Problem set 2

Difference equations and the sustainability of fiscal stance

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The sum

- Consider the following sum

$$S_n = \sum_{i=0}^{n-1} \alpha^i \beta. \tag{1}$$

- What is the result S_n for this sum?
- The solution will be given by function $S_n(n, \alpha, \beta)$.
- The derivation depends on the value of α.
- We consider two cases
  - $\alpha = 1$,
  - $\alpha \neq 1$.

$\alpha \neq 1$

- For $\alpha \neq 1$ we can derive a general formula.
- Therefore we take the original series

$$S_n = \beta \sum_{i=0}^{n-1} \alpha^i = \beta \left(\alpha^0 + \alpha^1 + \alpha^2 + \cdots + \alpha^{n-1} \right). \tag{2}$$

- We multiply the original series by α

$$\alpha S_n = \beta \sum_{i=0}^{n-1} \alpha^{i+1} = \beta \left(\alpha^1 + \alpha^2 + \cdots + \alpha^{n-1} + \alpha^n \right). \tag{3}$$

$\alpha \neq 1$

- Now substract (3) from (2) to get

$$(1-\alpha)S_n = \beta\left(\alpha^0 + \alpha^1 - \alpha^1 + \alpha^2 - \alpha^2 + \cdots + \alpha^{n-1} - \alpha^{n-1} - \alpha^n\right).$$

- Many terms cancel. What is left over is

$$(1-\alpha)S_n = \beta\left(1 - \alpha^n\right).$$

- Dividing by $1-\alpha$ yields the result

$$S_n = \sum_{i=0}^{n-1} \alpha^i \beta = \beta \frac{1-\alpha^n}{1-\alpha}. \tag{4}$$

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$\alpha = 1$

- For $\alpha = 1$ we can also derive a simple solution

$$
\begin{aligned}
S_n &= \sum_{i=0}^{n-1} 1^i \beta \\
&= \underbrace{(1 + 1 + 1 + \cdots + 1)}_{n \text{ times}} \beta \\
&= n\beta. \qquad (5)
\end{aligned}
$$

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$n \rightarrow \infty$

- The infinite sum ($n \rightarrow \infty$) can be derived from the results we obtained before.
- For $\alpha \neq 1$ we have derived the formular

$$S_n = \sum_{i=0}^{n-1} \alpha^i \beta = \frac{1-\alpha^n}{1-\alpha}. \tag{4}$$

- We now let $n \rightarrow \infty$, for $|\alpha| \geq 1$ this equation is not bounded (tends to $\pm\infty$).

$\Rightarrow$ We cannot derive a finite solution for this case.

- For $|\alpha| < 1$ the solution is

$$\lim_{n \rightarrow \infty} = S_\infty = \beta \frac{1}{1-\alpha}.$$

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Homogenous difference equations

- We only consider *linear* difference equations.
- This implies that a and b are constant over time.
- The first difference equation is called *homogenous* difference equation because $b = 0$

$$y_{t+s} = ay_{t+s-1}. \tag{6}$$

- We can shift the equation one period backwards such that

$$y_{t+s-1} = ay_{t+s-2}. \tag{7}$$

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Homogenous difference equations

- Plugging (7) into (6) yields

$$y_{t+s} = a\underbrace{ay_{t+s-2}}_{=y_{t+s-1}} = a^2 y_{t+s-2}.$$

- Doing the same for y_{t+s-2} and for y_{t+s-3} and so on gives

$$y_{t+s} = a^s y_t. \tag{8}$$

Stability of homogenous difference equations

- In order to examine stability properties of the difference equation we let $s \to \infty$.
- We say that a difference equation is stable if it decays to zero.
- Formally

$$\lim_{s\to\infty} y_{t+s} = \begin{cases} 0 & \Leftrightarrow \text{difference equation is stable} \\ \pm\infty & \Leftrightarrow \text{difference equation is unstable} \end{cases}.$$

- Thus, a homogenous difference equation is stable if and only if $|a| < 1$.
- Otherwise it is unstable.

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Non-homogenous difference equations

- Now we turn to the more complicated case, i.e. non-homogenous difference equations

$$y_{t+s} = ay_{t+s-1} + b.$$

- We use the same solution method to solve the equation.
- We write the equation for $t + s - 1$

$$y_{t+s-1} = ay_{t+s-2} + b.$$

- This is replaced into the original equation

$$y_{t+s} = a^2 y_{t+s-2} + ab + b.$$

Non-homogenous difference equations

- Doing the same calculation repeatedly we get

$$y_{t+s} = a^s y_t + \sum_{i=0}^{s-1} a^i b.$$

- The second part of this equation we have analyzed in problem 1.
- Assuming $|a| \neq 1$ we can rewrite the equation to

$$y_{t+s} = a^s y_t + b\frac{1-a^s}{1-a}.$$

Stability of inhomogenous difference equations

- In order to examine stability properties of the difference equation we let $s \to \infty$.
- We say that a difference equation is stable if it decays to some constant value.
- This value will be the *steady state* or *fix point* y^*.
- Formally

$$\lim_{s\to\infty} y_{t+s} = \begin{cases} y^* & \Leftrightarrow \text{difference equation is stable} \\ \pm\infty & \Leftrightarrow \text{difference equation is unstable} \end{cases}.$$

- Thus, a homogenous difference equation is stable if and only if $|a| < 1$.
- Otherwise it is unstable.

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Stability of non-homogenous difference equations

- If the non-homogenous difference equation is stable, it converges to its steady state.

$$\lim_{s\to\infty} y_{t+s} = \frac{b}{1-a} = y^*.$$

- The steady state is characterized by the fact that y_t does not change anymore, i.e. $y_t = y_{t+1} \equiv y^*$.
- To see that, consider the difference equation again

$$y_{t+s} = ay_{t+s-1} + b.$$

- Set $y_{t+s} = y_{t+s-1} = y^*$

$$y^* = ay^* + b$$
$$y^* = \frac{b}{1-a}.$$

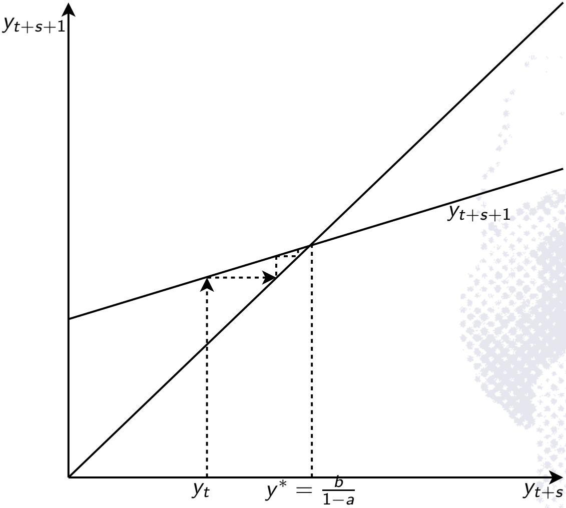


Figure: $0 < a < 1$

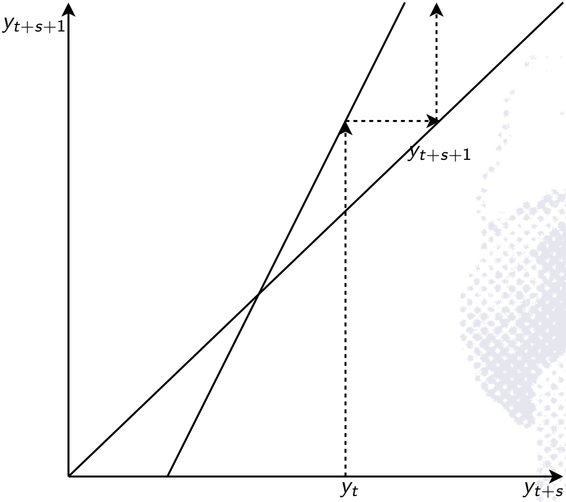
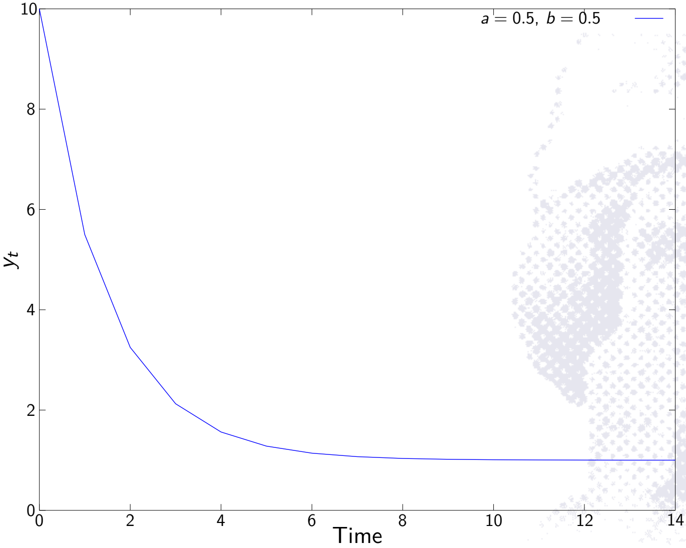


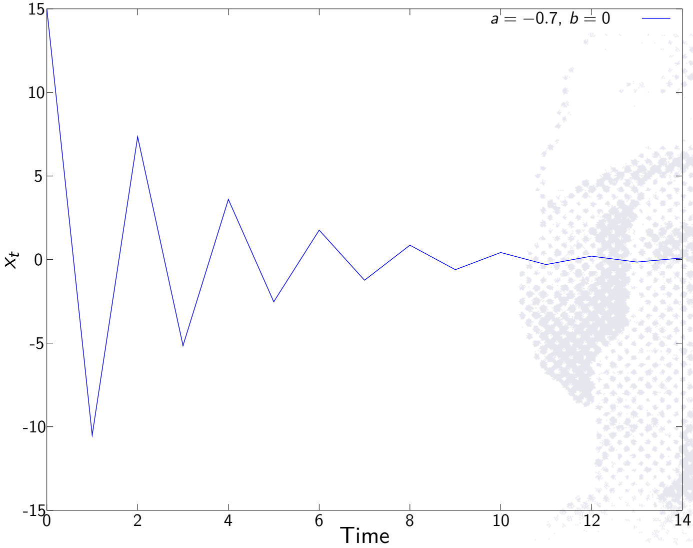
Figure: $a > 1$

Additional information

- The following slides contain two additional graphs.
- The graphs illustrate the dynamic behaviour of difference equations.
- At this point, we consider only stable difference equations ($|a| < 1$).
- We call such dynamic graphs *impulse response functions*.
- They describe the time path of y_{t+s} when we start at $y_t \neq y^*$.
- The reason why $y_t \neq y^*$ could be a shock ε_t to y_t.



$a = 0.5,\ b = 0.5$
10
8
6
4
2
0
y_t
0
2
4
6
8
10
12
14
Time


$a = -0.7,\ b = 0$
15
10
5
0
-5
-10
-15
x_t
0
2
4
6
8
10
12
14
Time

Contents

The equations

- In the lecture we derived the following equation

$$\frac{b_{t+1}}{y_{t+1}} = \frac{1}{\pi+\gamma}\frac{b_t}{y_t} + \frac{1}{\pi+\gamma}\frac{D_t}{y_t}, \tag{9}$$

  where we assumed a constant growth rate of GDP.
- We assume that the budget will be balanced from today on

$$\frac{D_{t+s}}{y_{t+s}} = 0 \quad \forall s \geq 0.$$

- The equation becomes

$$\frac{b_{t+1}}{y_{t+1}} = \frac{1}{\pi+\gamma}\frac{b_t}{y_t}, \tag{10}$$

  which is a homogenous difference equation.
- Note that $1/(\pi+\gamma)$ corresponds to a in problem 2.

The dynamics

- How long does it take until we reach $b_{t+s}/y_{t+s} = 0.6$?
- We forward the difference equation one period

$$\frac{b_{t+2}}{y_{t+2}} = \frac{1}{\pi+\gamma}\frac{b_{t+1}}{y_{t+1}}.$$

- Then, we substitute the original equation

$$\frac{b_{t+2}}{y_{t+2}} = \left(\frac{1}{\pi+\gamma}\right)^2 \frac{b_t}{y_t}.$$

- Doing this repeatedly we arrive at

$$\frac{b_{t+s}}{y_{t+s}} = \left(\frac{1}{\pi+\gamma}\right)^s \frac{b_t}{y_t}. \quad (11)$$

Calculating s

- Now, we are interested in how long it takes to come back to $b_{t+s}/y_{t+s} = 0.6$.
- This means, we want to compute s for this case

$$\begin{aligned}
\frac{b_{t+s}}{y_{t+s}} &= \left(\frac{1}{\pi+\gamma}\right)^s \frac{b_t}{y_t} \\
0.6 &= \left(\frac{1}{\pi+\gamma}\right)^s \frac{b_t}{y_t} \\
\log 0.6 &= s \log\left(\frac{1}{\pi+\gamma}\right) + \log\left(\frac{b_t}{y_t}\right) \\
s &= \frac{\log 0.6 - \log\left(\frac{b_t}{y_t}\right)}{\log\left(\frac{1}{\pi+\gamma}\right)}.
\end{aligned}$$

Example: Germany

- Consider the example of Germany.
- We have a GDP growth rate of $\pi + \gamma = 1.019$.
- The debt to GDP ratio is $b_t / y_t = 0.791$.
- Calculating s with this values gives

$$s = \frac{\log 0.6 - \log\left(\frac{b_t}{y_t}\right)}{\log\left(\frac{1}{\pi+\gamma}\right)} = \frac{\log 0.6 - \log\left(0.791\right)}{\log\left(\frac{1}{1.019}\right)} = 14.683 \rightarrow 15.$$

- It will take 15 years until Germany arrives at $b_{t+s} / y_{t+s} = 0.6$.

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Calculating s

- We can also calculate s by using Excel or Octave/Matlab.
- In order to do so we simulate the difference equation.
- The parameters of the difference equation are calibrated using actual data.
- From the simulated series we can easily see how long it takes until we arrive at $b_{t+s}/y_{t+s} \leq 0.6$.

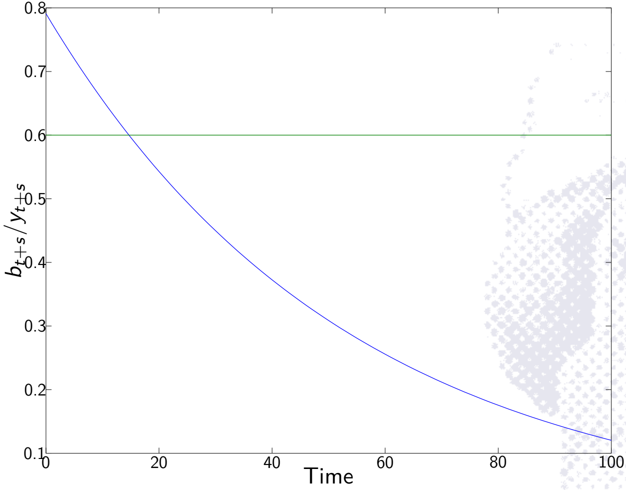

Figure: Evolution of b_{t+s}/y_{t+s} for Germany.

References

de la Fuente, Á. (2000).
Mathematical Methods and Models for Economists.
Cambridge University Press.

Wälde, K. (2009).
Applied Intertemporal Optimization.
Lecture Notes, University of Mainz, Available at
http://www.waelde.com/aio.

Wickens, M. (2008).
Macroeconomic Theory: A Dynamic General Equilibrium Approach.
Princeton University Press.