Problem set 5

Asset pricing

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Purpose of this exercise

- This exercise gives an introduction into the C-CAPM for specific income processes.
- We want to show the relationship between asset prices and consumption growth.
- Intuitively, when the consumption path (or consumption growth) is very volatile agents want to invest to smooth consumption.
- This higher demand for assets makes them more expensive (prices increase).
- Similarly, a larger coefficient of β means that we are more likely to invest (give up some of today‘s consumption), thus asset prices will increase.
- Recall that $0 < \beta < 1$ and that larger values of β mean that the agent is less impatient.

General remarks on the C-CAPM

- We solve a household maximization problem similar to what we have seen before.
- However, the direction of our arguments is reversed in the C-CAPM.
- We do not want to determine the consumption path over time given some interest rates.
- We want to determine the interest rate (or sometimes the asset price) given a consumption path.
- We find that the covariance of the asset return with the *stochastic discount factor* determins the risk premium.
- Sometimes the stochastic discount factor is also called “pricing kernel”.

This problem

- In the first problem we also deal with the issue of asset pricing by the C-CAPM.
- However, we are given a specific function for the consumption growth process and want to give explicit solutions.
- The next problem will then be more general again.

Maximization problem

- The problem is

$$\max_{c_t, c_{t+1}, a} U(c_t) + \beta \mathbb{E}_t U(c_{t+1}) \tag{1}$$

subject to

$$\begin{aligned} c_t + p_t a &= y_t \\ c_{t+1} &= y_{t+1} + \underbrace{(p_{t+1} + d_{t+1})}_{\equiv x_{t+1}} a \end{aligned}$$

- Where we defined $x_{t+1} \equiv p_{t+1} + d_{t+1}$.
- By employing the usual steps we come up with the Euler equation.

Deriving the optimality condition

- The Lagrangian is

$$\begin{aligned}\mathcal{L} =& U(c_t) + \beta \mathbb{E}_t U(c_{t+1}) \\ &+ \lambda_t (y_t - c_t - p_t a) + \mathbb{E}_t \left[\lambda_{t+1} (y_{t+1} + x_{t+1} a - c_{t+1}) \right]. \end{aligned} \tag{2}$$

- The FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t} = U'(c_t) - \lambda_t \overset{!}{=} 0 \tag{I}$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \mathbb{E}_t \left[\beta U'(c_{t+1}) - \lambda_{t+1} \right] \overset{!}{=} 0 \tag{II}$$

$$\frac{\partial \mathcal{L}}{\partial a} = \mathbb{E}_t \left[-\lambda_t p_t + \lambda_{t+1} x_{t+1} \right] \overset{!}{=} 0. \tag{III}$$

The optimality condition

- Rearranging the FOCs gives

$$\lambda_t = U'(c_t) \tag{3}$$
$$\mathbb{E}_t \lambda_{t+1} = \beta \mathbb{E}_t U'(c_{t+1}) \tag{4}$$
$$p_t \lambda_t = \mathbb{E}_t \left[\lambda_{t+1} x_{t+1}\right]. \tag{5}$$

- Plugging (3) and (4) into (5) gives the Euler equation

$$p_t U'(c_t) = \mathbb{E}_t \left[\beta U'(c_{t+1}) x_{t+1}\right]. \tag{6}$$

Unsing the period utility function

- We rewrite it to

$$p_t = \mathbb{E}_t\left(\beta \frac{U'(c_{t+1})}{U'(c_t)} x_{t+1}\right).$$

- Using the given period utility function we get

$$p_t c_t^{-\gamma} = \beta \mathbb{E}_t\left(c_{t+1}^{-\gamma} x_{t+1}\right).$$

- Interpretation:

  LHS: Costs of buying one more asset (p_t valued by marginal utility of consumption)

  RHS: Benefits of buying one more asset (payoff x_{t+1} discounted and weighted by marginal utility)

  - Costs equal benefits in the optimum.

One period bond

- We set $p_t = q_t$, the equation becomes

$$q_t = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right].$$

- We now use the assumption that consumption growth is log normally distributed $\log \left(\frac{c_{t+1}}{c_t} \right) \sim \mathcal{N}(\mu, \sigma^2)$.
- Using the hint on the problem set we can compute

$$q_t = \beta \mathrm{e}^{-\gamma\mu + \frac{\gamma^2}{2}\sigma^2}$$

or

$$\log q_t = \log \beta - \gamma\mu + \frac{\gamma^2}{2}\sigma^2.$$

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Interpretation

$$\log q_t = \log \beta - \gamma\mu + \frac{\gamma^2}{2}\sigma^2.$$

- We have derived the price of a risk free bond.
- For this assumption about consumption growth prices are constant over time.
- The higher the impatience the lower is the price of the bond.
  - $\Rightarrow$ If everyone wants to consume today, it takes lower prices to induce the agents to buy the bond.
- Prices are low if consumption growth is high.
  - $\Rightarrow$ It pays agents to consume less today

  in order to invest today and to consume more tomorrow.

Consumption growth

- Now assume that consumption growth is characterized by

$$\log\left(\frac{c_{t+1}}{c_t}\right) = (1-\rho)\mu + \rho\log\left(\frac{c_t}{c_{t-1}}\right) + \varepsilon_{t+1},$$

with $\varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

- Using this we get

$$q_t = \beta \mathrm{e}^{-\gamma\left[(1-\rho)\mu+\rho\log\left(\frac{c_t}{c_{t-1}}\right)\right]+\frac{\gamma^2}{2}\sigma^2}.$$

or

$$\log q_t = \log\beta - \gamma\left[(1-\rho)\,\mu + \rho\log\left(\frac{c_t}{c_{t-1}}\right)\right] + \frac{\gamma^2}{2}\sigma^2.$$

Interpretation

- Time varying bond price.
  - $\Rightarrow$ This is more realistic.
- If current consumption growth is high then q_t is low.
  - $\Rightarrow$ High consumption growth today means expected high growth tomorrow (we see this from the specified consumption growth process).
  - $\Rightarrow$ High growth means that consumption in the future will be larger.
  - $\Rightarrow$ Since agents want to smooth consumption we want to borrow today.
  - $\Rightarrow$ Borrowing means that we sell the asset in order to consume.
- Due to smoothing motives agents borrow against future growth thus q_t decreases.
- Interpretation of the remaining parameters is the same as under log normality of consumption growth.

Stochastic discount factor

- Next we define the stochastic discount factor to be

$$M_{t+1} \equiv \beta \frac{U'(c_{t+1})}{U'(c_t)}. \tag{7}$$

- Sometimes it is also called "pricing kernel".
- Hence, we can express the optimality condition as

$$p_t = \mathbb{E}_t \left[M_{t+1} x_{t+1} \right].$$

- We rewrite this to

$$p_t = \mathbb{E}_t(M_{t+1})\mathbb{E}_t(x_{t+1}) + Cov(M_{t+1}, x_{t+1}).$$

The risky asset

- Replacing the riskless asset‘s price we get

$$p_t = q_t \mathbb{E}_t(x_{t+1}) + Cov(M_{t+1}, x_{t+1}).$$

- Substituting again the expression for M_{t+1} yields

$$p_t = q_t \mathbb{E}_t(x_{t+1}) + \frac{Cov(\beta U'(c_{t+1}), x_{t+1})}{U'(c_t)}.$$

- p_t is lowered (increased) if its payoff covaries positively (negatively) with consumption.
- An asset whose return covaries positively with consumption makes the consumption stream more volatile. It requires a lower price to induce the agent to buy such an asset.
- Agents want assets that covariate negatively with consumption.

Why all this?

- In this problem we have seen how to price assets according to the C-CAPM.
- We do this because we want to structurally explain asset prices and returns.
- In this exercise we did not explain the risk premium.
- However, this is also possible using the C-CAPM.
- But the empirical evidenve is weak.
- Example: the *equity premium puzzle*.

Equity premium puzzle

- The puzzle is that...
  - ... the equity premium is too large to be explained by the covariance of consumption growth with stock returns (which is quite low).
- The risk premium can only be explained when assuming a degree of risk aversion which is too big to be plausible.
- For illustration recall the example from the lecture...
  - ... Investors would have to be indifferent between a lottery equally likely to pay \$50,000 or \$100,000 (an expected value of \$75,000) and a certain payoff between \$51,209 and \$51,858 (the two last numbers correspond to measures of risk aversion equal to 30 and 20).
- There are some approaches that try to solve the puzzle but none of them can solve it fully.
- For example *habit formation* or *Epstein Zin* preferences...

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Purpose of this exercise: Expectation hypothesis of TS

- Why do we consider this exercise?
- We want to deal with the issue of the *term structure*.
- Therefore we need to know, what we mean by "upward- or downward sloping" term structure.
- To make the point we choose the simplest example of two periods and two assets.
- Consider first the (unrealistic) case of certainty.
- We could either invest in a two period asset which yields $1 + r^l$ or we could invest two times in the one period asset which yields $1 + r_1^s$ in the first period and $1 + r_2^s$ in the second period.

Perfect capital market

- Since we have assumed certainty and the capital market is perfect both investment opportunities must yield the same return, this means that
$$1 + r^l = (1 + r_1^s)(1 + r_2^s). \tag{8}$$
- Why must this be the case?
- Suppose both expressions would not be equal, then nobody would invest in the asset with the lower return.
- They must be equal, otherwise arbitrage would be possible.
- Note that in the following we assume that there is no risk premium.

Uncertainty

- Now suppose that there is uncertainty in the economy.
- The returns r^l and r_1^s are known in period 1, but the return r_2^s is unknown, thus our condition above changes to

$$1 + r^l = (1 + r_1^s)\mathbb{E}_1(1 + r_2^s). \tag{9}$$

- Of course we always have that $1 + r^l > 1 + r_1^s$ and $1 + r^l > 1 + r_2^s$.
- However, we can construct some artificial short term interest rate that is implied by $1 + r^l$.
- This means that we search for a short term interest rate $\overline{r}$ that is constant over both periods and yields the same compounded return as r^l.
- Hence, the following condition must hold

$$1 + r^l = (1 + \overline{r})(1 + \overline{r}) = (1 + \overline{r})^2. \tag{10}$$

Uncertainty

- Since this constructed interest rate is a one period interest rate we can compare it to the other short term interest rates.
- What does it mean when $\overline{r} > r_1^s$?
- If this is the case we must have that $\overline{r} < \mathbb{E}_1 r_2^s$.
  (you can actually derive the result explicitly)
- And in turn this means that $\mathbb{E}_1 r_2^s > r_1^s$
- This would be the case of an upward sloping term structure curve.
- This means that markets expect interest rate to increase.
- Of course in the opposite case we would have a downward sloping term structure.

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Risk premium

- However, recall that we have assumed that there is no risk premium.
- This makes a difference because investing in the two period asset is riskless and investing in the two one-period assets is risky.
- Hence, in order to invest in the risky alternative agents demand for a risk premium on the return r_2^s.
- Note that empirically there are deviations from that hypothesis.
- One important point are bubbles.
- The discussion above is very stylized, in reality there are more than just two assets.
- The following problem tries to explain the risk premium in the above sense.

Maximization problem

- The maximization problem is

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{11}$$

subject to

$$c_t + L_{1,t} + L_{2,t} \leq y_t + L_{1,t-1} R_{1,t-1} + R_{2,t-2}. \tag{12}$$

- We set up the Lagrangian to solve this problem

$$\begin{aligned}\mathcal{L} = & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ U(c_t) \\ & + \lambda_t \left(y_t + L_{1,t-1} R_{1,t-1} + R_{2,t-2} - c_t + - L_{1,t} - L_{2,t} \right) \Big\}\end{aligned} \tag{13}$$

FOCs

- The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \left[U'(c_t) - \lambda_t \right] \overset{!}{=} 0 \tag{I}$$

$$\frac{\partial \mathcal{L}}{\partial L_{1,t}} = -\beta^t \lambda_t + \mathbb{E}_t \beta^{t+1} \lambda_{t+1} R_{1,t} \overset{!}{=} 0. \tag{II}$$

- Solving (I) for λ yields

$$\lambda_t = U'(c_t).$$

- Plugging this into (II) gives

$$-\beta^t U'(c_t) + \mathbb{E}_t \beta^{t+1} U'(c_t + 1) R_{1,t} = 0$$

or

$$U'(c_t) = \beta R_{1,t} \mathbb{E}_t U'(c_t + 1). \tag{14}$$

Optimality condition $L_{1,t}$

- Note that we can do the last step because $R_{1,t}$ is known as of period t, thus $\mathbb{E}_t(R_{1,t}) = R_{1,t}$.
- (14) is the Euler equation for the short-term interest rate.
- It equates marginal utility today with marginal utility tomorrow discounted by β and multiplied by the short term rate.
- The household cannot improve its utility in the optimum by shifting consumption intertemporally.

Optimality condition $L_{2,t}$

- In order to get the optimality condition for the two period bond we calculate the following derivative

$$\frac{\partial \mathcal{L}}{\partial L_{2,t}} = -\beta^t \lambda_t + \beta^{t+2} \mathbb{E}_t \left[\lambda_{t+2} R_{2,t} \right] \stackrel{!}{=} 0. \qquad \text{(III)}$$

- We rewrite this into

$$\lambda_t = \beta^2 R_{2,t} \mathbb{E}_t(\lambda_{t+2}).$$

- Substituting (I) gives

$$U'(c_t) = \beta^2 R_{2,t} \mathbb{E}_t U'(c_{t+2}). \qquad (15)$$

Optimality condition $L_{2,t}$

- (15) is the Euler equation for the long-term interest rate.
- It equates marginal utility today with marginal utility tomorrow discounted by β and multiplied by the long-term rate.
- The household cannot improve its utility in the optimum by shifting consumption intertemporally.

Asset pricing

- We want to derive an expression for $\frac{1}{R_{1,t}}$ and $\frac{1}{R_{2,t}}$.
- Using the optimality conditions this is straight forward.
- In both cases we just divide by the gross return and by the marginal utility of consumption today.
- The results are

$$\frac{1}{R_{1,t}} = \beta \mathbb{E}_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \right] \tag{OPB}$$

$$\frac{1}{R_{2,t}} = \beta^2 \mathbb{E}_t \left[\frac{U'(c_{t+2})}{U'(c_t)} \right]. \tag{TPB}$$

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Relationship between one- and two period bonds

- Next we show that we can derive a relationship between the one- and the two- period bonds.
- Therefore we use (OPB) and forward it one period

$$\frac{1}{R_{1,t}} = \beta \mathbb{E}_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \right] \tag{OPB}$$

$$\frac{1}{R_{1,t+1}} = \beta \mathbb{E}_{t+1} \left[\frac{U'(c_{t+2})}{U'(c_{t+1})} \right].$$

We rewrite this expression to

$$\mathbb{E}_{t+1} \left[\frac{U'(c_{t+2})}{U'(c_t)} \right] = \frac{1}{\beta R_{1,t+1}} \mathbb{E}_{t+1} \left[\frac{U'(c_{t+1})}{U'(c_t)} \right]. \tag{16}$$

Relationship between one- and two period bonds

- Note that the crucial step was to multiply by $\mathbb{E}_{t+1}\left[\frac{U'(c_{t+1})}{U'(c_t)}\right]$ on both sides of the equation.
- Apply expectation at date t to (16)

$$\mathbb{E}_t\left[\frac{U'(c_{t+2})}{U'(c_t)}\right] = \mathbb{E}_t\left[\frac{1}{\beta R_{1,t+1}}\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)\right]. \tag{17}$$

- Note that here we have used the law of iterated expectations ($\mathbb{E}_t\mathbb{E}_{t+1}x_{t+2} = \mathbb{E}_t x_{t+2}$).
- We now substitute (16) into (TPB), this yields

$$\frac{1}{R_{2,t}} = \beta^2\mathbb{E}_t\left[\frac{U'(c_{t+2})}{U'(c_t)}\right] \tag{TPB}$$

$$\frac{1}{R_{2,t}} = \beta^2\mathbb{E}_t\left[\frac{1}{\beta R_{1,t+1}}\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)\right]. \tag{18}$$

Relationship between one- and two period bonds

- Rewriting (18) yields the result

$$
\begin{aligned}
\frac{1}{R_{2,t}} &= \beta^2 \mathbb{E}_t \left[\frac{1}{\beta R_{1,t+1}} \left(\frac{U'(c_{t+1})}{U'(c_t)} \right) \right] \\
\frac{1}{R_{2,t}} &= \mathbb{E}_t \left[\beta \frac{1}{R_{1,t+1}} \left(\frac{U'(c_{t+1})}{U'(c_t)} \right) \right] .
\end{aligned}
\tag{19}
$$

Covariance expression

- Recall that

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y),$$

where X and Y are arbitrary random variables.

- Rewriting this gives

$$\mathbb{E}(XY) = Cov(X,Y) + \mathbb{E}(X)\mathbb{E}(Y).$$

- Hence we can rewrite (19) to

$$\frac{1}{R_{2,t}} = \mathbb{E}_t\left[\beta\frac{1}{R_{1,t+1}}\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)\right] \tag{19}$$

$$\frac{1}{R_{2,t}} = \mathbb{E}_t\left(\frac{1}{R_{1,t+1}}\right)\mathbb{E}_t\left(\beta\frac{U'(c_{t+1})}{U(c_t)}\right) + Cov_t\left[\frac{1}{R_{1,t+1}}, \beta\frac{U'(c_{t+1})}{U'(c_t)}\right]. \tag{20}$$

Relationship between one- and two period bonds

- The last step is to replace $\mathbb{E}_t\left(\beta\frac{U'(c_{t+1})}{U'(c_t)}\right)$ by (OPB)

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}}\mathbb{E}_t\left(\frac{1}{R_{1,t+1}}\right) + Cov_t\left[\frac{1}{R_{1,t+1}}, \beta\frac{U'(c_{t+1})}{U'(c_t)}\right]. \qquad (21)$$

Risk neutral households

- Suppose now that the household is risk neutral.
- Thus, we could come up with our solution immediately.
- We know that the household is indifferent between lotteries with an expected outcome of μ and a certain outcome μ.
- For this reason the household should also be indifferent between holding a one- or two-period asset.
- However, here we will show the result explicitly.
- Risk neutral households are expressed by a linear utility function, for example

$$U(c_t) = a + bc_t.$$

Risk neutral households and marginal utility

- Marginal utilities for this household is thus

$$U'(c_t) = b$$
$$U'(c_{t+1}) = b.$$

- Hence, we have found that the ratio between the utilities equals unity

$$\frac{U'(c_{t+1})}{U'(c_t)} = \frac{b}{b} = 1.$$

- Note that the covariance between a random variable and a constant is equal to zero.

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Risk neutral households and marginal utility

- Using that finding with our final result (21) we get

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}}\mathbb{E}_t\left(\frac{1}{R_{1,t+1}}\right) + \tag{21}$$

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}}\mathbb{E}_t\left(\frac{1}{R_{1,t+1}}\right) + Cov_t\left[\frac{1}{R_{1,t+1}}, \beta 1\right]$$

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}}\mathbb{E}_t\left(\frac{1}{R_{1,t+1}}\right). \tag{22}$$

- Our result shows that the term structure of risk neutral agents is flat.

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Risk premium

- The risk premium in this case would be the difference between the return of this asset and the return for this asset when the agent is risk neutral.
- This is because when the agent is risk neutral she/he does not care about risk and thus does not ask for a premium.
- Thus we can compute it as (21) minus (22).
- The result is just

$$Cov_t\left[\frac{1}{R_{1,t+1}}, \beta\frac{U'(c_{t+1})}{U'(c_t)}\right]. \tag{23}$$

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