Problem set 3

The decentralized economy and the permanent income hypothesis

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Constant interest rate

- In the first part of the question we discuss budget constraints when interest rates are asumed to be constant through time.
- This means that the interest rate r does not have a time subscript t.
- Note that this makes the analysis easier.
- Unlike the case of time varying interest rates we will not have to use products $\prod$.
- We start with the period budget constraint which says that we can either consume c_t or invest a_{t+1}.
- Recall that from consumption the agent gains utility whereas from investment she/he does not.
- For the consumption/investment decision the household uses income x_t and the return from selling last period's asset $(1+r)a_t$.

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From period to lifetime budget constraint

- The period budget constraint is given by

$$c_t + a_{t+1} = (1+r)a_t + x_t. \tag{PB}$$

- We rearrange this equation

$$a_t = \frac{1}{1+r}\left(c_t - x_t + a_{t+1}\right). \tag{1}$$

- Forwarding this expression one period yields

$$a_{t+1} = \frac{1}{1+r}\left(c_{t+1} - x_{t+1} + a_{t+2}\right).$$

- We now plug this equation into (1)

$$a_t = \frac{1}{1+r}\left[c_t - x_t + \frac{1}{1+r}\left(c_{t+1} - x_{t+1} + a_{t+2}\right)\right].$$

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Iterated substitution

- Rewriting this gives

$$a_t = \frac{1}{1+r}(c_t - x_t) + \left(\frac{1}{1+r}\right)^2 (c_{t+1} - x_{t+1} + a_{t+2}). \tag{2}$$

- We can then forward (1) one more period to substitute a_{t+2}.

$$a_{t+2} = \frac{1}{1+r}(c_{t+2} - x_{t+2} + a_{t+3}).$$

- Substituting this reltionship in (2) again gives

$$a_t = \frac{1}{1+r}(c_t - x_t) + \left(\frac{1}{1+r}\right)^2 \times \\ \times \left[c_{t+1} - x_{t+1} + \frac{1}{1+r}(c_{t+2} - x_{t+2} + a_{t+3})\right]. \tag{3}$$

Iterated substitution

- This we can write as

$$a_t = \frac{1}{1+r}(c_t - x_t) + \left(\frac{1}{1+r}\right)^2 (c_{t+1} - x_{t+1}) \\ + \left(\frac{1}{1+r}\right)^3 (c_{t+2} - x_{t+2} + a_{t+3}) .$$

- We could do this substitution for a_{t+3} one more time but we can already see how the expression evolves when we repeat this procedure an infinite number of times, the result is

$$a_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} (c_{t+s} - x_{t+s}) + \underbrace{\lim_{s\to\infty} \left(\frac{1}{1+r}\right)^{s+1} a_{t+s+1}}_{=0 \text{ (by assumption)}} .$$

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No-ponzi game condition

- The condition

$$\lim_{s\to\infty}\left(\frac{1}{1+r}\right)^{s+1} a_{t+s+1} = 0$$

is called *no-ponzi game condition*.

- Multiplying by $1+r$ and rearranging yields the lifetime budget constraint

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} c_{t+s} = (1+r)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} x_{t+s}. \qquad \text{(LB1)}$$

- From the lifetime budget constraint we can derive the period budget constraint again.

The lifetime budget constraint

- Now consider the lifetime budget constraint (LB1).

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} c_{t+s} = (1+r)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} x_{t+s}. \tag{LB1}$$

- The left hand side represents the present discounted value of consumption expenditures over the whole lifecycle of the agent.
- The right hand side consists of the present discounted value of income over the whole lifecycle of the agent plus $(1+r)a_t$.
- We can interpret the right hand side of this equation as household‘s (lifetime) wealth.

From lifetime- to period budget constraint

- We rewrite the lifetime budget contraint (LB1) as

$$a_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+r} \right)^{s+1} (c_{t+s} - x_{t+s}) .$$

- Now, we "extract" $c_t - x_t$ from the infinite sum

$$a_t = \frac{1}{1+r} (c_t - x_t) + \sum_{s=1}^{\infty} \left(\frac{1}{1+r} \right)^{s+1} (c_{t+s} - x_{t+s}) .$$

- Rewriting this gives

$$a_t = \frac{1}{1+r} (c_t - x_t) + \frac{1}{1+r} \underbrace{\sum_{s=0}^{\infty} \left(\frac{1}{1+r} \right)^{s+1} (c_{t+1+s} - x_{t+1+s})}_{=a_{t+1}} .$$

Back to the period budget constraint

- Substituting a_{t+1} gives
$$a_t = \frac{1}{1+r}(c_t - x_t) + \frac{1}{1+r}a_{t+1}.$$
- Multiplying by $1+r$ and rearranging yields the period budget constraint
$$c_t + a_{t+1} = (1+r)a_t + x_t. \tag{PB}$$

Non-constant interest rate

- Next, consider the more general case where the interest rates are not time-constant.
- A time subscript t as to be assigned to the interest rate r such that the period budget constraint becomes

$$c_t + a_{t+1} = (1 + r_t)a_t + x_t. \tag{4}$$

- With a non-constant interest rate mathematics become a bit more tedious but the general procedure stays the same.
- We first rewrite the period beudget constraint as

$$a_t = \frac{1}{1 + r_t}(c_t - x_t + a_{t+1}).$$

- Forward it one period

$$a_{t+1} = \frac{1}{1 + r_{t+1}}(c_{t+1} - x_{t+1} + a_{t+2}).$$

Non-constant interest rate PB to LB

- Substitute a_{t+1}

$$a_t = \frac{1}{1+r_t}\left[c_t - x_t + \frac{1}{1+r_{t+1}}(c_{t+1} - x_{t+1} + a_{t+2})\right]$$
$$\Leftrightarrow a_t = \frac{1}{1+r_t}(c_t - x_t) + \frac{1}{1+r_t}\frac{1}{1+r_{t+1}}(c_{t+1} - x_{t+1} + a_{t+2}).$$

- Similar to the case before we could do this repeatedly until we get

$$a_t = \sum_{s=0}^{\infty}\left(\prod_{j=0}^{s}\frac{1}{1+r_{t+j}}\right)(c_{t+s} - x_{t+s}) + \underbrace{\lim_{s\to\infty}\prod_{j=0}^{s}\frac{1}{1+r_{t+j}}a_{t+s+1}}_{=0}.$$

- Again we assume that the no-ponzi game condition holds.

Non-constant interest rate PB to LB

- Finally we arrive at the lifetime budget constraint by multiplying by $1 + r_t$ and adding the discounted income stream on both sides

$$a_t(1+r_t) + \sum_{s=0}^{\infty} \left(\prod_{j=0}^{s} \frac{1}{1+r_{t+j}} \right) x_{t+s} = \sum_{s=0}^{\infty} \left(\prod_{j=0}^{s} \frac{1}{1+r_{t+j}} \right) c_{t+s}.$$

- Of course the interpretation of this equation is the same as for constant interest rate.
- However, it is a more general expression for the lifetime budget constraint.
- Note also that assuming $r_t = r$ for all $t \geq 0$ in the above expression we get back to the time-constant interest rate representation above.

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Maximization problem

- The representative household maximizes

$$\max_{\{c_{t+s}\}_{s=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$$

subject to

$$c_t + a_{t+1} = (1+r)a_t + x_t. \tag{PB}$$

with period utility function

$$U(c_{t+s}) = c_{t+s} - \frac{\alpha}{2} c_{t+s}^2.$$

- The Lagrangian to this problem is

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ \beta^s U(c_{t+s}) + \lambda_{t+s} \left[(1+r)a_{t+s} + x_{t+s} - c_{t+s} - a_{t+s+1} \right] \right\}.$$

FOCs

- The first order conditions to the problem are

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \mathbb{E}_t \left[\beta^s \left(1 - \alpha c_{t+s}\right) - \lambda_{t+s}\right] \overset{!}{=} 0 \tag{I}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+s+1}} = \mathbb{E}_t \left[(1+r)\lambda_{t+s+1} - \lambda_{t+s}\right] \overset{!}{=} 0. \tag{II}$$

- Substituting the λs in (II) by an expression obtained from (I) gives

$$\mathbb{E}_t \left[\beta^s \left(1 - \alpha c_{t+s}\right)\right] = (1+r)\mathbb{E}_t \left[\beta^{s+1} \left(1 - \alpha c_{t+s+1}\right)\right]$$

$$\Leftrightarrow \mathbb{E}_t \left[\left(1 - \alpha c_{t+s}\right)\right] = (1+r)\beta\mathbb{E}_t \left[\left(1 - \alpha c_{t+s+1}\right)\right].$$

Euler equation

- Writing the expression for period t yields the Euler equation

$$(1-\alpha c_t) = (1+r)\beta \mathbb{E}_t\left[(1-\alpha c_{t+1})\right].$$

- When does expected consumption rise, i.e. when is the gross growth rate $\mathbb{E}_t c_{t+1}/c_t > 1$?
- Note that (omitting the expectations operator for a moment)

$$\frac{c_{t+1}}{c_t} = (1+g_c) > 1 \text{ if } g_c > 0$$
$$\frac{c_{t+1}-c_t}{c_t} = \frac{c_{t+1}}{c_t} - 1 = g_c,$$

  where g_c is the growth rate and $1+g$ is the gross growth rate of consumption.

- g_c is positive if $c_{t+1} > c_t$.

When does expected consumption rise?

- The Euler equation in rewritten form is

$$\mathbb{E}_t \frac{1-\alpha c_{t+1}}{1-\alpha c_t} = \frac{1}{(1+r)\beta} \lesseqqgtr 1.$$

- The expected growth rate is positive if $(1+r)\beta > 1$.
- The expected growth rate is negative if $(1+r)\beta < 1$.
- The expected growth rate is zero if $(1+r)\beta = 1$.
- Note that this result is valid for concave utility functions in general.

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Euler equation

- For simplicity we set $(1+r)\beta = 1$, the Euler equation as of period t is
$$c_t = \mathbb{E}_t c_{t+1}.$$
- Iterating forward (and using the *LIE*) we have
$$c_t = \mathbb{E}_t c_{t+1} = \mathbb{E}_t \mathbb{E}_{t+1} c_{t+2} = \mathbb{E}_t c_{t+2} = \cdots = \mathbb{E}_t c_i = \cdots .$$
- As we have already found in problem 2, expected consumption is constant over time.
- The lifetime budget constraint is
$$\mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i c_i = (1+r) a_0 + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i x_i. \tag{LB2}$$
- Substituting the Euler equation gives
$$\sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i c_t = (1+r) a_0 + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i x_i.$$

Solving for c_t

- Note that c_t does not depend on i, thus we can pull it out of the sum

$$c_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i = (1+r)a_0 + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i x_i.$$

$$\Leftrightarrow c_t \frac{1}{1 - \frac{1}{1+r}} = (1+r)a_0 + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i x_i.$$

$$\Leftrightarrow c_t \frac{1+r}{r} = (1+r)a_0 + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i x_i.$$

$$\Leftrightarrow c_t = \underbrace{r a_0 + \underbrace{\frac{r}{1+r} \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i x_i}_{\text{annuity value of lifetime income}}}_{\text{lifetime wealth}} .$$

Change in consumption

- Note that $r/(1+r)$ is the *marginal propensity to consume*.
- It tells us by how much current consumption is increased when lifetime wealth changes.
- Compare it to c_1 in the traditional Keynesian consumption function
$$C = c_0 + c_1 Y.$$
- We derive the change in consumption simply be substracting c_{t-1} from c_t
$$\Delta c_t = \frac{r}{1+r}\left[\mathbb{E}_t \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i x_i - \mathbb{E}_{t-1}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i x_i\right].$$

Change in the information set

- We have derived

$$\Delta c_t = \frac{r}{1+r}\left[\mathbb{E}_t \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i x_i - \mathbb{E}_{t-1}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i x_i\right].$$

- Consumption changes ($\Delta c_t \neq 0$) if an *unexpected* change in lifetime income has ocurred.
- This means if the lifetime income expected as of period t is different from the same lifetime income expected as of period $t-1$ the household changes consumption.
- Expected changes in income do *not* influence household‘s consumption decision.

Random walk income

- Assume income follows a random walk

$$x_t = x_{t-1} + \varepsilon_t.$$

- Start with period 0

$$x_1 = x_0 + \varepsilon_1.$$

- Iterating forward and plugging one equation into the other yields

$$x_2 = x_0 + \varepsilon_2 + \varepsilon_1.$$

- Doing this repeatedly gives

$$x_i = x_0 + \sum_{j=1}^{i} \varepsilon_j.$$

Expected change in lifetime in income

- Now, we compute expectations as of period t and $t-1$

$$\mathbb{E}_t x_i = x_0 + \sum_{j=1}^{t} \varepsilon_j$$

and

$$\mathbb{E}_{t-1} x_i = x_0 + \sum_{j=1}^{t-1} \varepsilon_j.$$

- Note that $\mathbb{E}_t \varepsilon_t = \varepsilon_t$, $\mathbb{E}_{t-1} \varepsilon_t = 0$.
- Plugging this result into the expression for Δc_t gives

$$\Delta c_t = \frac{r}{1+r} \left[\sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i \varepsilon_t \right].$$

The expression for Δc_t

- Since ε_t does not depend on i, we can write this as

$$\Delta c_t = \frac{r}{1+r}\frac{1}{1-\frac{1}{1+r}}\varepsilon_t = \frac{r}{1+r}\frac{1+r}{r}\varepsilon_t = \varepsilon_t.$$

- If income follows a random walk, a shock to income has a one to one impact on consumption.
- Note however, that we assumed that income follows a random walk, i.e. that shocks to income last forever and are thus very persistent.

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Distortionary taxes

- A tax is said to be distortionary if it changes the consumption decision.
- This means that a tax is distortionary if the relationship of consumption levels between two periods is affected by the tax.
- Thus, we analyze the Euler equation to decide if the tax is distortionary.
- In principle we could tax many things such as assets, the interest rate, or income.
- In this problem we consider a consumption tax τ_c.

The problem

- The consumer maximizes

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$(1+\tau_c)c_t + a_{t+1} = (1+r)a_t + x_t. \tag{TPB}$$

- The Lagrangian to this problem is

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \frac{c_{t+s}^{1-\sigma} - 1}{1-\sigma} + \cdots \right.$$

$$\left. \cdots + \lambda_{t+s} \left[(1+r)a_{t+s} + x_{t+s} - (1+\tau_c)c_{t+s} - a_{t+s+1} \right] \right\}$$

FOCs

- The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \mathbb{E}_t \beta^s \left[c_{t+s}^{-\sigma} - \lambda_{t+s}(1+\tau_c) \right] \stackrel{!}{=} 0 \qquad \text{(I)}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+s+1}} = \mathbb{E}_t \left(-\beta^s \lambda_{t+s} + \beta^{s+1}(1+r)\lambda_{t+s+1} \right) \stackrel{!}{=} 0 \qquad \text{(II)}$$

- Rewriting (I) gives

$$\mathbb{E}_t \lambda_{t+s} = \mathbb{E}_t \frac{c_{t+s}^{-\sigma}}{1+\tau_c} \Leftrightarrow \mathbb{E}_t \lambda_{t+s+1} = \mathbb{E}_t \frac{c_{t+s+1}^{-\sigma}}{1+\tau_c}$$

- Plugging this into (II) gives

$$\mathbb{E}_t \frac{c_{t+s}^{-\sigma}}{1+\tau_c} = (1+r)\beta \mathbb{E}_t \frac{c_{t+s+1}^{-\sigma}}{1+\tau_c}.$$

Euler equation

- $1 + \tau_c$ cancels on both sides.
- We write the Euler equation for period t (where $s = 0$)

$$c_t^{-\sigma} = (1 + r)\beta \mathbb{E}_t c_{t+1}^{-\sigma}.$$

- The tax τ_c does not appear.
- The Euler equation does not change compared to a situation without the consumption tax τ_c.
- Thus, the consumption tax is not distortionary.

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