Problem set 5

The decentralized economy and log-linearization

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General remarks

- In this problem set we deal with the issue of uncertainty.
- In combination with utility functions we need to impose an additional restriction.
- We assume that preferences over uncertain outcomes can be represented by an *expected utility form*.
- We call these functions von Neumann-Morgenstern expected utility functions.
- Although this is a crucial assumption, in our computations we do not have to care about it.
- You should have covered this in your micro lecture.



Why do we care about uncertainty?

- The first reason why to care about uncertainty is that it makes our models more realistic.
- Particularly agents seem to be *risk averse* which means that they dislike risk.
- Consider the following alternatives
 - 1 You get 500.000 with certainty.
 - 2 You get a lottery that pays 1.000.000 with probability 0.5 and 0 else.
- Most of us will prefer the first alternative...
 - ... although both have the same expected payoff (namely 500.000).
- Since in our macroeconomic models we deal with agents' behavior under uncertainty we have to think about this issue.



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Asset pricing

- Another reason closely linked to the first one is that we want to give explanations to empirical observations of financial markets.
- For example, we find that risky assets (take stocks for example) have a higher expected return compared to riskless assets (e.g. bonds).
- We want to explain this phenomenon using our structural models of the economy.
- We deduce implications from basic assumptions about agents preferences including their attitude towards risk.
- Generally we expect that agents demand a higher return if they are confronted with risk compared to a riskless asset.



Overview

- We want to characterize agents' attitude towards risk using utility functions.
- Part of the material you will see here should be familiar to you from basic microeconomics or finance courses.
- We will relate risk aversion to the *certainty equivalent* and give an intuition about risk averse utility functions.
- Using this we will price assets using the *consumption capital asset pricing model* (C-CAPM).
- Note that this asset pricing model is different from the "ordinary" CAPM you know from finance.



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Prerequisites

We will see that in order to compute the limit of the function we need L'Hôspital's rule:
Let g(x₀) and f(x₀) be both differentiable at x₀ and let g(x₀) = f(x₀) = 0. The limit of the quotient of both functions is then given by

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

• This means in plain words that if the limit of the nominator and the denominator are both zero, we can compute the limit of the ratio of the derivatives instead.



Limit of power utility

• We want to compute the limit



- Since the nominator and the denominator approach zero we cannot determine the limit in one step.
- Therefore we have to use L'Hôspital's rule.

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} \stackrel{L'H}{=} \lim_{\sigma \to 1} \frac{-1c^{1-\sigma} \ln c}{-1} = \ln c.$$

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Differentiation rule

• Note that we used the following differentiation rule

$$f(x) = a^{g(x)} \Rightarrow f'(x) = g'(x)a^{g(x)}\ln a.$$

- To get an intuition for the limit of the function we have just computed you can plot the power utility function for values that approach unity.
- You will find that indeed the power utility becomes log utility.
- In the next problem we will see what is captured by σ .



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Problem 2 (Risk aversion)

CRRA utility function

• For the first utility function the relevant derivatives are

$$U'(W) = W^{-\sigma}$$
$$U''(W) = -\sigma W^{-\sigma-1}$$

The coefficient of absolute risk aversion is then

$$ARA = -\frac{-\sigma W^{-\sigma-1}}{W^{-\sigma}} = \frac{\sigma}{W}$$

• The coefficient of relative risk aversion is

$$RRA = -\frac{-\sigma W^{-\sigma-1}}{W^{-\sigma}}W = \sigma.$$



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CRRA

- As we have seen from the derivation above this utility function exhibits a constant measure of relative risk aversion.
- This means that the degree of relative risk aversion does not depend on the function argument (in this case *W*).
- We often refer to this kind of utility functions as constant relative risk aversion or CRRA utility functions.
- Note that risk aversion requires $\sigma > 0$.
- The next function we will analyze is the logarithmic function.
- We have seen in the problem before that the logarithmic function is a limiting case of the power utility function for the case *σ* → 1.
- Thus, we could guess that the measure of relative risk aversion is equal to 1.



Log CRRA

• The relevant derivatives for this function are

$$U'(W) = \frac{1}{W}$$
$$U''(W) = -\frac{1}{W^2}.$$

• The coefficient of absolute risk aversion is then

$$ARA = -\frac{-\frac{1}{W^2}}{\frac{1}{W}} = \frac{1}{W}.$$

• The coefficient of relative risk aversion is

$$RRA = -\frac{-\frac{1}{W^2}}{\frac{1}{W}}W = 1 > 0.$$

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CRRA=1

- As we already guessed before, the relative risk aversion measure for the log utility function equals unity.
- Thus, we have again a coefficient of relative risk aversion that is constant.
- We also call this kind of functions CRRA utility function.
- This emphasizes that this function is just the limiting case of the power utility function where *σ* = 1.



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CARA

- Next we consider the CARA utility function.
- The relevant derivatives are

$$U'(W) = \rho e^{-\rho W}$$
$$U''(W) = -\rho^2 e^{-\rho W}$$

• The coefficient of absolute risk aversion is then

$$ARA = -\frac{-\rho^2 e^{-\rho W}}{\rho e^{-\rho W}} = \rho.$$

• The coefficient of relative risk aversion is

$$RRA = -\frac{-\rho^2 e^{-\rho W}}{\rho e^{-\rho W}} W = \rho W.$$



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CARA

- Hence, this exponential utility function exhibits a constant absolute risk aversion.
- The short form for those functions is CARA.
- The coefficient of absolute risk aversion is constant (does not vary with *W*) and equal to *ρ*.
- For risk aversion we must then have that $\rho > 0$.



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Problem 2 (Risk aversion)



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Problem 2 (Risk aversion)

Properties of utility functions

- In order to have risk averse agents the utility function must be concave.
- This means that the second derivative must be negative.
- If a function is concave we know by Jensen's inequality that

$$U\left[\mathbb{E}(W)\right] \ge \mathbb{E}\left[U(W)\right]. \tag{JI}$$

and vice versa.

- This means that the agent would get a higher utility level when consuming the certain expected value of a lottery compared to consuming the lottery consisting of the two utility values.
- We will examine this graphically.
- Consider a lottery where the outcome is $W + \varepsilon$ with probability p or $W \varepsilon$ with probability 1 p.

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Why ARA and RRA?

- Thus, one could argue that we only need to look at concavity, which is determined by the second derivative.
- However, we want to have a measure which is independent of positive linear transformations.
- This means that the function

$$V(W) = a + bU(W)$$

with b > 0 should exhibit the same degree of risk aversion as U(W).

• For this reason we apply the concept of ARA and RRA.



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Remarks

- In our cases we consider utility functions with U'(W) > 0.
- This property is called strict monotonicity ("more is always better").
- Strictly concave utility functions are then characterized by U''(W) < 0.
- Strictly convex utility functions are then characterized by U''(W) > 0.
- Linear utility functions are then characterized by U''(W) = 0.



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Non concave utility

- Think about the implications of convexity and linearity for...
 - ... the certainty equivalent (we will cover this in problem 3).
 - ... the measures of absolute and relative risk aversion.
- Note that concave utility function are also necessary for consumption smoothing.
- If a utility function is concave we prefer consuming some intermediate level today and tomorrow to consuming a low level today and a low level tomorrow.
- If utility would be linear we would not care if we consume today or tomorrow.
- Convince yourself about this with a graph of a utility function.
- Hence, concave utility functions are also necessary to explain consumption smoothing in the case of certainty.



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Coefficient of risk aversion

- In order to show that the utility function exhibits risk aversion we derive the coefficient of absolute risk aversion.
- The relevant derivatives are

$$U'(W) = \frac{1}{2}W^{-\frac{1}{2}}$$
$$U''(W) = -\frac{1}{4}W^{-\frac{3}{2}}$$

• The coefficient of absolute risk aversion is

ARA =
$$-\frac{-\frac{1}{4}W^{-\frac{3}{2}}}{-\frac{1}{2}W^{\frac{1}{2}}} = \frac{1}{2}\frac{1}{W}.$$

• The coefficient of relative risk aversion is

$$RRA = -\frac{-\frac{1}{4}W^{-\frac{3}{2}}}{\frac{1}{2}W^{-\frac{1}{2}}}W = \frac{1}{2}$$



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ARA, RRA and risk aversion

- Both expressions are positive (as long as *W* is positive), thus the agent is risk averse.
- Now we want to compute the certainty equivalent.
- In order to do this we think of the definition of the certainty equivalent.
- The certainty equivalent of a lottery *L* is the riskless payoff that makes the agent indifferent between this payoff and lottery *L*.
- In our case we must have that

$$U(C) \stackrel{!}{=} \mathbb{E}[U(W)] = \frac{1}{2}U(4) + \frac{1}{2}U(16).$$



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Computing the certainty equivalent

• Thus the certainty equivalent is

$$\sqrt{C} = \frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{16}$$
$$\Leftrightarrow \sqrt{C} = 1 + 2$$
$$\Leftrightarrow C = 9.$$

- By definition of the certainty equivalent the agent is indifferent between the certainty equivalent and the lottery.
- We can show that the agent is risk averse by comparing the certainty equivalent *C* to the expected value *μ*.
- If *C* is smaller than μ the agent is risk averse.
- If *C* is larger than μ the agent is risk seeking.
- If *C* is equal to μ the agent is risk neutral.



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Comparing *C* with μ

• The expected value is

$$\mu = \frac{1}{2}16 + \frac{1}{2}4 = 10.$$

- In our case we have $C = 9 < 10 = \mu$.
- Hence, the agent is risk averse.
- Recall the agent is indifferent between getting 9 without uncertainty or getting a lottery which has expected payoff of 10.
- Note that we have seen the case of a discrete probability function with only two outcomes.
- We could make this more realistic (and maybe more difficult) by assuming some continuous density function.



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Purpose of this exercise

- This exercise gives an introduction into the C-CAPM for specific income processes.
- We want to show the relationship between asset prices and consumption growth.
- Intuitively, when the consumption path (or consumption growth) is very volatile agents want to invest to smooth consumption.
- This higher demand for assets makes them more expensive (prices increase).
- Similarly, a larger coefficient of *β* means that we are more likely to invest (give up some of today's consumption), thus asset prices will increase.
- Recall that 0 < β < 1 and that larger values of β mean that the agent is less impatient.

General remarks on the C-CAPM

- We solve a household maximization problem similar to what we have seen before.
- However, the direction of our arguments is reversed in the C-CAPM.
- We do not want to determine the consumption path over time given some interest rates.
- We want to determine the interest rate (or sometimes the asset price) given a consumption path.
- We find that the covariance of the asset return with the *stochastic discount factor* determins the risk premium.
- Sometimes the stochastic discount factor is also called "pricing kernel".



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This problem

- In the first problem we also deal with the issue of asset pricing by the C-CAPM.
- However, we are given a specific function for the consumption growth process and want to give explicit solutions.
- The next problem will then be more general again.



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Maximization problem

• The problem is

$$\max_{c_t,c_{t+1},a} U(c_t) + \beta \mathbb{E}_t U(c_{t+1})$$

subject to

$$c_{t+1} = y_t$$

$$c_{t+1} = y_{t+1} + \underbrace{(p_{t+1} + d_{t+1})}_{\equiv x_{t+1}} a_{t+1}$$

- Where we defined $x_{t+1} \equiv p_{t+1} + d_{t+1}$.
- By employing the usual steps we come up with the Euler equation.



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Deriving the optimality condition

• The Lagrangian is

$$\mathcal{L} = U(c_t) + \beta \mathbb{E}_t U(c_{t+1}) + \lambda_t (y_t - c_t - p_t a) + \mathbb{E}_t \left[\lambda_{t+1} (y_{t+1} + x_{t+1} a - c_{t+1}) \right].$$
(3)

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t} = U'(c_t) - \lambda_t \stackrel{!}{=} 0 \tag{I}$$
$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \mathbb{E}_t \left[\beta U'(c_{t+1}) - \lambda_{t+1} \right] \stackrel{!}{=} 0 \tag{II}$$
$$\frac{\partial \mathcal{L}}{\partial a} = \mathbb{E}_t \left[-\lambda_t p_t + \lambda_{t+1} x_{t+1} \right] \stackrel{!}{=} 0. \tag{III}$$



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The optimality condition

• Rearranging the FOCs gives

$$\lambda_{t} = U'(c_{t})$$

$$\mathbb{E}_{t}\lambda_{t+1} = \beta \mathbb{E}_{t}U'(c_{t+1})$$

$$p_{t}\lambda_{t} = \mathbb{E}_{t} [\lambda_{t+1}x_{t+1}].$$
(4)
(5)
(6)

• Plugging (4) and (5) into (6) gives the Euler equation

$$p_t U'(c_t) = \mathbb{E}_t \left[\beta U'(c_{t+1}) x_{t+1} \right].$$



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Unsing the period utility function

• We rewrite it to

$$p_t = \mathbb{E}_t \left(\beta \frac{U'(c_{t+1})}{U'(c_t)} x_{t+1} \right).$$

• Using the given period utility function we get

$$p_t c_t^{-\gamma} = \beta \mathbb{E}_t \left(c_{t+1}^{-\gamma} x_{t+1} \right)$$

• Interpretation:

- LHS: Costs of buying one more asset (p_t valued by marginal utility of consumption)
- RHS: Benefits of buying one more asset (payoff x_{t+1} discounted and weighted by marginal utility)
 - Costs equal benefits in the optimum.



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One period bond

• We set $p_t = q_t$, the equation becomes

$$q_t = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$$

- We now use the assumption that consumption growth is log normally distributed $\ln \left(\frac{c_{t+1}}{c_t}\right) \sim \mathcal{N}(\mu, \sigma^2)$.
- Using the hint on the problem set we can compute

$$q_t = \beta \mathrm{e}^{-\gamma \mu + \frac{\gamma^2}{2}\sigma^2}$$

or

$$\ln q_t = \ln \beta - \gamma \mu + \frac{\gamma^2}{2} \sigma^2.$$



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Interpretation

$$\ln q_t = \ln \beta - \gamma \mu + \frac{\gamma^2}{2} \sigma^2.$$

- We have derived the price of a risk free bond.
- For this assumption about consumption growth prices are constant over time.
- The higher the impatience the lower is the price of the bond.
 - \Rightarrow If everyone wants to consume today, it takes lower prices to induce the agents to buy the bond.
- Prices are low if consumption growth is high.
 - \Rightarrow It pays agents to consume less today

in order to invest today and to consume more tomorrow.



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Consumption growth

• Now assume that consumption growth is characterized by

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = (1-\rho)\mu + \rho \ln\left(\frac{c_t}{c_{t-1}}\right) + \varepsilon_{t+1},$$

with $\varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

• Using this we get

$$q_t = \beta \mathrm{e}^{-\gamma \left[(1-\rho)\mu + \rho \ln \left(\frac{c_t}{c_{t-1}} \right) \right] + \frac{\gamma^2}{2} \sigma^2}$$

or

$$\ln q_t = \ln \beta - \gamma \left[(1 - \rho) \, \mu + \rho \ln \left(\frac{c_t}{c_{t-1}} \right) \right] + \frac{\gamma^2}{2} \sigma^2$$

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Interpretation

- Time varying bond price.
 - \Rightarrow This is more realistic.
- If current consumption growth is high then q_t is low.
 - ⇒ High consumption growth today means expected high growth tomorrow (we see this from the specified consumption growth process).
 - \Rightarrow High growth means that consumption in the future will be larger.
 - ⇒ Since agents want to smooth consumption we want to borrow today.
 - \Rightarrow Borrowing means that we sell the asset in order to consume.
- Due to smoothing motives agents borrow against future growth thus *q*_t decreases.
- Interpretation of the remaining parameters is the same as under log normality of consumption growth.



Stochastic discount factor

• Next we define the stochastic discount factor to be

$$M_{t+1} \equiv \beta \frac{U'(c_{t+1})}{U'(c_t)}.$$

- Sometimes it is also called "pricing kernel".
- Hence, we can express the optimality condition as

$$p_t = \mathbb{E}_t \left[M_{t+1} x_{t+1} \right].$$

• We rewrite this to

$$p_t = \mathbb{E}_t(M_{t+1})\mathbb{E}_t(x_{t+1}) + Cov(M_{t+1}, x_{t+1}).$$



(8)

The risky asset

• Replacing the riskless asset's price we get

$$p_t = q_t \mathbb{E}_t(x_{t+1}) + Cov(M_{t+1}, x_{t+1}).$$

• Substituting again the expression for M_{t+1} yields

$$p_t = q_t \mathbb{E}_t(x_{t+1}) + \frac{Cov(\beta U'(c_{t+1}), x_{t+1})}{U'(c_t)}$$

- *p_t* is lowered (increased) if its payoff covaries positively (negatively) with consumption.
- An asset whose return covaries positively with consumption makes the consumption stream more volatile. It requires a lower price to induce the agent to buy such an asset.
- Agents want assets that covariate negatively with consumption

Why all this?

- In this problem we have seen how to price assets according to the C-CAPM.
- We do this because we want to structurally explain asset prices and returns.
- In this exercise we did not explain the risk premium.
- However, this is also possible using the C-CAPM.
- But the empirical evidence is weak.
- Example: the *equity premium puzzle*.



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Equity premium puzzle

- The puzzle is that...
 - ... the equity premium is too large to be explained by the covariance of consumption growth with stock returns (which is quite low).
- The risk premium can only be explained when assuming a degree of risk aversion which is too big to be plausible.
- For illustration recall the example from the lecture...
 - ... Investors would have to be indifferent between a lottery equally likely to pay \$50,000 or \$100,000 (an expected value of \$75,000) and a certain payoff between \$51,209 and \$51,858 (the two last numbers correspond to measures of risk aversion equal to 30 and 20).
- There are some approaches that try to solve the puzzle but none of them can solve it fully.
- For example habit formation or Epstein Zin preferences...



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