Robust Recovery Risk Hedging: Only the First Moment Matters

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Motivation

- *Probability of default* and *recovery payment* (or *recovery rate* defined as the payback quota of the borrower) are crucial quantities in credit risk models.
- Both quantities are important when pricing and hedging credit derivatives written on the underlying loans and bonds.
- This paper's focus is on hedging, e.g. when a risk manager has to hedge a short position in a credit derivative (e.g. a defaultable senior bond).

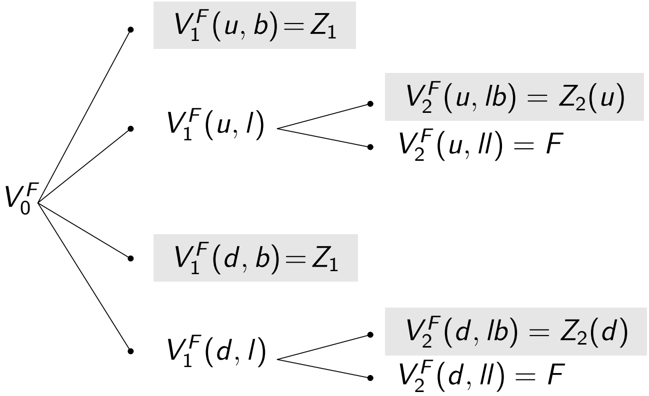
Related Literature

- Most pricing and hedging models for credit derivatives assume that the recovery payment is a known quantity.
- Bielecki, Jeanblanc and Rutkowski (2007, 2008) provide explicit hedge ratios only when the default time is the only random quantity.
- Biagini and Cretarola (2007, 2009, 2012) calculate the hedge ratio explicitly only when the recovery payment is constant conditional on default time and do not consider coupon payments.
- None of these papers examines the impact of the shape of the recovery payment distribution on hedging strategies.

Single- vs. Doubly-Stochastic Recovery

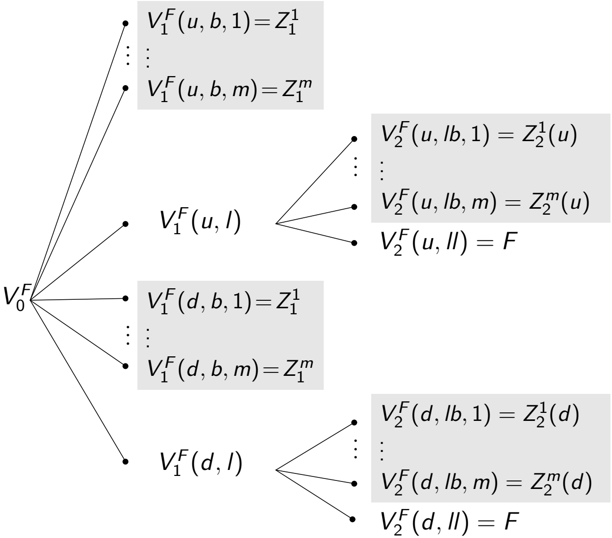
- Two recovery models are distinguished: single-stochastic and doubly-stochastic recovery models.
- A recovery model is called *single-stochastic* (SSR) if the recovery payment only depends on the default time (and the interest rate level).
- In *doubly-stochastic* recovery (DSR) models the recovery payment does not only depend on the default time (and the interest rate level) but also on another source of uncertainty, e.g. bankruptcy costs, time delay of the promised recovery payment, etc.

Single-Stochastic Recovery of a Zero Bond (T=2)



(See, e.g., Jarrow and Turnbull, 1995)

Doubly-Stochastic Recovery of a Zero Bond (T=2)



Main Findings

- Only the first moment matters: Hedging approaches minimizing the quadratic hedging error do only depend on the *expected* recovery payment at default (and not on the whole shape of the recovery payment distribution).
- For hedging purposes, a doubly-stochastic recovery payment model can be replaced by a single-stochastic recovery payment model where the recovery payment equals the expected value of the doubly-stochastic recovery payment.
- This result also holds when interest rates and default intensities are stochastic.

Hedging Concepts: An Overview

	Complete Financial Market	Incomplete Financial Market	
No Shortfall	*Delta-Hedging* Black, Merton, Scholes (1973)	*Superhedging* Naik and Uppal (1992)	No Restriction on Initial Costs
Shortfall Risk		*Risk- & Variance-Minimizing Hedging* Föllmer and Sondermann (1986) *Locally Risk-Minimizing Hedging* Föllmer and Schweizer (1989)	
	Globally Risk- and Variance-Minimizing Hedging Schweizer (1995)		
	Shortfall-Hedging Föllmer and Leukert (1999) *(Global) Expected Shortfall-Hedging* Föllmer and Leukert (2000) *Local Expected Shortfall-Hedging* Schulmerich (2001), Schulmerich and Trautmann (2003)		Restriction on Initial Costs

Locally Risk-Minimizing (LRM) Hedging

Problem

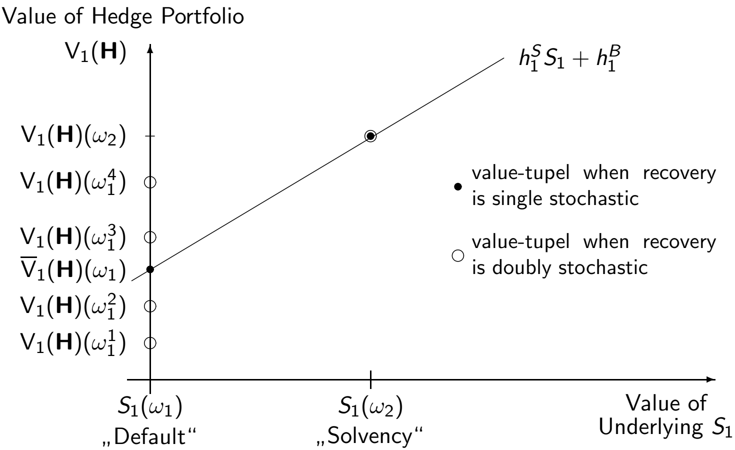
- Given two hedging instruments, a money market account with time-t price B_t, and an underlying instrument (say, a defaultable junior bond) with time-t price S_t
- one looks for a trading strategy $\mathbf{H} = (h^S, h^B)$ which exactly replicates the credit derivative V_T^F (say, a defaultable (senior) bond) at maturity T and in addition
- minimizes the expected quadratic growth of the hedging costs $\Delta\mathcal{C}_t(\mathbf{H}) = V_t^F - (h_t^S S_t + h_t^B B_t)$ at every point in time t :

$$\mathrm{E}_P\Big[(\Delta\mathcal{C}_t(\mathbf{H}))^2 \,|\mathcal{F}_{t-1}\Big] \to \min \text{ for all } t \text{ and } \mathbf{H} \text{ with } \mathrm{V}_T(\mathbf{H}) = V_T^F.$$

Solution: Linear Regression

- The LRM-Hedging problem is equivalent to a sequential regression problem and can be solved by backwards induction for $t = T, T-1, \ldots, 1$.
- We obtain the hedge ratio h^S as the *slope* and the share in the money market account h^B as the *intercept* of the regression line.
- The following figure illustrates this proposition and the paper's main finding.

LRM-Hedging of a Defaultable Bond (T=1): Key Insight



Basic Model

- Credit derivative to be hedged is a defaultable coupon bond ("*senior bond*") characterized by the triple (Z, C, F), where
  - Z is the recovery payment,
  - C is a stream of (cumulative) coupon payments,
  - F is the senior bond's face value.
- The cumulative value $V^{F,cum}$ of the senior bond at maturity T is given by

$$V_T^{F,cum} = \begin{cases} B_T \int_0^T \frac{1}{B_t}\, dC_t + F\ , & \text{if } \tau > T, \\ B_T \int_0^\tau \frac{1}{B_t}\, dC_t + B_T \cdot \frac{Z(\tau)}{B_\tau}\ , & \text{if } \tau \leq T. \end{cases}$$

- The default time τ of the senior bond is modelled by an inhomogeneous Poisson process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ with deterministic intensity function $\hat{\lambda}$ and corresponding *survival probability*

$$G_t = \widehat{P}(\tau > t) = \widehat{P}(H_t = 0) = \exp\left\{-\int_0^t \hat{\lambda}(s)ds\right\}.$$

- Hedging instrument is a defaultable zero coupon bond with maturity T, face value 1, and total loss in case of default ("*junior bond*") of the same firm whose price process S is given by

$$S_t = (1 - H_t)\frac{B_t}{B_T}\exp\left\{-\int_t^T \hat{\lambda}(s)ds\right\} = (1 - H_t)\frac{B_t}{B_T} \cdot \frac{G_T}{G_t}.$$

- Assuming deterministic interest rates we get the time-t value $V_t^F = g_t^Z + g_t^C + g_t^F$ of the senior bond (Z, C, F), where

$$\begin{aligned} g_t^Z &= \int_t^T \frac{B_t}{B_u} \exp\left\{-\int_t^u \hat{\lambda}(s)\,ds\right\} \hat{\lambda}(u)\,\mu^Z(u)\,du\,, \\ g_t^C &= \int_t^T \frac{B_t}{B_u} \exp\left\{-\int_t^u \hat{\lambda}(s)\,ds\right\}\, dC_u\,, \\ g_t^F &= \frac{B_t}{B_T}\, \exp\left\{-\int_t^T \hat{\lambda}(s)\,ds\right\} \cdot F\,, \end{aligned}$$

- $\mu^Z(t)$ denotes the expected recovery payment when default occurs at time $\tau = t$.

LRM-Hedge

- The LRM-hedge ratio h^S is given by

$$h_t^S = \frac{\mathrm{d}\langle V^{F,cum}, S\rangle_t^{\widehat{P}}}{\mathrm{d}\langle S, S\rangle_t^{\widehat{P}}} = \frac{V_{t-}^F - \mu^Z(t)}{S_{t-}}, \quad \text{if } \tau > t,$$
$$h_t^S = 0, \quad \text{if } \tau \leq t.$$

- The number of money market accounts h^B is given by

$$h_t^B = \int_0^t \frac{1}{B_s} dC_s + \frac{\mu^Z(t)}{B_t}, \quad \text{if } \tau > t,$$
$$h_t^B = \int_0^\tau \frac{1}{B_s} dC_s + \frac{Z(\tau)}{B_\tau}, \quad \text{if } \tau \leq t.$$

Extensions

Stochastic Default Intensity

- The default time τ is modelled by a doubly-stochastic Poisson process with intensity process $\hat{\lambda}$ under the minimal martingale measure $\widehat{P}$.
- The time-t information is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{F}_t$ contains the information about the diffusion risk and $\mathcal{H}_t$ about the jump risk.
- Denote by G the conditional survival probability

$$G_t = \widehat{P}(\tau > t | \mathcal{F}_t).$$

- The cumulative value $V_T^{F,cum}$ of the senior bond at maturity T admits the representation, see Bielecki et al. (2008),

$$\frac{V_T^{F,cum}}{B_T} = \frac{V_0^F}{B_0} + \int_0^T (1-H_t)\frac{1}{G_t}dm_t + \int_0^T \left(\frac{Z_t - V_t^F}{B_t}\right) d\widetilde{H}_t.$$

- dm_t describes the change of the senior bond's value that is solely due to the diffusion risk with

$$\begin{aligned} m_t &= \widehat{E}\left[\int_0^T \frac{G_u}{B_u}\,\hat{\lambda}_u\,\mu^Z(u)\,du + \frac{G_T}{B_T}F + \int_0^T \frac{G_u}{B_u}dC_u \middle| \mathcal{F}_t\right] \\ &= m_0 + \int_0^t \xi_s^{m,r}\,d\widehat{W}_s^r + \int_0^t \xi_s^{m,\hat{\lambda}}\,d\widehat{W}_s^{\hat{\lambda}}. \end{aligned}$$

- When using CIR dynamics to model r and $\hat{\lambda}$,

$$\begin{aligned} dr_t &= \kappa^r(\theta^r - r_t)dt + \sigma^r\sqrt{r_t}d\widehat{W}_t^r, \\ d\hat{\lambda}_t &= \kappa^{\hat{\lambda}}(\theta^{\hat{\lambda}} - \hat{\lambda}_t)dt + \sigma^{\hat{\lambda}}\sqrt{\hat{\lambda}_t}d\widehat{W}_t^{\hat{\lambda}}, \end{aligned}$$

there exists a *deterministic* function u such that

$$m_t = u(t,r_t,\hat{\lambda}_t).$$

- Applying a result from Heath (1995), we then can get the integrands in the martingale representation:

$$\xi_t^{m,r} = \sigma^r\sqrt{r_t}\,\frac{\partial}{\partial r}u(t,r_t,\hat{\lambda}_t) \quad \text{and} \quad \xi_t^{m,\hat{\lambda}} = \sigma^{\hat{\lambda}}\sqrt{\hat{\lambda}_t}\,\frac{\partial}{\partial \hat{\lambda}}u(t,r_t,\hat{\lambda}_t).$$

- The LRM-hedge ratio is now given by

$$h_t^S = \frac{V_{t-}^F - \mu^Z(t)}{S_{t-}} + \frac{\xi_t^{m,r} + \xi_t^{m,\hat{\lambda}}}{G_t \sigma_t S_t}$$

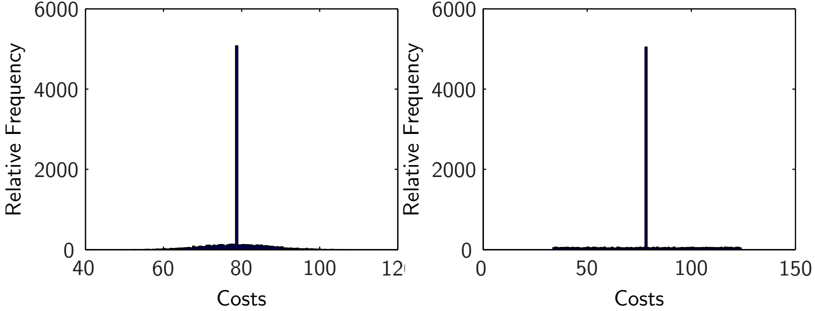
where σ denotes the junior bond's volatility.

- The first term of the RHS equals the LRM-hedge ratio in our basic model.
- The second term on the RHS of the above equation is therefore solely due to the assumed additional diffusion risk.

Hedging Costs Distribution: r and $\hat{\lambda}$ deterministic

Parameters: $N = 10000$ (runs), $r_0 = 0.05$, $\kappa^r = 2.5$, $\theta^r = 0.05$, $\sigma^r = 0.2$, $\hat{\lambda}_0 = 0.35$, $\kappa^{\hat{\lambda}} = 0.5$, $\theta^{\hat{\lambda}} = 0.35$, $\sigma^{\hat{\lambda}} = 0.4$, $F = 100$, $C = 0.08$, $T = 2$.

(a) $Z \sim Beta(12,12)$ (b) $Z \sim U(0,1)$



Hedging Costs Distribution: r and $\hat{\lambda}$ stochastic

Parameters: $N = 10000$ (runs), $r_0 = 0.05$, $\kappa^r = 2.5$, $\theta^r = 0.05$, $\sigma^r = 0.2$, $\hat{\lambda}_0 = 0.35$, $\kappa^{\hat{\lambda}} = 0.5$, $\theta^{\hat{\lambda}} = 0.35$, $\sigma^{\hat{\lambda}} = 0.4$, $F = 100$, $C = 0.08$, $T = 2$.

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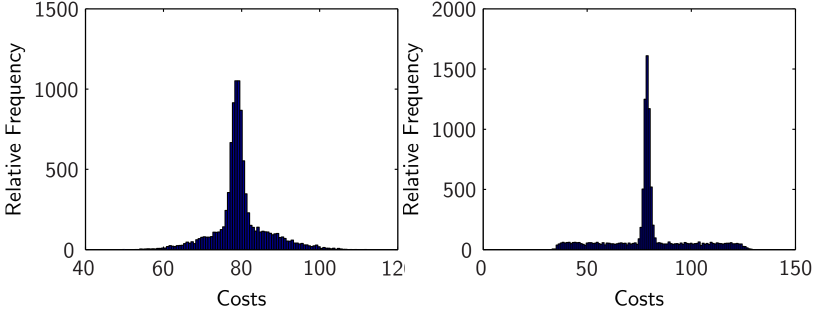

Conclusion

- *Quadratic hedging* approaches depend *only* on the *first moment* of the recovery payment distribution (conditional on the default time and/or interest rate development).
- Therefore single- and doubly-stochastic recovery modeling result in the *same quadratic hedging* strategy, if the expected recovery payment in a DSR model coincides with the recovery payment in a SSR model.
- The paper provides *explicit* hedge ratios even when all relevant quantities are stochastic.

- The time-t values of the senior bond's three components are now given by

$$
\begin{aligned}
g_t^Z &= \frac{B_t}{G_t} \cdot \widehat{E}\left[\int_t^T \frac{G_s}{B_s} \hat{\lambda}(s)\, \mu^Z(s)\, ds \middle| \mathcal{F}_t\right] , \\
g_t^C &= \frac{B_t}{G_t} \cdot \widehat{E}\left[\int_t^T \frac{G_s}{B_s}\, dC_s \middle| \mathcal{F}_t\right] , \\
g_t^F &= \frac{B_t}{G_t} \cdot \widehat{E}\left[\frac{G_T}{B_T} \middle| \mathcal{F}_t\right] \cdot F .
\end{aligned}
$$

  which can be explicitly calculated (in the general case, admittedly, with the help of numerical integraton).

- The time-t value of the junior bond is

$$
S_t = (1 - H_t) \cdot \frac{B_t}{G_t} \cdot \widehat{E}\left[\frac{G_T}{B_T} \;\middle|\; \mathcal{F}_t\right] \cdot 1.
$$