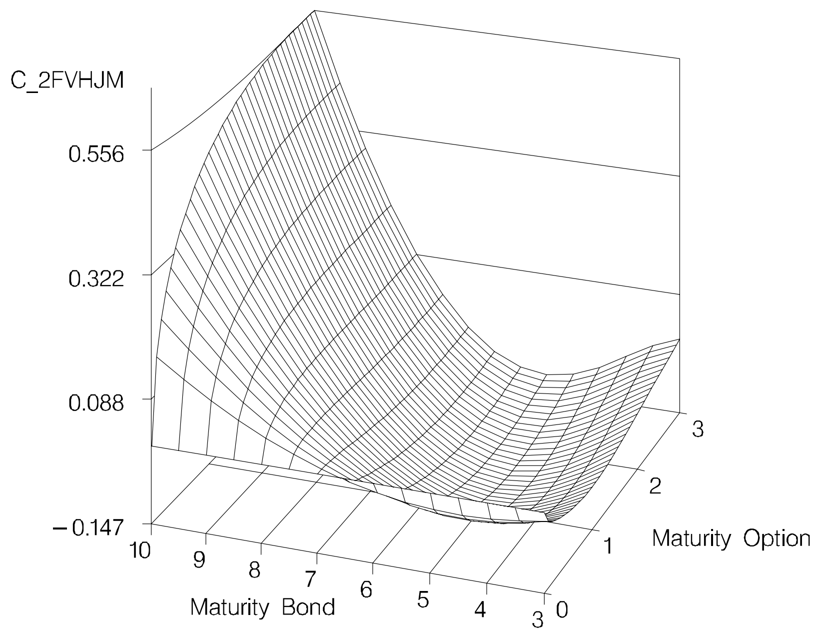
Bond Maturity	C_{BS}	C_{HL}	C_{Vas}	C_{HJM}
3.0	0.91584	0.30529	0.38601	0.42535
3.5	0.88434	0.44218	0.52618	0.56254
4.0	0.85393	0.56930	0.63842	0.66486
4.5	0.82456	0.68714	0.72717	0.74062
5.0	0.79620	0.79620	0.79620	0.79620
5.5	0.76881	0.89694	0.84869	0.83650
6.0	0.74237	0.98980	0.88734	0.86527
6.5	0.71684	1.07521	0.91445	0.88535
7.0	0.69218	1.15357	0.93195	0.89892
7.5	0.66837	1.22525	0.94147	0.90759
8.0	0.64538	1.29064	0.94441	0.91261
8.5	0.62319	1.35007	0.94193	0.91487
9.0	0.60175	1.40388	0.93501	0.91506
9.5	0.58106	1.45238	0.92449	0.91367
10.0	0.56107	1.49588	0.91108	0.91108

to the Black/Scholes, Ho/Lee, Vasicek, and HJM model are summarized in table 7. The volatility parameters for the Black/Scholes and HJM model are again implicitly calculated to fit the two-year option on the five-year bond. The call prices for the Black/Scholes model show clearly that the pull-to-par-effect is not captured because options on bonds with a maturity close to the maturity of the option are strongly overvalued and options on bonds with a maturity far larger than the option maturity are undervalued. The HJM model with three volatility parameters is fitted not only to the five-year bond but also to the ten-year bond. Price differentials are fairly small for all options.

Finally we examine pricing differences between our two-factor models. A comparison of the HJM and 2FV model makes sense only for a subperiod with a significant smile effect because otherwise the 2FV model reduces to the HJM model. Hence, we use the parameters estimates for the subperiod 1/90–12/91 and compare the call prices for the same set of options as before. The call price differences are plotted in figure 13. Obviously the HJM model misprices calls on bonds with large maturities

Figure 13: **2FV-Call Price versus HJM-Call Price**

This figure shows call price differences $C_{2FV} - C_{HJM}$ for at-the-money options with a maturity of 0–3 years. The maturity of the underlying bonds with a face value of 100 varies between 3 and 10 years. The initial term structure is flat at 7%. The volatility parameters are $\sigma_{1,HJM} = 0.0076, \sigma_{2,HJM} = 0.0161, \kappa_{HJM} = 2.7859$, bzw. $\sigma_{1,2FV} = 0.0064, \sigma_{2,2FV} = 0.0301, \kappa_{1,2FV} = 0.2307, \kappa_{2,2FV} = 0.0.0193$.



when a smile effect is present. The absolute (relative) price difference for the two-year option on the ten-year bond is 0.53488 (27.14%). Besides the undervaluation of calls on long term bonds, the HJM model also overvalues calls on medium term bonds (3–7 years maturity). This mispricing is caused by our estimation procedure which tries to fit the variance function. Since the u-shaped historical variance function cannot be fitted in the HJM model, the estimated variance is too large for maturities between 2 and 7 years and too small thereafter (see figure 10).[27]

[27]Call prices for the HJM and 2FV model are summarized in table 10 in the appendix.

6 Conclusions

Gaussian models of term structure movements are attractive because of their simplicity and analytical tractability. The contribution of this paper is threefold. First, we examine the properties of four Gaussian models, the popular one-factor models of Ho/Lee and Vasicek, respectively, and a two-factor model suggested by Heath/Jarrow/Morton. The fourth model is a two-factor Vasicek model, which we propose to capture the smile effect in the volatility structure.

Second, we examine the models' ability to explain the historical volatility structure for the German bond market. We find three basic patterns in the historical yield curve movement: a parallel *shift*, a *reversion* and a *twist*. The parallel shift implies a spot rate variance which is constant across all maturities, as in the Ho/Lee model. The reversion of the short rate to a long run average implies higher spot rate volatilities for short time to maturities. This monotonically decreasing variance function is modeled by Vasicek and Heath/Jarrow/Morton. But only the two-factor models are able to incorporate twists of the term structure. Despite this similarity, the HJM and 2FV model differ in their ability to explain a smile effect in the volatility structure which is generally associated with a twist of the term structure. Since the 2FV model includes the HJM model as a special case and a smile effect is present in two of seven analyzed subperiods we prefer the 2FV model. The application of the 2FV model is only limited by the probability of negative spot rates which increases sharply with the time to maturity of the contingent claim and should be analyzed carefully when the maturity of the contingent claim is larger than three years.

Finally, the relevance of these findings is judged by their implications for option pricing. The purpose of a term structure model like the four under consideration is the valuation of a broad variety of interest rate contingent claims in a consistent manner. We use options on zero bonds for which analytical solutions in all these models exist, and vary only the maturity of the option and the underlying bond. Even this simple set of interest rate contingent claims yields significant price differences which can be attributed to the model characteristics. An open question remaining for future research is the predictability of the twist and reversion effect. Moreover it remains to show that the proposed two-factor Vasicek model enables its user to earn abnormal profits from apparent mispricings.

A Proof of Theorem 2.1

The equivalent martingale measures $\tilde{Q}$ and Q^* are defined through

$$d\tilde{Q}/dQ(t) = \exp\left\{-\sum_{k=1}^{K}\int_0^t \lambda_k(v)dW_k(v) - \frac{1}{2}\sum_{k=1}^{K}\int_0^t \lambda_k(v)^2 dv\right\}$$

and

$$\begin{aligned} dQ^*/dQ(t) &= \exp\left\{-\sum_{k=1}^{K}\int_0^t [\lambda_k(v) - \sigma_k^p(v,t^*)]dW_k(v)\right. \\ &\quad \left. -\frac{1}{2}\sum_{k=1}^{K}\int_0^t [\lambda_k(v) - \sigma_k^p(v,t^*)]^2 dv\right\}. \end{aligned}$$

The corresponding Brownian motions are given by Girsanov's theorem:[28]

$$\tilde{W}_k(t) = W_k(t) + \int_0^t \lambda_k(v)dv$$

$$W_k^*(t) = W_k(t) + \int_0^t [\lambda_k(v) - \sigma_k^p(v,t^*)]dv$$

(A-1) ⇔ (A-2): Application of Ito's Lemma gives the relative bond prices process

$$dZ(t,T) = [\mu^p(t,T) - r(t)]Z(t,T)dt + \sigma^p(t,T)Z(t,T)dW(t)$$

In terms of the Brownian motion $\tilde{W}(t)$ with respect to the risk neutral measure $\tilde{Q}$ the relative bond price process is

$$\begin{aligned} dZ(t,T) &= [\mu^p(t,T) - r(t)]Z(t,T)dt - \sigma^p(t,T)Z(t,T)\lambda(t)dt \\ &\quad +\sigma^p(t,T)Z(t,T)d\tilde{W}(t) \\ &= [\mu^p(t,T) - r(t) - \sigma^p(t,T)\lambda(t)]Z(t,T) \\ &\quad +\sigma^p(t,T)Z(t,T)d\tilde{W}(t). \end{aligned}$$

Relative bond price are thus martingales with respect to $\tilde{Q}$ if and only if

$$\mu^p(t,T) - r(t) = \sigma^p(t,T)\lambda(t).$$

(A-2) ⇔ (A-3): The equivalence of (A-2) and (A-3) follows immediately from the relation of the forward rate and bond price process. Substituting the bond price drift and diffusion rate in condition (A-2) with (5) and (4) gives

$$r(t) - \int_t^T \mu(t,y)dy + \frac{1}{2}\left[\int_t^T \sigma(t,y)dy\right]^2 - r(t) = -\int_t^T \sigma(t,y)dy\lambda(t).$$

[28]To apply Girsanov's theorem we need to assume that the market prices of risk $\lambda_k(t)$ satisfy the Novikov condition or condition C.4 of HEATH/JARROW/MORTON (1992). With this assumption and certain regularity conditions regarding the drift and diffusion coefficients the existence of an equivalent martingale measure is ensured, see HEATH/JARROW/MORTON (1992, Proposition 1).

Taking the partial derivative with respect to T yields

$$\begin{aligned} -\mu(t,T) + \sigma(t,T)\int_t^T \sigma(t,y)dy &= -\sigma(t,T)\lambda(t) \\ \Leftrightarrow \mu(t,T) &= \sigma(t,T)\left[\lambda(t) + \int_t^T \sigma(t,y)dy\right] \end{aligned}$$

Analogously condition (A-2) follows by integrating (A-3) with respect to T and substitution of the forward rate drift and diffusion rate according to (4) and (5).

(A-3) $\Leftrightarrow$ (A-4): The forward rate process

$$df(t,t^*) = \mu(t,t^*)dt + \sigma(t,t^*)dW(t)$$

in terms of the Brownian motion $W^*(t)$ with respect to the forward risk adjusted measure Q^* is

$$df(t,t^*) = \mu(t,t^*)dt - \sigma(t,t^*)[\lambda(t) - \sigma^p(t,t^*)]dt + \sigma(t,t^*)dW^*(t).$$

Forward rates are thus martingales with respect to Q^* if and only if

$$\mu(t,t^*) = \sigma(t,t^*)[\lambda(t) - \sigma^p(t,t^*)]. \qquad \blacksquare$$

B Tables and Figures

Table 8: **Empirical Studies**

Author	Market	Period	Results
HJM (1990)	U.S. T-Bonds	May 1989	Two factors which can be identified as shift and twist.
Kahn (1991)	U.S. T-Bonds	1980s	Two factors which explain 82.4% (Shift) and 4.6% (Twist) of the total variance.
Steeley (1990,1992)	UK government gilts	10/85 – 10/87	Three factors, which explain 86.8%, 6.4% and 4.4% of the total variance. The first factor is a parallel shift, the second factor determines the slope and the third the convexity of the yield curve.
Bühler, Schulze (1993)	German bonds issued by Bund, Bahn, and Post	1/68–12/88	Two factors which explain 86.5% and 10.9% of the total variance. The first factor is explained best by a medium term spot rate, the second factor by a spread between a long and a short rate.
Beckers (1993)	Different Countries	1/86–6/92	Two factors (Shift und Twist) in Australia, Belgium, Denmark, France, Germany, Italy, Netherlands, Spain. Three factors (Shift, Twist und Butterfly) in Canada, Japan, UK, USA.

Figure 14: **Probability of negative spot rates**

This figure shows the probability of negative spot rates with maturities of 1-10 years in the four Gaussian models as time elapses. The probabilities are computed based on an initially flat term structure at 4% and the parameters

Ho/Lee:	$\sigma = 0.0075$
Vasicek:	$\sigma = 0.0121, \kappa = 0.2564$
HJM:	$\sigma_1 = 0.0045, \sigma_2 = 0.0122, \kappa = 0.4416$
2FV:	$\sigma_1 = 0.0035, \kappa_1 = 0.1859, \sigma_2 = 0.0129, \kappa_2 = 0.7662$

The parameters for the Ho/Lee, Vasicek, and HJM model are the historical estimates for the subperiod 1/92–12/93, which does not reveal a smile effect. Since the 2FV model reduces in this case to the HJM model, we use for the 2FV model the historical parameter estimates for the subperiod 1/90-12/91 exhibiting a strong smile effect.

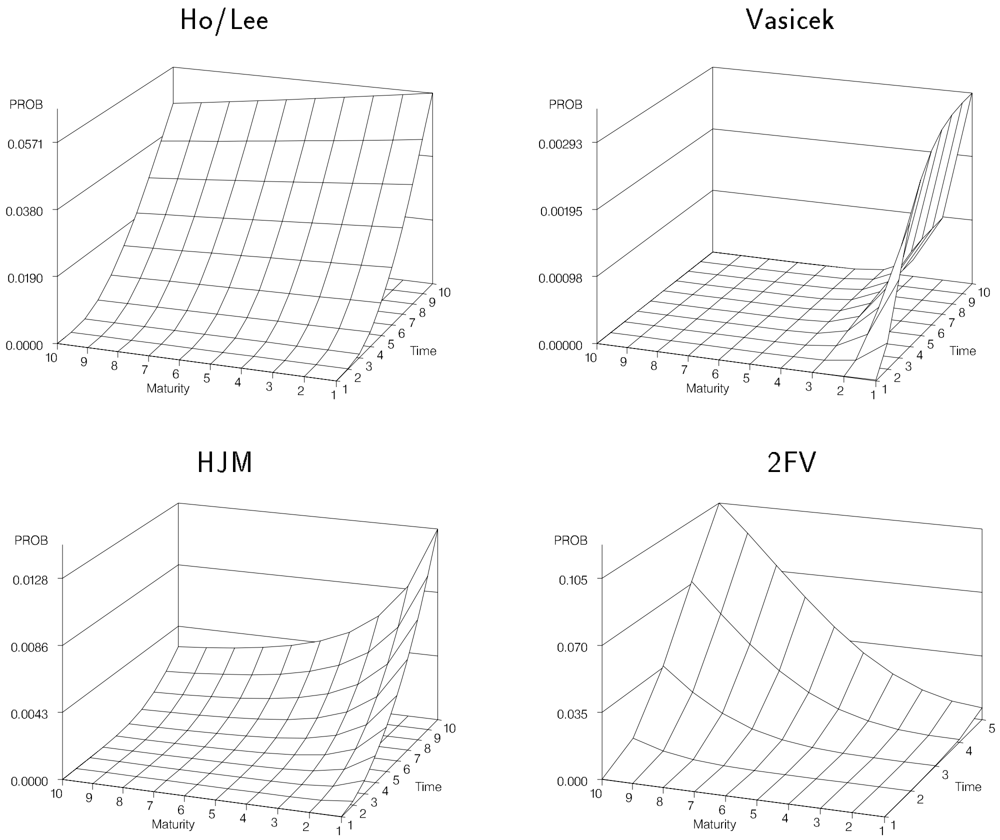


Table 9: **Identification of Factors 1/90–12/91**

This table reports the results of the regressions

$$F_{i,n} = \alpha_n + \beta_n \Delta R_n(\tau) + \epsilon_n$$

where $F_{i,n}$ denotes the factor score of the i-th factor with respect to the n-th observation. As independent variables the spot rate changes for the maturities $\frac{1}{2}, 1, 2, 3, 4, 5, 7, 9$ years and the changes of two spreads between a long and a short rate are chosen.

explanatory variable	factor 1			factor 2		
	R^2	β	t-Wert	R^2	β	t-Wert
$\Delta R(\frac{1}{2})$	0.3808	393.0	7.920	0.5524	473.4	11.220
$\Delta R(1)$	0.4690	502.5	9.491	0.4944	515.9	9.987
$\Delta R(2)$	0.6828	758.1	14.817	0.3160	515.7	6.865
$\Delta R(3)$	0.8721	981.7	26.374	0.1139	354.9	3.622
$\Delta R(4)$	**0.9487**	1063.9	43.434	0.0071	92.0	0.854
$\Delta R(5)$	0.9369	1029.6	38.915	0.0164	–136.3	–1.305
$\Delta R(7)$	0.8726	881.7	26.437	0.1271	–336.5	–3.854
$\Delta R(9)$	0.7584	673.2	17.896	0.1221	–270.1	–3.766
$\Delta(R(9\frac{1}{2}) - R(\frac{1}{2}))$	0.0209	91.0	1.477	**0.9586**	–616.0	–48.599
$\Delta(R(9\frac{1}{2}) - R(1))$	0.0295	119.2	1.760	0.9080	–661.4	–31.719

Table 10: **Call Prices HJM and 2FV**

This table shows call prices of at-the-money options with a maturity of 2 years for the HJM and 2FV model. The volatility parameters are ($\sigma_{1,HJM} = 0.0076, \sigma_{2,HJM} = 0.0161, \kappa_{HJM} = 2.7859, \sigma_{1,2FV} = 0.0035, \sigma_{2,2FV} = 0.0129, \kappa_{1,2FV} = 0.1859, \kappa_{2,2FV} = 0.7662$. The maturity of the underlying bonds varies between 3 and 10 years and the initial term structure is flat at 7%.

Bond Maturity	C_{HJM}	C_{2FV}
3.0	0.35541	0.31574
3.5	0.50901	0.43424
4.0	0.65228	0.54463
4.5	0.78552	0.65382
5.0	0.90905	0.76567
5.5	1.02328	0.88224
6.0	1.12866	1.00459
6.5	1.22563	1.13330
7.0	1.31463	1.26873
7.5	1.39606	1.41114
8.0	1.47036	1.56080
8.5	1.53789	1.71801
9.0	1.59904	1.88310
9.5	1.65416	2.05644
10.0	1.70359	2.23847

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