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Gaussian Multi-factor Interest Rate Models: Theory, Estimation, and Implications for Option Pricing

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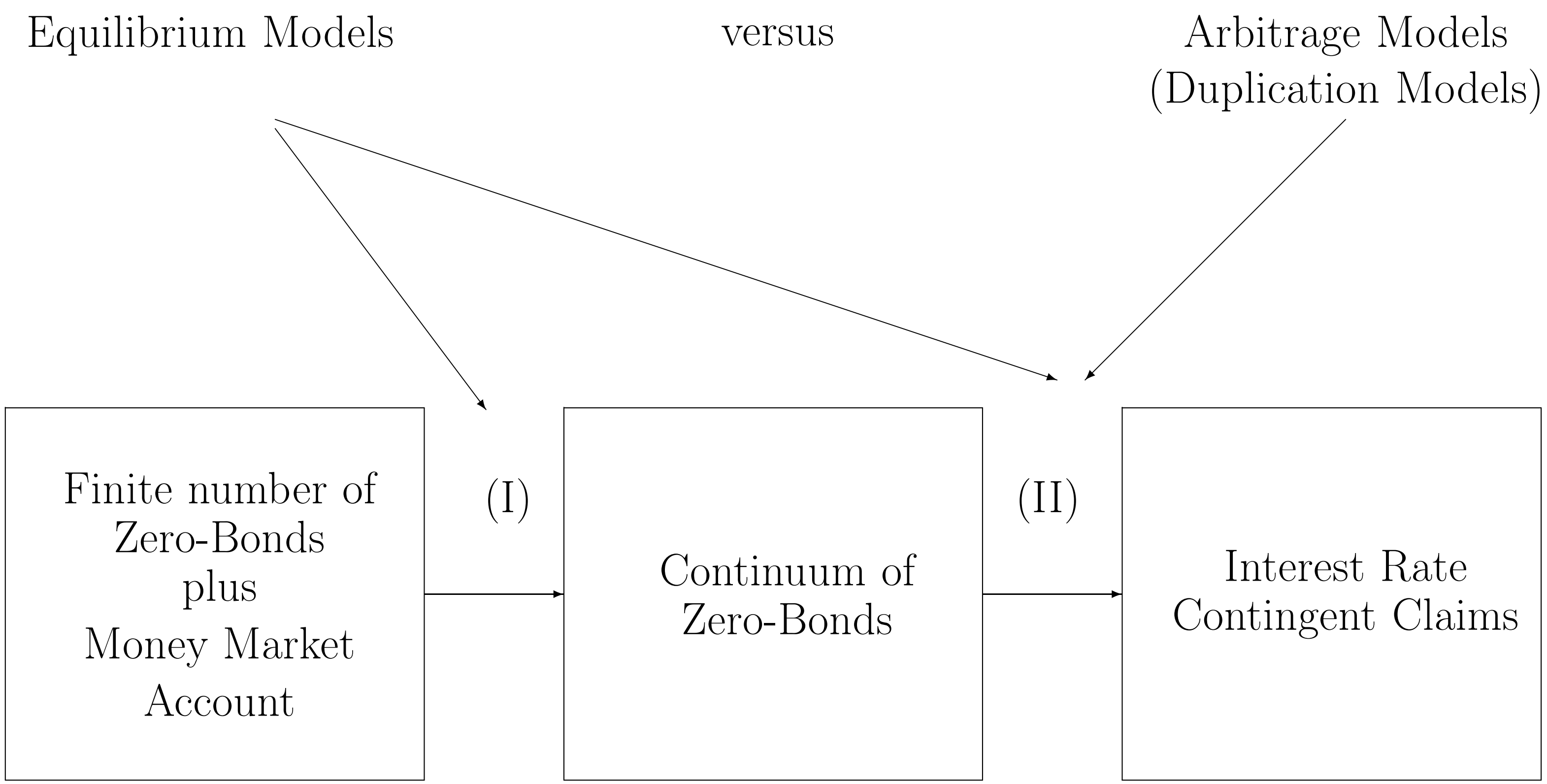
Outline

1. Introduction
2. The HJM Approach
3. Model Selection
4. Volatility Estimation
5. Implications for Option Pricing
6. Conclusions

1 Introduction



Equilibrium Models (schwach vollständig)

(z.B. (Cox / Ingersoll / Ross (1985), Brennan / Schwartz (1979))

- Anzahl der handelbaren Null-Kuponanleihen (NKA) reicht nicht aus, um ein Geldmarktkonto synthetisch erzeugen zu können.
- Ein exogen spezifizierter 'Spot rate process' übernimmt die Funktion eines handelbaren Geldmarktkontos.

  $\Rightarrow$ Zinsderivate hängen vom Marktpreis des Risikos ab!
- Probleme:
  - Marktpreis des Risikos ist stochastisch und nichtstationär.
  - Modellpreise können Marktpreise nur unvollkommen erklären.

Arbitrage Models (streng vollständig)

(z.B. Ho / Lee, Heath / Jarrow / Morton, Black / Derman / Toy, Hull / White)

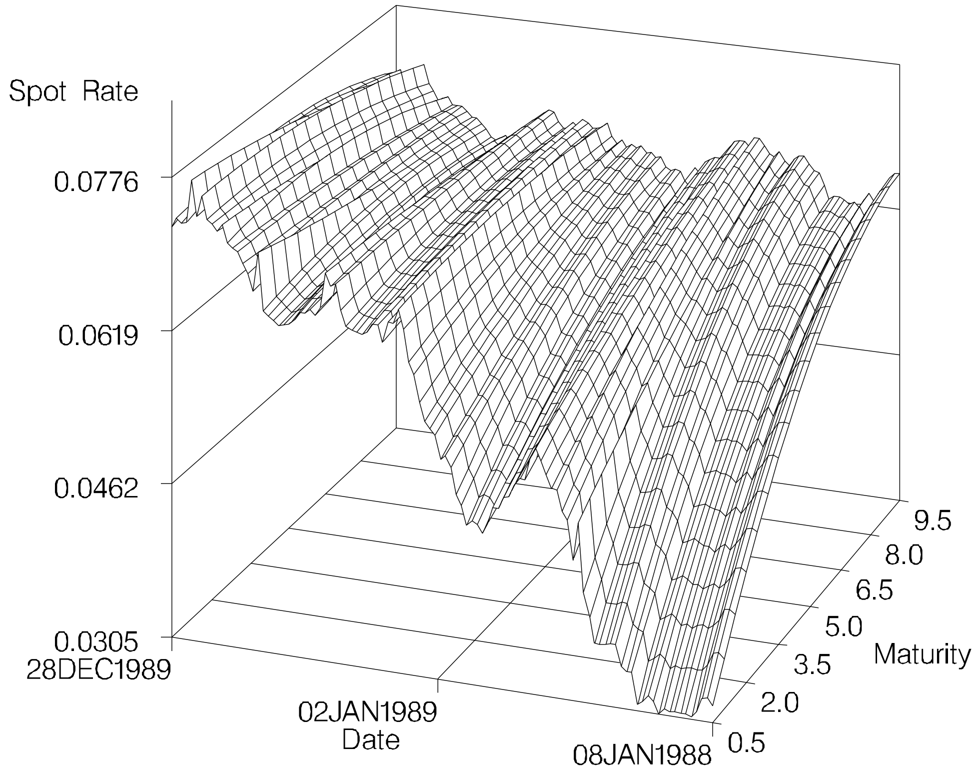
- Anzahl der handelbaren NKA (im zeitstetigen Fall **unendlich** viele) reicht aus, um ein Geldmarktkonto synthetisch erzeugen zu können.
- Der 'spot rate process' wird durch die Spezifikation der NKA-Prozesse spezifiziert.

  ⇒ Zinsderivate hängen nicht (explizit) vom Marktpreis des Risikos ab! (in Analogie zur Bewertung von Aktienderivaten gemäß Black / Scholes)

Patterns in Historical Term Structure Movements

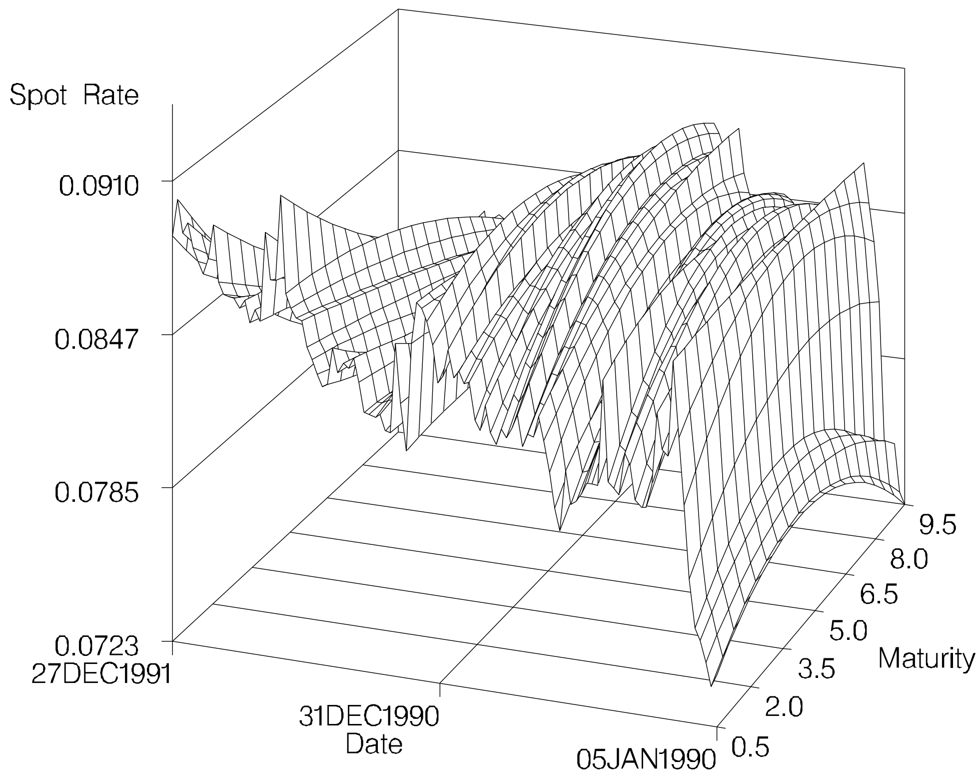
Yield Curves 1/88 – 12/89

This figure shows weekly estimated yield curves from 1/88 to 12/89. In this subperiod a strong *mean reversion* is observable.



Yield Curves 1/90 – 12/91

This figure shows weekly estimated yield curves from 1/90 to 12/91. In this subperiod a *twist* is observable.



2 The HJM Approach

- Modelling the stochastic dynamics of the term structure of forward rates.
- Take the initial forward rate curve as given (consistency with initial term structure by construction).
- No arbitrage condition restricts the drift of the forward rates.
- Preference free valuation (contingent claim valuation formulae which do not explicitly depend on the market prices of risk).

Definitions

$$
\begin{aligned}
P(t,T) &\equiv \text{Price of a Zero Bond with maturity } T \text{ at time } t.\\
F(t,t^*,T) &\equiv \text{Forward Price of a Zero Bond with maturity } T \text{ at time } t.\\
r(t) &\equiv \text{Instantaneous Short Rate at time } t \text{ for the period } [t, t+dt].\\
f(t,T) &\equiv \text{Forward Rate at time } t \text{ for the period } [T, T+dt].\\
R(t,T) &\equiv \text{Spot Rate at time } t \text{ for the period } [t, T].\\
f(t,t^*,T) &\equiv \text{Forward Rate at time } t \text{ for the period } [t^*, T].
\end{aligned}
$$

Relationships

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,s)ds\right)\\
F(t,t^*,T) &= \frac{P(t,T)}{P(t,t^*)}\\
r(t) &= f(t,t)\\
R(t,T) &= -\frac{\ln P(t,T)}{T-t}\\
f(t,t^*,T) &= -\frac{\ln F(t,t^*,T)}{T-t^*}
\end{aligned}
$$

Stochastic Dynamics

Let the stochastic evolution of *forward rates* follow a diffusion process of the form

$$f(t,T) - f(0,T) = \int_0^t \mu(v,T)dv + \sum_{k=1}^{K} \int_0^t \sigma_k(v,T)dW_k(v), \qquad (1)$$

where $\mu(v,T)$ is the drift of the forward rate with maturity T while $\sigma_k(v,T)$ for $k = 1,\ldots,K$ are its volatility coefficients, and $W_k(v)$ are independent standard Brownian motions. Since bond prices depend on forward rates, the drift and volatility of the *bond price process*

$$P(t,T) - P(0,T) \ = \ \int_0^t \mu^p(v,T)P(v,T)dv + \sum_{k=1}^{K} \int_0^t \sigma_k^p(v,T)P(v,T)dW_k(v).$$

must be connected to the drift and volatility coefficients of the forward rate process. HJM (1992) show using Ito's lemma and a generalized version of Fubini's theorem that

$$\sigma_k^p(t,T) \ = \ -\int_t^T \sigma_k(t,y)dy \qquad (2)$$

$$\mu^p(t,T) \ = \ r(t) - \int_t^T \mu(t,y)dy + \frac{1}{2}\sum_{k=1}^{K} \sigma_k^p(t,T)^2. \qquad (3)$$

No Arbitrage Conditions

The following conditions are equivalent. A K-factor model satisfying these conditions is called locally arbitrage-free.

(A1) There exists a unique equivalent martingale measure $\tilde{Q}$ such that relative bond prices $Z(t,T)$ are martingales with respect to this measure.

(A2) There exist market prices of risk $\lambda_k(t)$ with

$$\mu^p(t,T) - r(t) = \sum_{k=1}^{K} \sigma_k^p(t,T)\lambda_k(t) \quad \text{for all} \quad T \in [t,\bar{T}]. \tag{4}$$

(A3) The forward rate drift is uniquely determined by the volatility structure and the market prices of risk:

$$\mu(t,T) = \sum_{k=1}^{K} \sigma_k(t,T) \left[\lambda_k(t) + \int_t^T \sigma_k(t,y)dy\right] \tag{5}$$

(A4) There exists a unique equivalent martingale measure Q^* such that forward rates $f(t,t^*)$ are martingales with respect to this measure.

Arbitrag-free Forward Price Process

Applying Ito's Lemma to the bond price process yields for the *forward price process*

$$F(t,t^*,T)-F(0,t^*,T)=\int_0^t \mu(v,t^*,T)F(v,t^*,T)dv+\sum_{k=1}^{K}\int_0^t \sigma_k(v,t^*,T)F(v,t^*,T)dW_k(v).$$

with

$$\begin{aligned}\sigma_k(t,t^*,T) &= \sigma_k^p(t,T)-\sigma_k^p(t,t^*)\\ \mu(t,t^*,T) &= [\mu^p(t,T)-\mu^p(t,t^*)]-\sum_{k=1}^{K}\sigma^p(t,t^*)[\sigma^p(t,T)-\sigma^p(t,t^*)]\end{aligned}$$

Substituting the no arbitrage condition (A-2) for the bond price drift gives

$$\mu(t,t^*,T)=\sum_{k=1}^{K}[\lambda_k(t)-\sigma_k^p(t,t^*)]\sigma_k(t,t^*,T)$$

There's no magic in forward rates

The stochastic *bond price* in t^* given the information $\mathcal{F}_t$ is

$$P(t^*,T) = \frac{P(t,T)}{P(t,t^*)} \exp\left\{ \sum_{k=1}^{K} \int_t^{t^*} [\lambda_k(v) - \sigma_k^p(v,t^*)]\sigma_k(v,t^*,T)dv \right.$$
$$\left. -\frac{1}{2}\sum_{k=1}^{K}\int_t^{t^*} \sigma_k(v,t^*,T)^2 dv + \sum_{k=1}^{K}\int_t^{t^*} \sigma_k(v,t^*,T)dW_k(v) \right\}.$$

Thus the bond price at any time t^* given the information $\mathcal{F}_t$ depends on the forward price of the bond at time t, the volatility structure and the market prices of risk.

Similarly, the stochastic *Spot Rate* $R(t^*,T) = -\frac{\ln P(t^*,T)}{T-t^*}$ in t^* given the information $\mathcal{F}_t$ depends on the forward rate $f(t,t^*,T)$ at time t for the period $[t^*,T]$, the volatility structure and the market prices of risk.

$$R(t^*,T) = f(t,t^*,T) - \sum_{k=1}^{K}\int_t^{t^*} \frac{[\lambda_k(v) - \sigma_k^p(v,t^*)]\sigma_k(v,t^*,T)}{T-t^*}dv$$
$$+\frac{1}{2}\sum_{k=1}^{K}\int_t^{t^*} \frac{\sigma_k(v,t^*,T)^2}{T-t^*}dv - \sum_{k=1}^{K}\int_t^{t^*} \frac{\sigma_k(v,t^*,T)}{T-t^*}dW_k(v)$$

For the purpose of contingent claim valuation we are only interested in the stochastic bond price or spot rate in terms of $\tilde{W}(t)$ and $W^*(t)$, which are Brownian motions with respect to the risk-neutral measure $\tilde{Q}$ and the forward-risk-adjusted measure Q^*.

$$\begin{aligned} P(t^*,T)\,|\,\mathcal{F}_t &= \frac{P(t,T)}{P(t,t^*)}\exp\Big\{-\sum_{k=1}^{K}\int_t^{t^*}\sigma_k^p(v,t^*)\sigma_k(v,t^*,T)dv \\ &\quad -\frac{1}{2}\sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)^2dv+\sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)d\tilde{W}_k(v)\Big\} \end{aligned} \tag{6}$$

$$\begin{aligned} P(t^*,T)\,|\,\mathcal{F}_t &= \frac{P(t,T)}{P(t,t^*)}\exp\Big\{-\frac{1}{2}\sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)^2dv \\ &\quad +\sum_{k=1}^{K}\int_t^{t^*}\sigma_k(v,t^*,T)dW_k^*(v)\Big\} \end{aligned} \tag{7}$$

$\tilde{Q}$ and Q^* are identified by Girsanov's theorem. The determination of $\tilde{Q}$ and Q^* requires the knowledge of the market prices of risk, but the stochastics of the bond price are completely specified by (6) and (7), which are independent of the market prices of risk. This allows for a preference-free valuation of contingent claims.

3 Model Selection

Within the HJM framework we need to specify the

(1) number of factors and

(2) the functional form of the volatility coefficients.

Objective: **Parsimonious model**

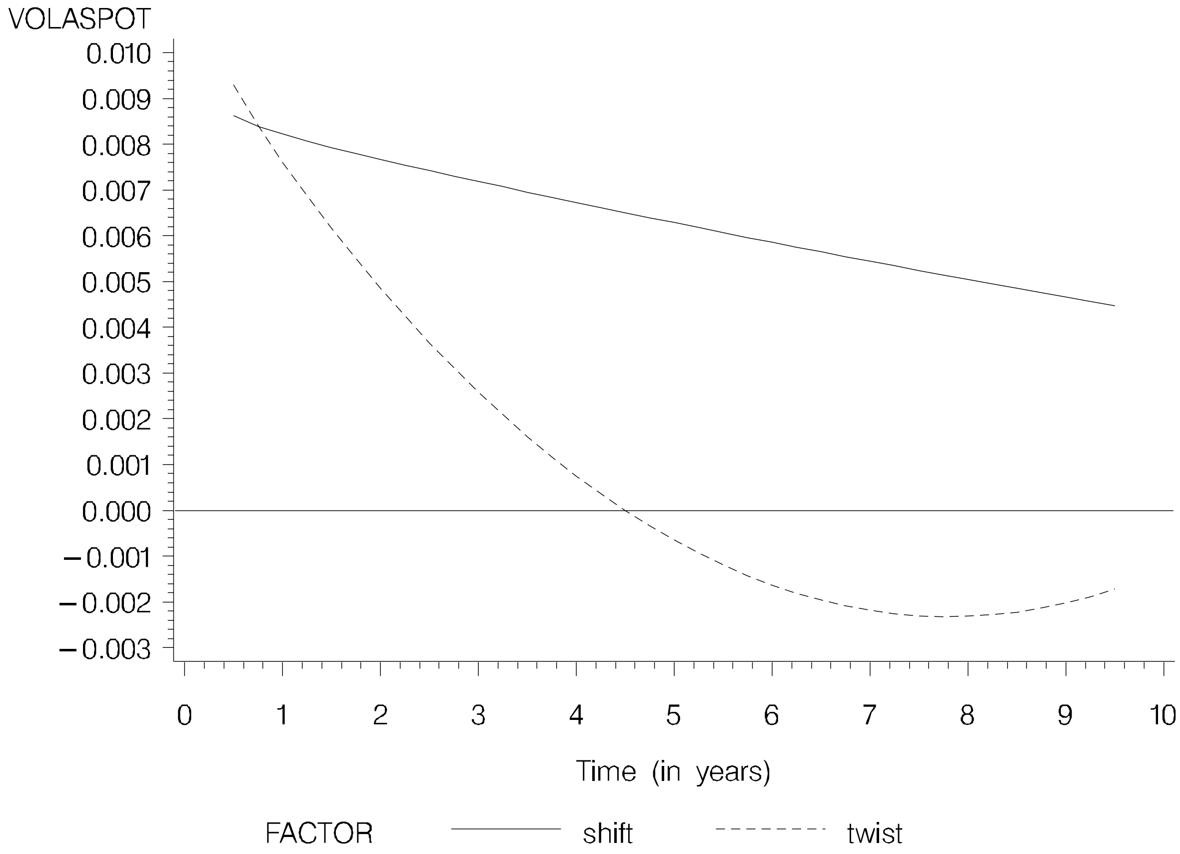
Suggestion: Gaussian 2-factor models

Results Principal Component Analysis

Period		factor 1	factor 2	factor 3
1/80–12/81	variance explained	0.8178	0.1447	0.0375
	cumulative	0.8178	0.9625	1.0000
1/82–12/83	variance explained	0.8345	0.1422	0.0234
	cumulative	0.8345	0.9766	1.0000
1/84–12/85	variance explained	0.7268	0.2322	0.0410
	cumulative	0.7268	0.9590	1.0000
1/86–12/87	variance explained	0.8056	0.1396	0.0548
	cumulative	0.8056	0.9452	1.0000
1/88–12/89	variance explained	0.8106	0.1593	0.0301
	cumulative	0.8106	0.9699	1.0000
1/90–12/91	variance explained	0.7999	0.1622	0.0379
	cumulative	0.7999	0.9621	1.0000
1/92–12/93	variance explained	0.8285	0.1175	0.0540
	cumulative	0.8285	0.9460	1.0000

Spot Rate Volatility Structure 1/88 – 12/89

This figure shows the volatility functions of the shift and twist factor estimated by the factor pattern of the first two principal components for the subperiod 1/88 – 12/89.



Gaussian Models

A *Gaussian model* is one in which the volatility structure $\sigma_k(v, T)$ is deterministic. This implies:

- Forward rates follow a Gaussian process and are normally distributed.
- Bond Prices are lognormally distributed.
- Spot Rates are normally distributed.

$\Rightarrow$ Positive probability of negative forward and spot rates.

One-factor models:

Ho/Lee: $\sigma(t, T) = \sigma$

Vasicek: $\sigma(t, T) = \sigma e^{-\kappa(T-t)}$

Two-factor models:

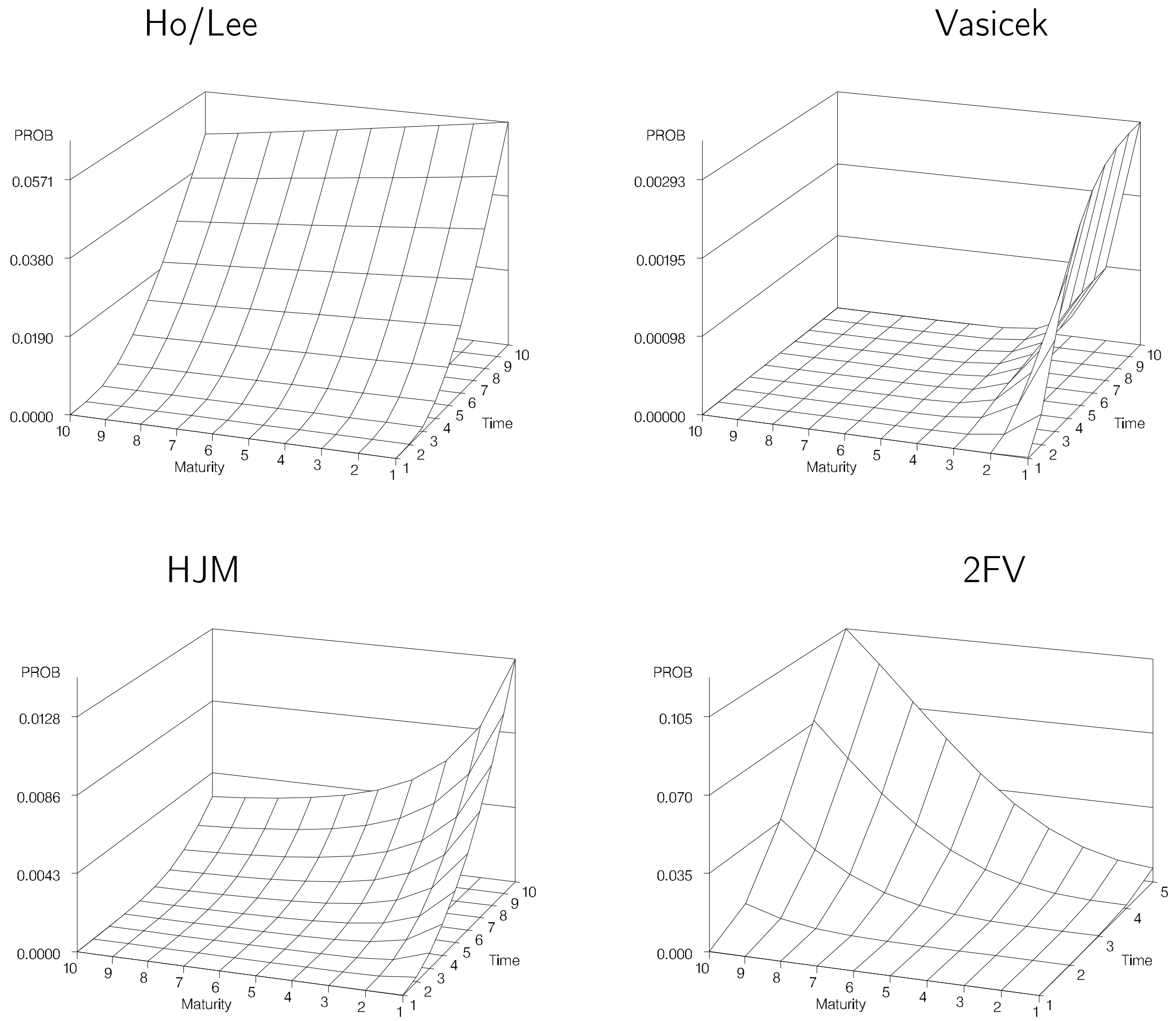
HJM: $\sigma_1(t, T) = \sigma_1$

$\sigma_2(t, T) = \sigma_2 e^{-\kappa(T-t)}$

2FV: $\sigma_1(t, T) = \sigma_1 e^{\kappa_1(T-t)}$

$\sigma_2(t, T) = \sigma_2 e^{-\kappa_2(T-t)}$

Probability of negative spot rates



The spot rate in the 2FV-model

$$\begin{aligned} R(t^*,T) &= f(t,t^*,T) + M(t^*,T) \\ &+\sigma_1 \frac{(e^{\kappa_1(T-t^*)}-1)}{\kappa_1(T-t^*)}\tilde{x}_1(t^*) + \sigma_2 \frac{(1-e^{-\kappa_2(T-t^*)})}{\kappa_2(T-t^*)}\tilde{x}_2(t^*) \end{aligned} \tag{8}$$

where $\tilde{x}_1(t^*), \tilde{x}_2(t^*)$ denote

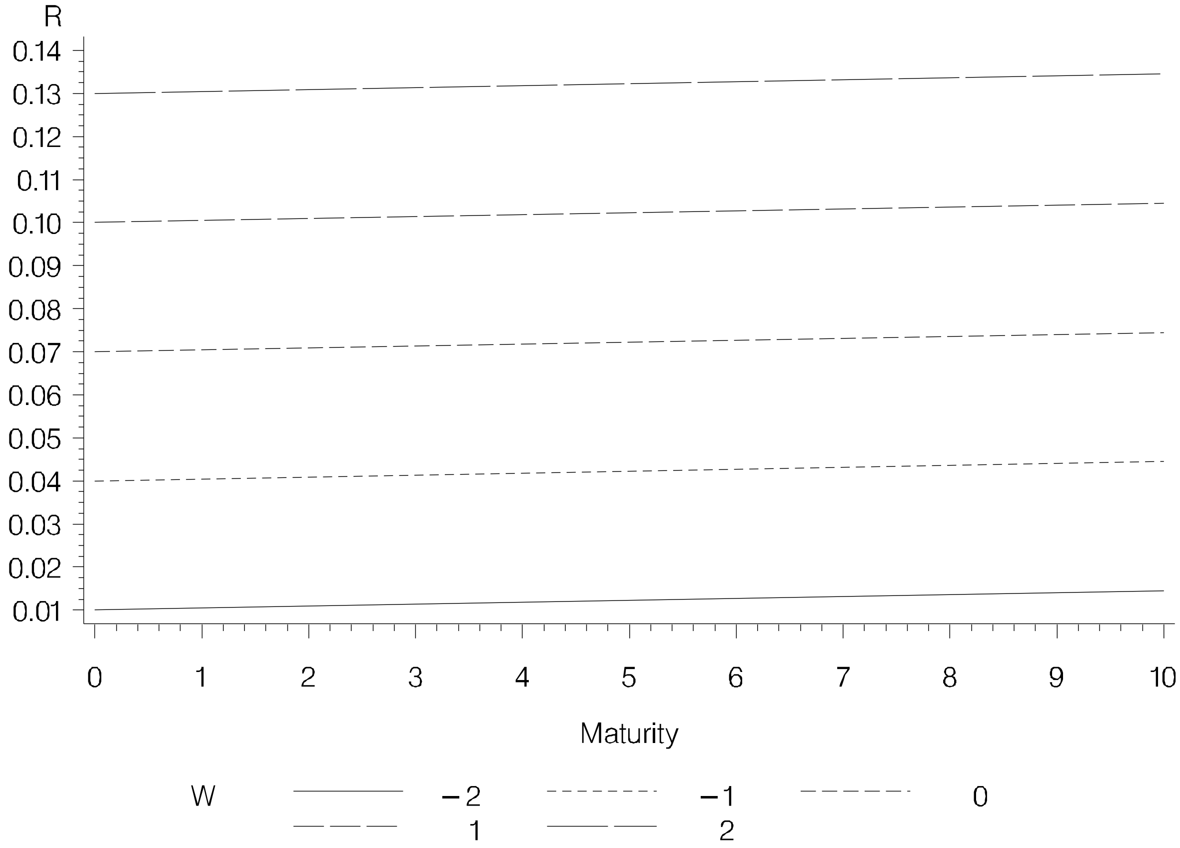
$$\begin{aligned} \tilde{x}_1(t^*) &= \tilde{x}_1(t) + \int_t^{t^*} e^{\kappa_1(t^*-v)} d\tilde{W}_1(v) \\ \tilde{x}_2(t^*) &= \tilde{x}_2(t) + \int_t^{t^*} e^{-\kappa_2(t^*-v)} d\tilde{W}_2(v) \end{aligned}$$

is normally distributed with

$$\begin{aligned} \mu^R(t^*,T) &= f(t,t^*,T) + M(t^*,T) \\ \sigma^R(t^*,T)^2 &= \frac{\sigma_1^2}{2\kappa_1^3}\left(\frac{e^{\kappa_1(T-t^*)}-1}{T-t^*}\right)^2 \left(e^{2\kappa_1 t^*}-1\right) \\ &+\frac{\sigma_2^2}{2\kappa_2^3}\left(\frac{1-e^{-\kappa_2(T-t^*)}}{T-t^*}\right)^2 \left(1-e^{-2\kappa_2 t^*}\right). \end{aligned}$$

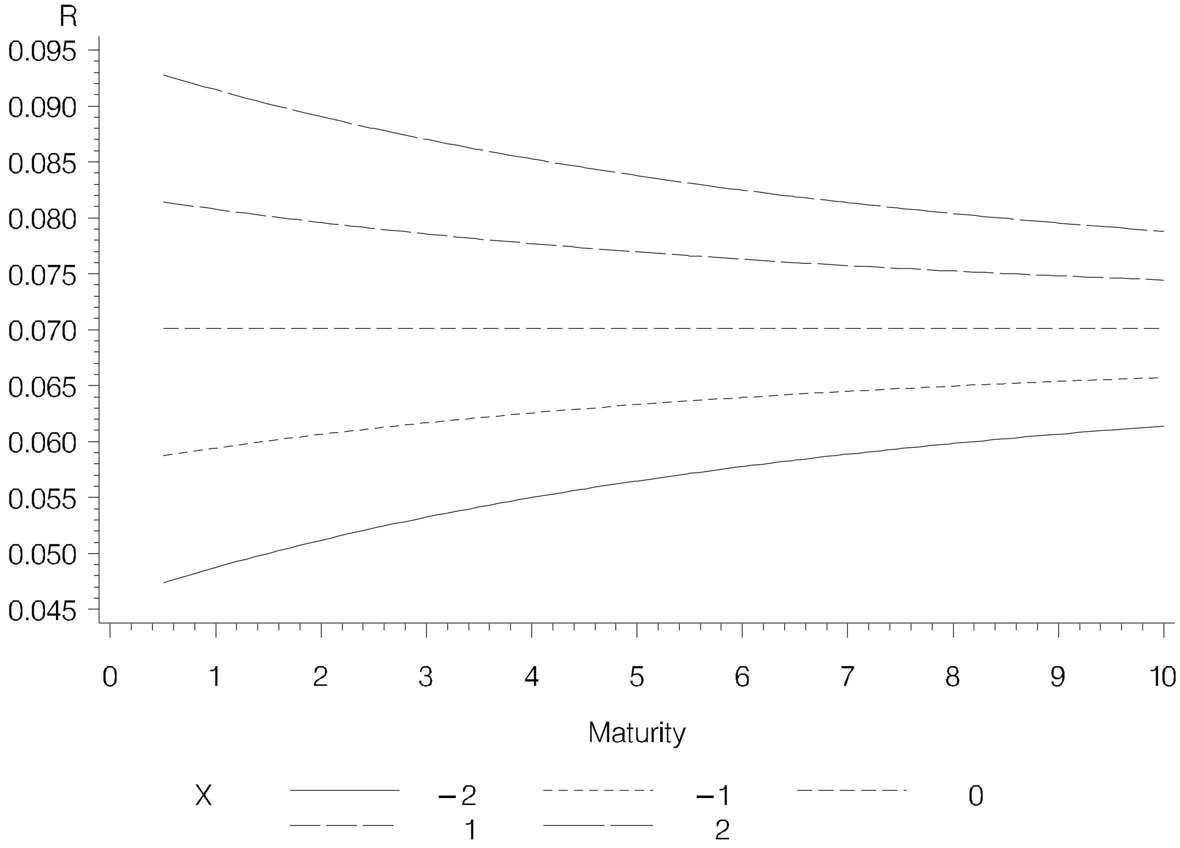
Yield Curves in the Ho/Lee-Model

This figure shows 5 possible yield curves after one year, when the initial term structure is flat at 7%. The 5 realizations of the Brownian motion are given by $\tilde{W}(t) = -2, -1, 0, 1, 2$. We choose a large volatility coefficient of $\sigma = 0.3$ to avoid the impression of a parallel shift in time.



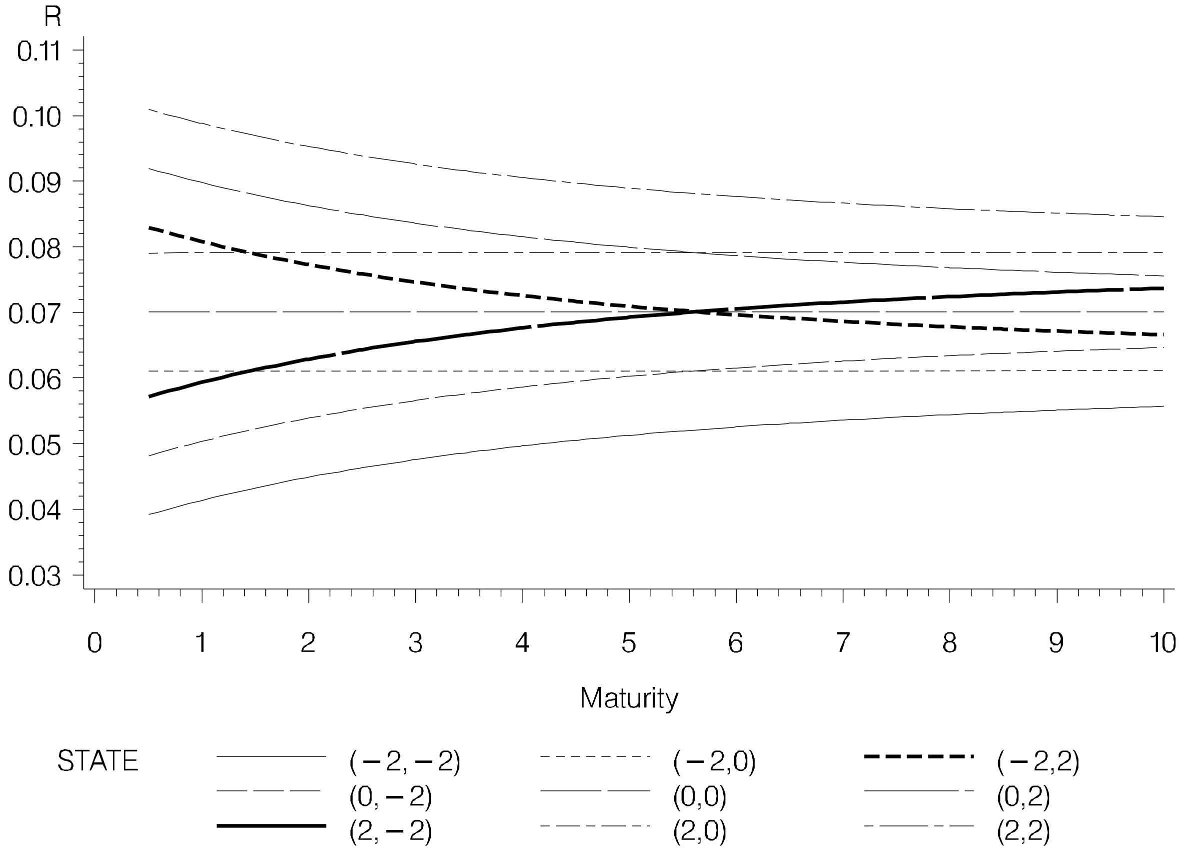
Yield Curves in the Vasicek-Model

This figure shows 5 possible yield curves after one year, when the initial term structure is flat at 7%. The 5 realizations of the Ornstein-Uhlenbeck process are given by $\tilde{x}(t) = -2, -1, 0, 1, 2$. The volatility and reversion parameter are $\sigma = 0.0121, \kappa = 0.2564$.



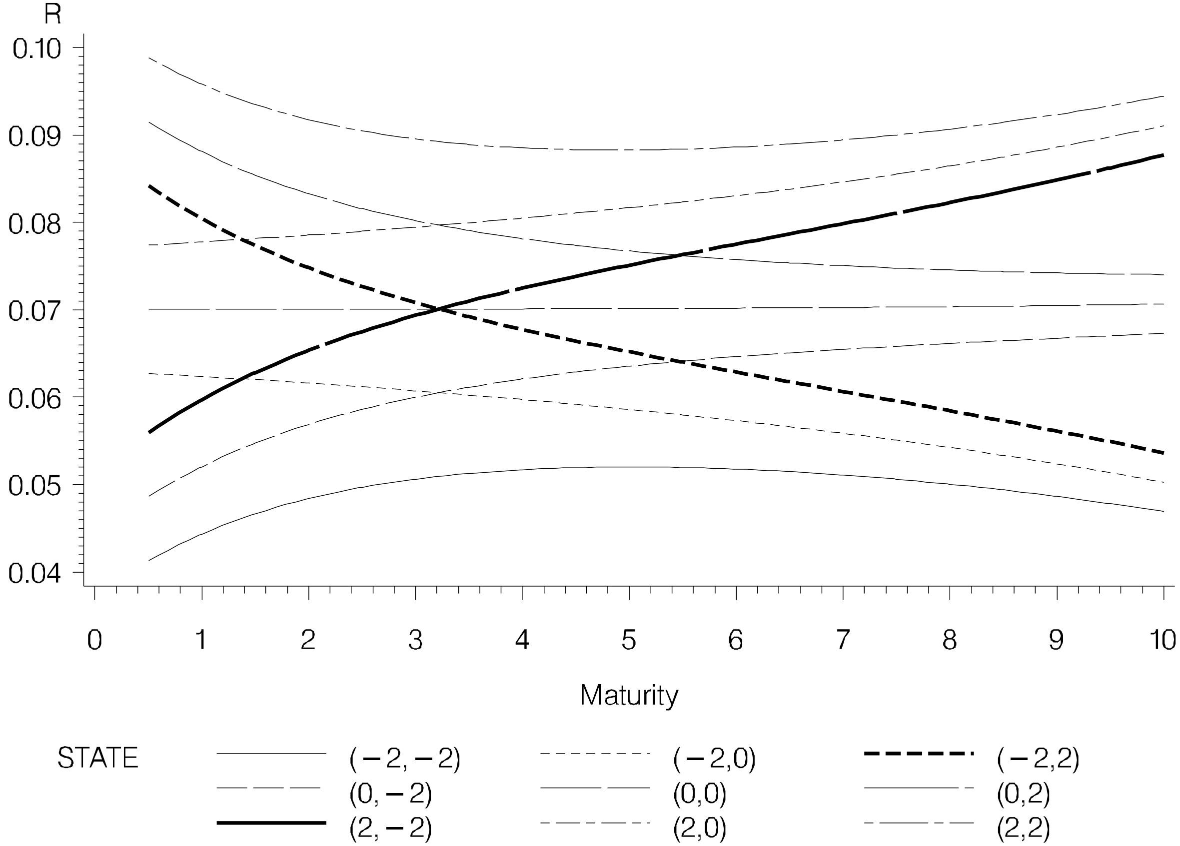
Yield Curves in the HJM model

This figure shows 9 possible yield curves after one year, when the initial term structure is flat at 7%. The 9 realizations of $(\tilde{W}_1(t), \tilde{x}(t))$ are given by $(-2,-2), (-2,0), \ldots, (2,0), (2,2)$. The volatility and reversion coefficients are $\sigma_1 = 0.0045, \sigma_2 = 0.0122, \kappa = 0.4416$.



Yield Curves in the 2FV-Model

This figure shows 9 possible yield curves after one year, when the initial term structure is flat at 7%. The 9 realizations of $(\tilde{x}_1(t), \tilde{x}_2(t))$ are given by $(-2,-2), (-2,0), \ldots, (2,0), (2,2)$. The volatility and reversion parameter are $\sigma_1 = 0.0035, \kappa_1 = 0.1859, \sigma_2 = 0.0129, \kappa_2 = 0.7662$ and yield a strong smile effect.

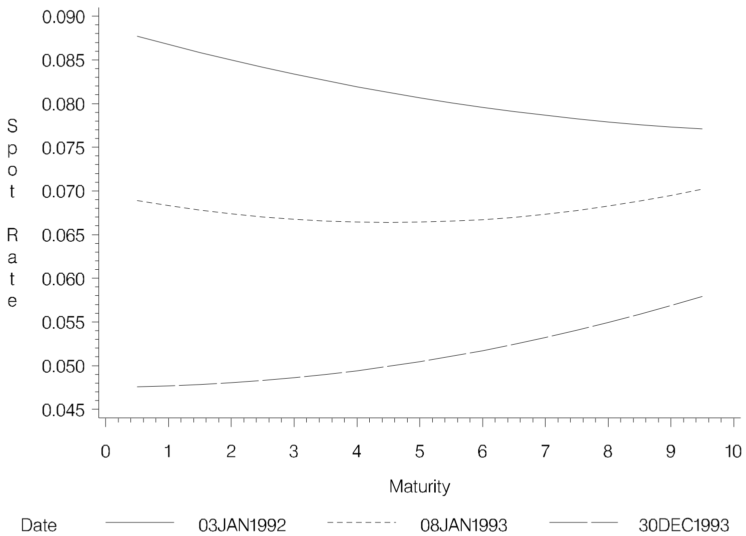


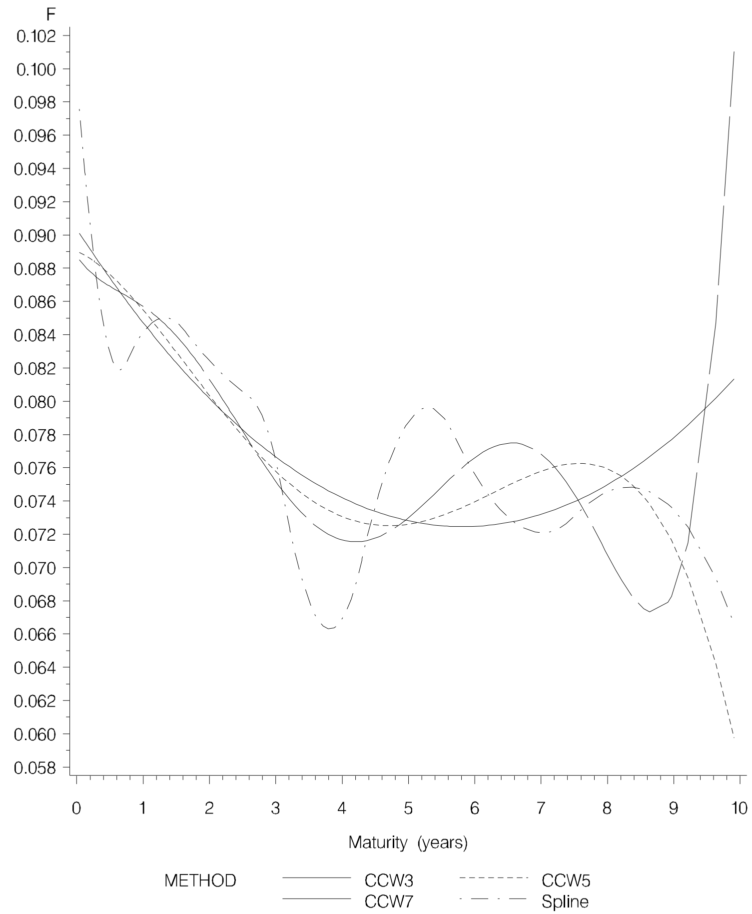
4 Volatility Estimation

Problems

- Term structure estimation problem.
- Using forward rates, bond prices or spot rates for the purpose of volatility estimation?
- Observable versus risk neutral distribution.
- Stationarity.

Term Structure of Spot Rates and Forward Rates on June 5, 92





Nonlinear Regression Approach

The absolute 'spot rate changes' in the 2FV-model

$$\Delta R(t,\tau) = R(t,t+\tau) - f(t-\Delta t,t,t+\tau) \sim \mathrm{N}\left(\mu^R(\Delta t,\tau), \sigma^R(\Delta t,\tau)^2\right)$$

are normally distributed with variance

$$\sigma^R(\Delta t,\tau)^2 = \frac{\sigma_1^2}{2\kappa_1^3}\left(\frac{e^{\kappa_1\tau}-1}{\tau}\right)^2\left(e^{2\kappa_1\Delta t}-1\right) + \frac{\sigma_2}{2\kappa_2^3}\left(\frac{1-e^{-\kappa_2\tau}}{\tau}\right)^2\left(1-e^{-2\kappa_2\Delta t}\right).$$

Based on a time series of N spot rate changes for M maturities $\tau_1,\ldots,\tau_M$ the volatility parameters $\sigma_1, \sigma_2, \kappa_1, \kappa_2$ can be estimated by the nonlinear regression

$$s^R(\Delta t,\tau_m)^2 = \frac{\sigma_1^2}{2\kappa_1^3}\left(\frac{e^{\kappa_1\tau_m}-1}{\tau_m}\right)^2\left(e^{2\kappa_1\Delta t}-1\right) + \frac{\sigma_2}{2\kappa_2^3}\left(\frac{1-e^{-\kappa_2\tau_m}}{\tau_m}\right)^2\left(1-e^{-2\kappa_2\Delta t}\right) + \epsilon_m, \qquad m = 1\ldots,M$$

where $s^R(\tau_m)^2$ denotes the sample variance of the spot rate changes with maturity τ_m. The computational method we use to determine the least squares estimators is the Gauss-Newton method due to Hartley (1961).

Data

- Period: January 1980 to December 1993
- 7 subperiods: 1/80–12/81, 1/82–12/83, ..., 1/92–12/93
- Weekly term structures of spot rates
- Estimated from prices of German T-Bonds and T-Notes
- Using the polynomial method of Chambers/Carleton/Waldman (1984) (Polynomial of order three)
- Spot rate maturities: $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, \ldots, 9\frac{1}{4}, 9\frac{1}{2}$ years

Parameter Estimates for the Ho/Lee and Vasicek-Model

Ho/Lee

Period	σ
1/80–12/81	0.0142 (0.0130;0.0155)
1/82–12/83	0.0095 (0.0086;0.0104)
1/84–12/85	0.0083 (0.0072;0.0095)
1/86–12/87	0.0079 (0.0075;0.0084)
1/88–12/89	0.0075 (0.0067;0.0083)
1/90–12/91	0.0080 (0.0076;0.0085)
1/92–12/93	0.0075 (0.0069;0.0082)

Vasicek

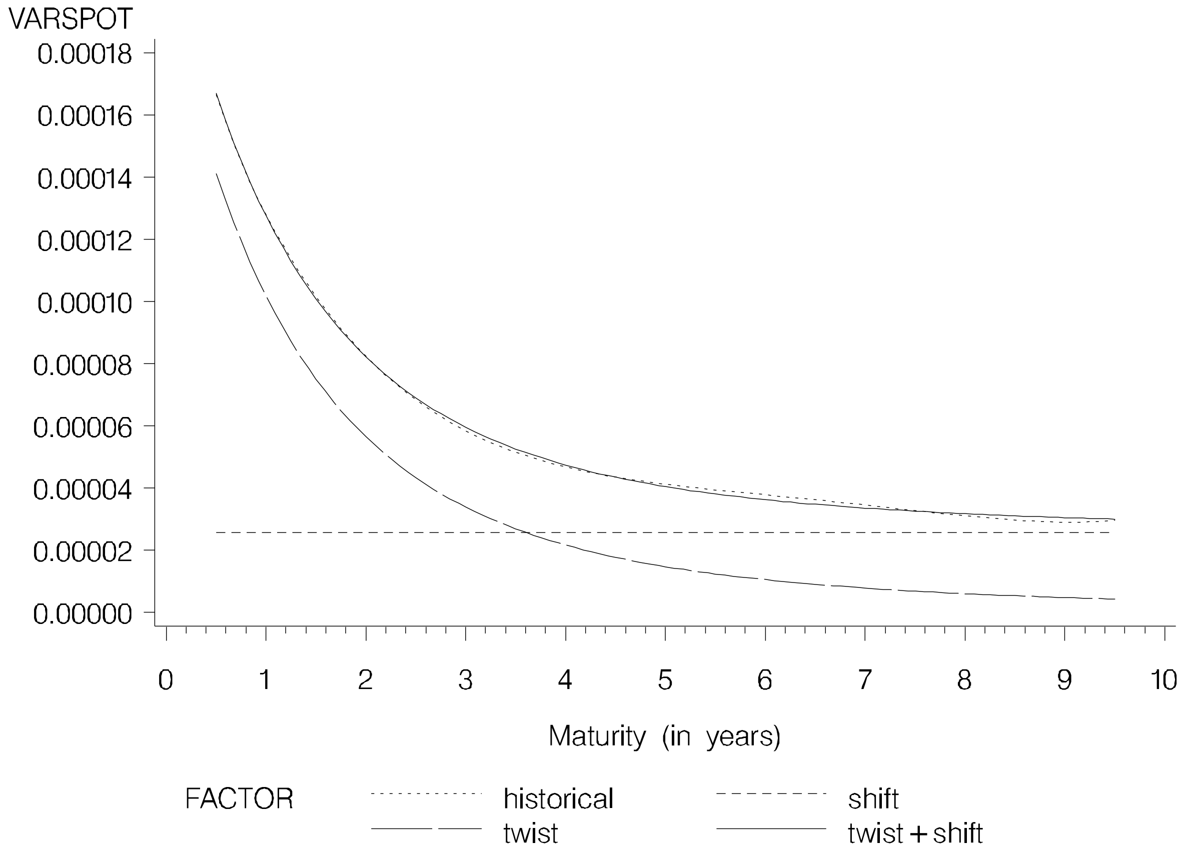
Period	σ	κ	R^2
1/80–12/81	0.0228 (0.0216;0.0239)	0.2583 (0.2181;0.2985)	0.8815
1/82–12/83	0.0162 (0.0155;0.0169)	0.3089 (0.2670;0.3478)	0.9277
1/84–12/85	0.0179 (0.0169;0.0188)	0.5482 (0.4735;0.6229)	0.9403
1/86–12/87	0.0084 (0.0075;0.0093)	0.0243 (-.0184;0.0670)	0.0262
1/88–12/89	0.0131 (0.0125;0.0137)	0.3281 (0.2873;0.3690)	0.9320
1/90–12/91	0.0081 (0.0071;0.0091)	0.0044 (-.0378;0.0466)	0.0009
1/92–12/93	0.0121 (0.0118;0.0124)	0.2564 (0.2361;0.2767)	0.9695

Parameter Estimates for the HJM-Model

Period	σ_1	σ_2	κ	R^2
1/80–12/81	0.0107 (0.0105;0.0109)	0.0255 (0.0250;0.0260)	0.8012 (0.7515;0.8509)	0.9962
1/82–12/83	0.0065 (0.0064;0.0066)	0.0174 (0.0172;0.0175)	0.6833 (0.6612;0.7054)	0.9990
1/84–12/85	0.0050 (0.0047;0.0054)	0.0193 (0.0186;0.0199)	0.9009 (0.812;40.9894)	0.9905
1/86–12/87	0.0074 (0.0070;0.0079)	0.0145 (0.0055;0.0235)	2.2911 (-.2212;4.8035)	0.5417
1/88–12/89	0.0051 (0.0050;0.0051)	0.0141 (0.0140;0.0142)	0.7149 (0.6946;0.7352)	0.9992
1/90–12/91	0.0076 (0.0071;0.0080)	0.0161 (0.0026;0.0297)	2.7859 (-.9698;6.5416)	0.4884
1/92–12/93	0.0045 (0.0043;0.0048)	0.0122 (0.0120;0.0124)	0.4416 (0.4022;0.4810)	0.9938

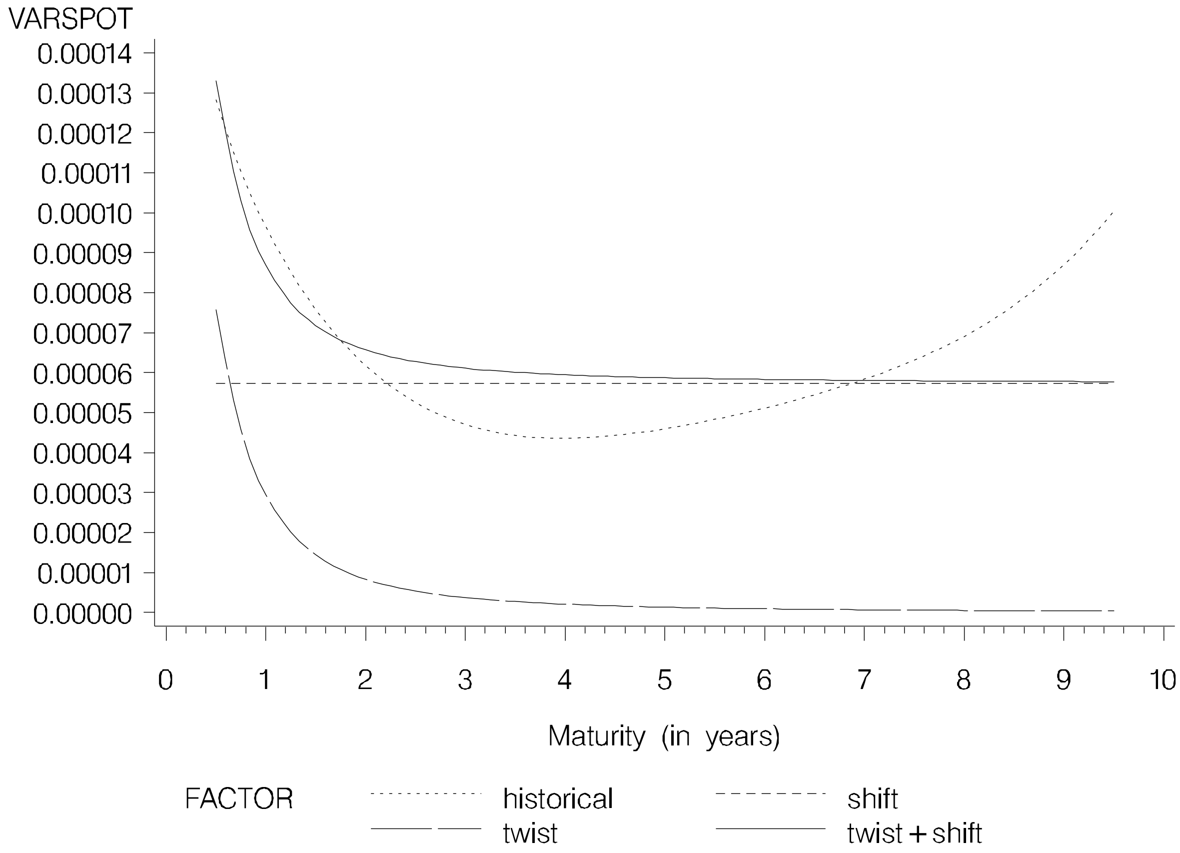
Explained Variance in the HJM model 1/88 – 12/89

This figure shows the spot rate variance explained by the HJM model compared to the historically spot rate variance for a subperiod with a reversion effect 1/88 – 12/89.



Explained Variance in the HJM model 1/90 – 12/91

This figure shows the spot rate variance explained by the HJM model compared to the historically spot rate variance for a subperiod with a strong smile effect 1/90 – 12/91.

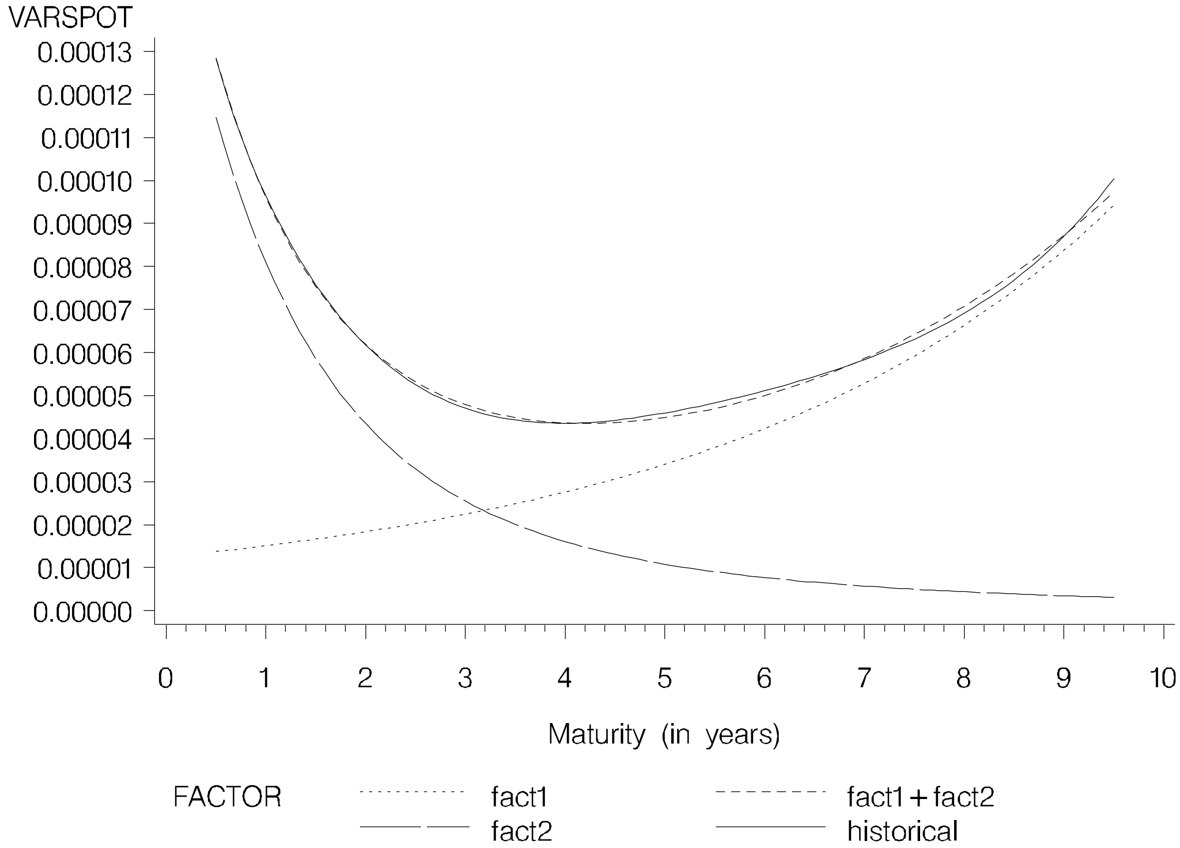


Parameter Estimates for the 2FV-Model

Period	σ_1	κ_1	σ_2	κ_2	R^2
1/80–12/81	0.0107 (0.0095;0.0119)	0.0000 (-.0247; 0.0247)	0.0255 (0.0250; 0.0260)	0.8012 (0.7048; 0.8976)	0.9962
1/82–12/83	0.0050 (0.0048; 0.0052)	0.0546 (0.0461; 0.0631)	0.0176 (0.0175; 0.0176)	0.5968 (0.5837; 0.6099)	0.9999
1/84–12/85	0.0019 (0.0016; 0.0022)	0.2088 (0.1834; 0.2342)	0.0191 (0.0190; 0.0193)	0.7225 (0.7006; 0.7444)	0.9991
1/86–12/87	0.0029 (0.0027; 0.0032)	0.2129 (0.1975; 0.2282)	0.0122 (0.0120; 0.0125)	0.5832 (0.5291; 0.6372)	0.9881
1/88–12/89	0.0051 (0.0047; 0.0055)	0.0000 (-.0181; 0.0181)	0.0141 (0.0140; 0.0143)	0.7152 (0.6734; 0.7570)	0.9992
1/90–12/91	0.0035 (0.0034; 0.0036)	0.1859 (0.1805; 0.1913)	0.0129 (0.0127; 0.0130)	0.7662 (0.7339; 0.7986)	0.9977
1/92–12/93	0.0033 (0.0018; 0.0054)	0.0546 (-.0539; 0.1631)	0.0124 (0.0120; 0.0128)	0.3949 (0.3177; 0.4721)	0.9939

Explained Variance in the 2FV-Model 1/90 – 12/91

This figure shows the spot rate variance explained by the two-factor Vasicek model (2FV) compared to the historically spot rate variance for a subperiod with a strong smile effect 1/90 – 12/91.



5 Implications for Option Pricing

Calls on Zero Bonds

In a locally arbitrage-free Gaussian K-factor model the price of a call on a zero bond $P(t,T)$ with strike price K and maturity t^* is:

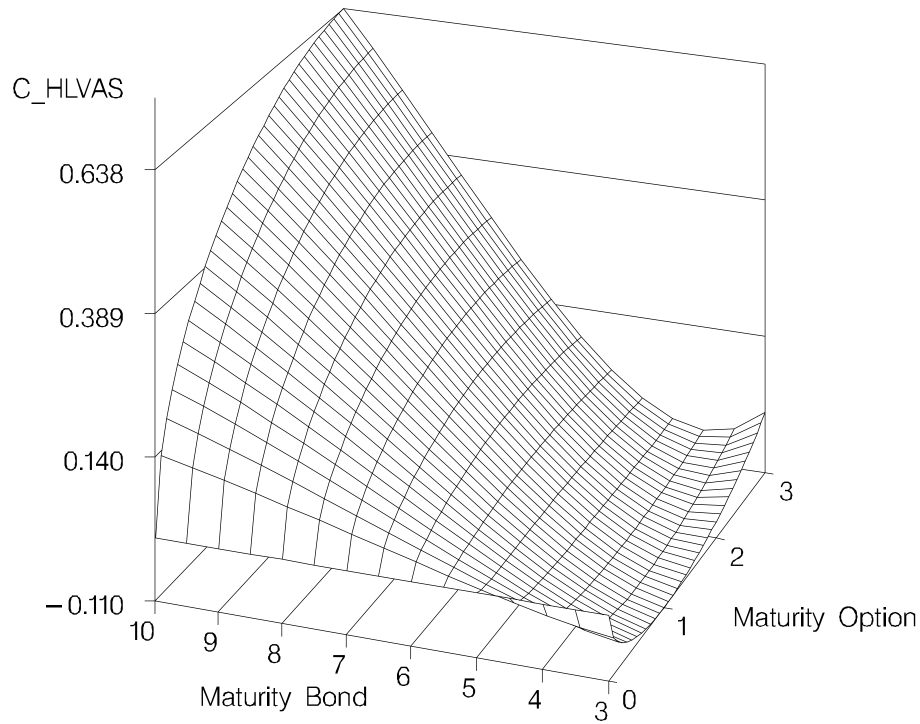
$$C(t,K,t^*,T) = P(t,T)N(d_1) - KP(t,t^*)N(d_2)$$

with

$$\begin{aligned} d_1 &\equiv \frac{1}{\nu(t,t^*,T)} \ln\left(\frac{P(t,T)}{P(t,t^*)K}\right) + \frac{1}{2}\nu(t,t^*,T) \\ d_2 &\equiv d_1 - \nu(t,t^*,T) \\ \nu(t,t^*,T)^2 &\equiv \sum_{k=1}^{K} \int_t^{t^*} \sigma_k(v,t^*,T)^2 dv = \sum_{k=1}^{K} \int_t^{t^*} [\sigma_k^p(v,T) - \sigma_k^p(v,t^*)]^2 dv \end{aligned}$$

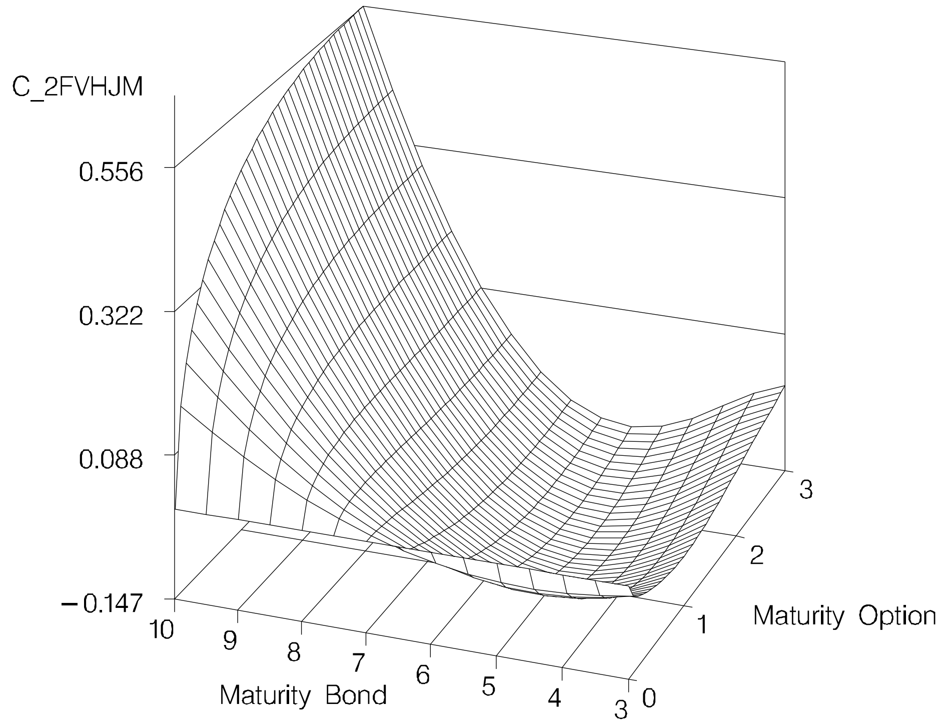
Ho/Lee-Call Price versus Vasicek-Call Price

This figure shows call price differences $C_{HL} - C_{Vas}$ for at-the-money options with a maturity of 0–3 years. The maturity of the underlying bonds with a face value of 100 varies between 3 and 10 years. The initial term structure is flat at 7%. The volatility parameters are $\sigma_{HL} = 0.0067, \sigma_{Vas} = 0.0121, \kappa = 0.2564$.



2FV-Call Price versus HJM-Call Price

This figure shows call price differences $C_{2FV} - C_{HJM}$ for at-the-money options with a maturity of 0–3 years. The maturity of the underlying bonds with a face value of 100 varies between 3 and 10 years. The initial term structure is flat at 7%. The volatility parameters are $\sigma_{1,HJM} = 0.0076, \sigma_{2,HJM} = 0.0161, \kappa_{HJM} = 2.7859$, bzw. $\sigma_{1,2FV} = 0.0064, \sigma_{2,2FV} = 0.0301, \kappa_{1,2FV} = 0.2307, \kappa_{2,2FV} = 0.0.0193$.



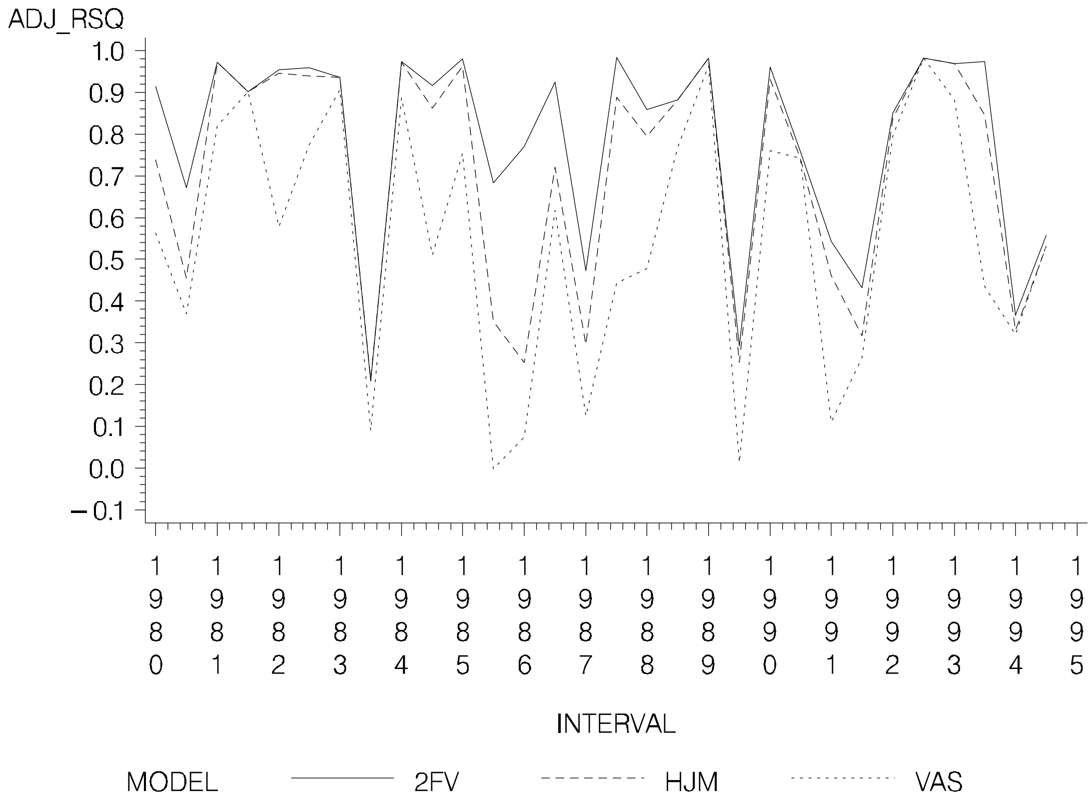
6 Conclusions

- Historical yield curve movement shows 3 basic patterns:
  a parallel *shift*, a *mean reversion*, and a *twist*.
- *One-factor models* such as Ho/Lee and Vasicek capture only the shift and mean reversion.
- *Two-factor models* are necessary to model the twist.
- Twists often induce a *smile effect* in the volatility structure.
- Smile effect can be explained by a *two-factor Vasicek model.*
- *Substantial option price differences* which can be attributed to the model characteristics.

A Modifications and Further Results

- Fitting not only spot rate variances, but also covariances
- Using exact theoretical covariances from integral equations rather than local covariances from differential equations
- Shorter estimation periods (1/2 year instead of 2 years) to improve predictable power

Goodness of Fit



Model	Mean(Adj-R^2)	STD(Adj-R^2)	Max(Adj-R^2)	Min(Adj-R^2)
2FV	0.7876	0.2349	0.9836	0.2090
HJM	0.7086	0.2763	0.9817	0.2121
VAS	0.5487	0.3104	0.9816	-.0017

Out-of-Sample-Test

Absolute Estimation Error (AE)

$$AE(\text{Cov}) = |\text{Cov}^{est}(\tau_1, \tau_2) - \text{Cov}^{true}(\tau_1, \tau_2)|$$
$$AE(\sigma) = |\sigma^{est}(\tau) - \sigma^{true}(\tau)|$$

- $\text{Cov}^{est}(\tau_1, \tau_2)$ and $\sigma^{est}(\tau)$ are based on the historical volatility structure of the 6 months prior to the estimation date
- $\text{Cov}^{true}(\tau_1, \tau_2)$ and $\sigma^{true}(\tau)$ are based on the spot rate changes in the 6-month-period following the estimation date

Variable	2FV	HJM	Vasicek	Ho/Lee
MAE(σ)	4.1672E-04	4.2099E-04	4.3029E-04	4.3640E-04
STDAE(σ)	3.6931E-04	3.7659E-04	3.8264E-04	3.7145E-04
MAE(Cov)	8.6713E-07	8.7000E-07	8.6951E-07	9.1639E-07
STDAE(Cov)	1.0265E-06	1.0281E-06	1.0267E-06	9.8780E-07

Modell	TYP	N	MC	MAE	STDAE	ME	STDE	MRAE	MRE	MIN	MAX
2FV	ALL	2494	3.336	0.275	0.368	-0.146	0.436	0.321	-0.201	0.000	2.805
2FV	CALL	1531	3.429	0.252	0.274	-0.087	0.362	0.299	-0.124	0.000	2.131
2FV	PUT	963	3.188	0.311	0.479	-0.240	0.518	0.357	-0.322	0.001	2.805
HJM	ALL	2494	3.336	0.280	0.370	-0.150	0.439	0.327	-0.200	0.000	2.878
HJM	CALL	1531	3.429	0.257	0.277	-0.090	0.367	0.306	-0.120	0.000	2.119
HJM	PUT	963	3.188	0.317	0.481	-0.245	0.521	0.361	-0.327	0.001	2.878
HL	ALL	2494	3.336	0.370	0.479	-0.089	0.599	0.374	-0.148	0.000	3.283
HL	CALL	1531	3.429	0.358	0.419	-0.025	0.550	0.320	-0.181	0.000	2.295
HL	PUT	963	3.188	0.390	0.561	-0.192	0.656	0.458	-0.096	0.000	3.283
VAS	ALL	2494	3.336	0.289	0.375	-0.169	0.442	0.337	-0.216	0.000	2.842
VAS	CALL	1531	3.429	0.264	0.286	-0.107	0.374	0.312	-0.131	0.000	2.155
VAS	PUT	963	3.188	0.328	0.481	-0.267	0.518	0.378	-0.352	0.001	2.842