1. **Power utility**

   Consider the power utility function

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}. \tag{1}$$

   Show that the power utility function (1) becomes the log-utility function ($U(c) = \log c$) as $\sigma \to 1$.

2. **Risk aversion**

   Consider the following utility functions

$$U(W) = \frac{W^{1-\sigma} - 1}{1 - \sigma} \tag{CRRA}$$

$$U(W) = \log(W) \tag{CRRA}$$

$$U(W) = -\mathrm{e}^{-\rho W}. \tag{CARA}$$

   (a) Compute the coefficients of absolute risk aversion

$$\mathrm{ARA}(W) = -\frac{U''(W)}{U'(W)} \tag{2}$$

   and the coefficient of relative risk aversion

$$\mathrm{RRA}(W) = -\frac{U''(W)}{U'(W)} W \tag{3}$$

   of each function.

   (b) Choose some paramter value $0 < \alpha, \beta, \gamma, \sigma, \rho < 1$ and plot the functions.

   (c) Which property of the utility functions ensures that the agent is risk avers? Why do we use the ARA and the RRA measures instead?

3. **Certainty equivalent and risk aversion**

   The utility function of an agent is of the form

$$U(W) = \sqrt{W} \tag{4}$$

(a) Show that the agent is risk averse by calculating either the coefficient of relative risk aversion or the coefficient of absolute risk aversion. Consider $W \geq 0$.

(b) What is the certainty equivalent? Assume that the agent faces a lottery with the following discrete probability distribution

$$\begin{aligned} P(W = 4) = P(W = 16) &= 0.5 \\ P(W \neq 4 \wedge W \neq 16) &= 0. \end{aligned}$$

Compute the certainty equivalent C given the distribution and the utility function. Show that the agent is risk averse using the certainty equivalent.

4. **Portfolio risk/diversification**

Consider a market with two assets A and B. Asset A pays an expected return of $\mathbb{E}(r_A) = \mu_A$ and asset B pays an expected return of $\mathbb{E}(r_B) = \mu_B$. Both assets are risky such that the variance of the first asset is $Var(r_A) = \sigma_A^2$ and the variance of the second asset is $Var(\sigma_B^2)$. There exists a third asset P (portfolio) that consists of both assets A and B, where λ is the fraction of asset A and $1 - \lambda$ is the fraction of asset B in the portfolio.

(a) Show that the return of the two assets equals a linear combination of the returns of both assets.

(b) Show that diversification lowers aggregate risk, i.e. show the linear combination of the standard deviation of the two assets is larger than the standard deviation of the portfolio.

(c) Show that if the two assets are perfectly negatively correlated ($\rho_{AB} = -1$) one can create the riskless asset by constructing a portfolio out of A and B. Derive the weights λ and $1 - \lambda$ for this portfolio as a function of the standard deviations σ_A and σ_B.

5. **Asset pricing**

Consider an agent with the following maximization problem.

$$\max_{c_t, c_{t+1}, a} U(c_t) + \beta \mathbb{E}_t U(c_{t+1}) \tag{5}$$

subject to

$$\begin{aligned} c_t + p_t a &= y_t \\ c_{t+1} &= y_{t+1} + (p_{t+1} + d_{t+1})a, \end{aligned}$$

Where c denotes consumption, p the price of the asset a, y is income, and d denote dividents from investin in the asset a. We define $x \equiv p+d$ to be the gross return of asset a.

Period utility is given by

$$U(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma},$$

with $\beta > 0$ and $\gamma > 0$. Furthermore, assume that consumption growth is log-normally distributed, i.e.

$$\log\left(\frac{c_{t+1}}{c_t}\right) \sim \mathcal{N}\left(\mu, \sigma^2\right).$$

Note that the expected value $\mathbb{E}(z^\alpha)$ of a log normally distributed variable z is

$$\mathbb{E}(z^\alpha) = \mathrm{e}^{\alpha\mu + \frac{\alpha^2}{2}\sigma^2},$$

where $\log z \sim \mathcal{N}(\mu, \sigma^2)$.

(a) Compute the first order condition.

(b) Assume that the agent invests in a riskless asset which has a price $p_t = q_t$. Use the consumption CAPM approach in order to derive an expression for the price q_t of the riskless asset. Derive also an expression for $\log q_t$.

(c) Assume now that

$$\log\left(\frac{c_{t+1}}{c_t}\right) = (1-\rho)\mu + \rho\log\left(\frac{c_t}{c_{t-1}}\right) + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2).$$

Using this assumption derive an expression for q_t.

(d) Assume now that the agent invests into a risky asset with price p_t and and payoff x_{t+1}. Use the first order condition to derive an expression for p_t dependent on q_t . How is the asset‘s price affected if its payoff covaries positively with consumption? Provide intuition for your findings.